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On Nikodým and Rainwater sets for $ba(\mathcal{R})$ and a Problem of M. Valdivia

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Abstract. If $\mathcal{R}$ is a ring of subsets of a set Ω and $ba(\mathcal{R})$ is the Banach space of bounded finitely additive measures defined on $\mathcal{R}$ equipped with the supremum norm, a subfamily Δ of $\mathcal{R}$ is called a *Nikodým set* for $ba(\mathcal{R})$ if each set $\{\mu_\alpha : \alpha \in \Lambda\}$ in $ba(\mathcal{R})$ which is pointwise bounded on Δ is norm-bounded in $ba(\mathcal{R})$. If the whole ring $\mathcal{R}$ is a Nikodým set, $\mathcal{R}$ is said to have property (N), which means that $\mathcal{R}$ satisfies the Nikodým-Grothendieck boundedness theorem. In this paper we find a class of rings with property (N) that fail Grothendieck's property (G) and prove that a ring $\mathcal{R}$ has property (G) if and only if the set of the evaluations on the sets of $\mathcal{R}$ is a so-called *Rainwater set* for $ba(\mathcal{R})$. Recalling that $\mathcal{R}$ is called a (wN)-ring if each increasing web in $\mathcal{R}$ contains a strand consisting of Nikodým sets, we also give a partial solution to a question raised by Valdivia by providing a class of rings without property (G) for which the relation $(N) \Leftrightarrow (wN)$ holds.

1. Preliminaries

Let $\mathcal{R}$ be a ring of subsets of a nonempty set Ω, χ_A the characteristic function of a set $A \in \mathcal{R}$ and $\ell_0^\infty(\mathcal{R}) := \operatorname{span}\{\chi_A : A \in \mathcal{R}\}$ the linear space of all $\mathbb{K}$-valued $\mathcal{R}$-simple functions, $\mathbb{K}$ being the scalar field of the real or complex numbers. Since $A \cap B \in \mathcal{R}$ and $A \,\Delta\, B \in \mathcal{R}$ whenever $A, B \in \mathcal{R}$, for each $f \in \ell_0^\infty(\mathcal{R})$ there are pairwise disjoint sets $A_1, \dots, A_m \in \mathcal{R}$ and nonzero $a_1, \dots, a_m \in \mathbb{K}$, with $a_i \neq a_j$ if $i \neq j$ such that $f = \sum_{i=1}^m a_i \chi_{A_i}$, with $f = \chi_\emptyset$ if $f = 0$. Unless otherwise stated, we assume $\ell_0^\infty(\mathcal{R})$ endowed with the supremum norm $\|f\| = \sup\{|f(\omega)| : \omega \in \Omega\}$. If $Q = \operatorname{acx}\{\chi_A : A \in \mathcal{R}\}$ is the absolutely convex hull of $\{\chi_A : A \in \mathcal{R}\}$ another equivalent norm is defined on $\ell_0^\infty(\mathcal{R})$ by the *gauge* of Q, namely $\|f\|_Q = \inf\{\lambda > 0 : f \in \lambda Q\}$. For if $f \in \ell_0^\infty(\mathcal{R})$ with $\|f\| \leq 1$ it can be shown by induction on the number of non-vanishing different values of f that $f \in 4Q$ (*cf.* [7, Proposition 5.1.1]), hence $\|\cdot\| \leq \|\cdot\|_Q \leq 4\|\cdot\|$. The dual of $\ell_0^\infty(\mathcal{R})$ is the Banach space $ba(\mathcal{R})$ of bounded finitely additive measures defined on $\mathcal{R}$ equipped with the supremum-norm, that is, with the dual norm of the gauge $\|\cdot\|_Q$. Each $A \in \mathcal{R}$ defines a continuous linear form on $ba(\mathcal{R})$ represented by δ_A, named the

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evaluation on A, given by $\langle \delta_A, \mu \rangle = \mu(A)$ for each $\mu \in ba(\mathcal{R})$. The completion of $\ell_0^\infty(\mathcal{R})$ is the Banach space $\ell_\infty(\mathcal{R})$ of all bounded $\mathcal{R}$-measurable functions endowed with the supremum-norm. The ring $\mathcal{R}$ is an algebra of subsets of Ω if $\Omega \in \mathcal{R}$ and the ring (resp. algebra) $\mathcal{R}$ is a σ-ring (σ-algebra) if $\cup\{A_n : n \in \mathbb{N}\} \in \mathcal{R}$ whenever $A_n \in \mathcal{R}$ for all $n \in \mathbb{N}$.

We say that a subfamily Δ of a ring $\mathcal{R}$ is a *Nikodým set* for $ba(\mathcal{R})$, or that Δ has property (N), if each set $\{\mu_\alpha : \alpha \in \Lambda\}$ in $ba(\mathcal{R})$ which is pointwise bounded on Δ is norm-bounded in $ba(\mathcal{R})$, i.e., if $\sup_{\alpha\in\Lambda} \left|\mu_\alpha(A)\right| < \infty$ for each $A \in \Delta$ implies that $\sup_{\alpha\in\Lambda} \sup_{A\in\mathcal{R}} \left|\mu_\alpha(A)\right| = \sup_{\alpha\in\Lambda} \left\|\mu_\alpha\right\| < \infty$. We say that a subfamily Δ of a ring $\mathcal{R}$ is a *strong Nikodým set* for $ba(\mathcal{R})$, or that it has property (sN), if each increasing covering $\{\Delta_m : m \in \mathbb{N}\}$ of Δ contains a Nikodým set Δ_n for $ba(\mathcal{R})$.

The Nikodým-Grothendieck boundedness theorem asserts that every σ-algebra Σ of subsets of a set Ω has property (N). This result has been improved by some authors, in particular by Manuel Valdivia, who proved in [26, Theorem 2] that each σ-algebra Σ has property (sN). Valdivia obtained this result in order to prove that if μ is a bounded additive vector-measure defined in a σ-algebra Σ with values in a inductive limit of Fréchet spaces $F(\tau) := \lim_n F_n(\tau_n)$, there exists $m \in \mathbb{N}$ such that μ is an $F_m(\tau_m)$-valued bounded finite additive measure [26, Theorem 4].

An *increasing web* $\left\{\Delta_{n_1,n_2,\ldots,n_p} : p, n_1, n_2, \ldots, n_p \in \mathbb{N}\right\}$ of subsets of a set Δ is a web on Δ such that $\Delta_{m_1} \subseteq \Delta_{n_1}$ whenever $m_1 \le n_1$ and $\Delta_{n_1,n_2,\ldots,n_p,m_{p+1}} \subseteq \Delta_{n_1,n_2,\ldots,n_p,n_{p+1}}$ whenever $m_{p+1} \le n_{p+1}$ for every $n_i \in \mathbb{N}$ and $i \le p$. A subset Δ of a ring $\mathcal{R}$ is a *web Nikodým set* for $ba(\mathcal{R})$, or has property (wN), if each *increasing web* $\left\{\Delta_{n_1,n_2,\ldots,n_p} : p, n_1, n_2, \ldots, n_p \in \mathbb{N}\right\}$ on Δ has a strand $\left\{\Delta_{m_1,m_2,\ldots,m_p} : p \in \mathbb{N}\right\}$ consisting of Nikodým sets. In particular, a ring $\mathcal{R}$ is called a (wN)-ring if each increasing web on $\mathcal{R}$ contains a *strand* $\left\{\mathcal{R}_{m_1,m_2,\ldots,m_p} : p \in \mathbb{N}\right\}$ consisting of Nikodým sets (see [15]). Valdivia's theorem concerning the (sN) property for σ-algebras was improved in [16, Theorem 2.7], where it was shown that each σ-algebra Σ of subsets of a set Ω has property (wN). This result also extends other strong Nikodým properties involving finite strands of increasing webs (see [7] and references therein).

The situation of rings and algebras with respect to properties (N), (sN) and (wN) is totally different. The algebra $\mathcal{A}$ of finite and cofinite subsets of $\mathbb{N}$ does not have property (N), for if δ_n is the point mass at $\{n\}$ then the measures $\mu_n \in ba(\mathcal{A})$ such that $\mu_n(A) = n(\delta_{n+1}(A) - \delta_n(A))$ for A finite and $\mu_n(A) = -n(\delta_{n+1}(A) - \delta_n(A))$ for A cofinite and $n \in \mathbb{N}$ are pointwise bounded, but $\{\mu_n : n \in \mathbb{N}\}$ is unbounded in $ba(\mathcal{A})$.

Several important classes of algebras of sets have been shown to have property (N), among them algebras with the following properties: *Interpolation Property* (Seever [24]), *Subsequential Interpolation Property* (Freniche [10]), *Weak Subsequential Interpolation Property* (Aizpuru [1]), *Property* (f) (Moltó [17]), *Property* (E) (Schachermayer [22]) and *Subsequential Completeness Property* (Haydon [11]). The last two properties are the same and they imply the well known Vitaly-Hans-Saks property, which is stronger than the Nikodým property. Koszmider and Shelah have shown in [13] that if an infinite algebra $\mathcal{A}$ has the so-called *Weak Subsequential Separation Property* then the cardinal of $\mathcal{A}$ is greater than or equal to the continuum $\mathfrak{c}$. Since all algebras considered here have the Weak Subsequential Separation Property, it arises the natural question whether there exist algebras with the Nikodým property with cardinality less than $\mathfrak{c}$. This question has been solved positively by Sobota in [25]. On the other hand, in [14, Theorem 1] it was proved that the algebra $\mathcal{J}(K)$ of Jordan measurable subsets of the compact interval $K = \prod_{i=1}^{k} [a_i, b_i]$ of $\mathbb{R}^k$, with $a_i < b_i$ for $1 \le i \le k$, has property (wN), extending preliminary results due to Schachermayer [22, Proposition 3.3], Ferrando [4, Corollary] and Valdivia [27, Theorem 4]. Note that $|\mathcal{J}(K)| = 2^{\mathfrak{c}}$, where $|A|$ denotes the cardinality of the set A. Valdivia asked in [27, Problem 1] whether the equivalence $(N) \Leftrightarrow (sN)$ holds for an algebra $\mathcal{A}$ of sets which is not a σ-algebra. Concerning this question, the first named author showed in [5, theorem 2.5] that the ring $\mathcal{Z}$ of subsets of density zero of $\mathbb{N}$ has property (wN), improving a previous result of Drewnowski, Florencio and Paúl [2] (see also [3] and [9]) stating that $\mathcal{Z}$ has property (N).

Let us recall that a ring $\mathcal{R}$ of subsets of a set Ω has *property* (G) if $\ell_\infty(\mathcal{R})$ is a Grothendieck space, i.e., if each weak* convergent sequence in $ba(\mathcal{R})$ is weak convergent in the Banach space $ba(\mathcal{R})$. In [22, equivalence $(G_1) \Leftrightarrow (G_2)$ of Definition 2.3] Schachermayer proved that an algebra $\mathcal{R}$ has property (G) if and only if a bounded sequence $\{\mu_n : n \in \mathbb{N}\}$ in $ba(\mathcal{R})$ which converges pointwise on $\mathcal{R}$ is uniformly exhaustive, i.e., for each sequence $\{A_i : i \in \mathbb{N}\}$ of pairwise disjoint elements of $\mathcal{R}$ it happens that $\lim_{i\to\infty} \sup_{n\in\mathbb{N}} \left|\mu_n(A_i)\right| = 0$.

The algebra $\mathcal{J}$ of Jordan subsets of the interval $[0,1]$ was the first example, due to Schachermayer [22, Propositions 3.2 and 3.3], of an algebra of subsets with property (N) that does not have property (G), answering in the negative the question $(N) \Rightarrow (G)$? stated by Seever in [24] (see [9] for more details). Let us finally recall that a subset X of the dual unit ball B_{E^*} of a Banach space E is called a *Rainwater set* for E if every *bounded* sequence $\{x_n\}_{n=1}^{\infty}$ of E that converges pointwise on X converges weakly in E (*cf.* [18]).

The rest of the paper is divided in three sections. In the second section we present a class of rings of sets with property (N) that fail property (G). In the third we get a partial solution to Valdivia's question with a class of rings without property (G) for which the equivalence $(N) \Leftrightarrow (wN)$ holds. In the last section we show that a ring $\mathcal{R}$ has property (G) if and only if the set of evaluations $\{\delta_A : A \in \mathcal{R}\}$ is a Rainwater set for $ba(\mathcal{R})$.

2. A class of rings with property (*N*) that fail property (*G*)

If Σ is a σ-algebra of subsets of a set Ω and $A \in \Sigma$, then $\Sigma_A := \{B \in \Sigma : B \subseteq A\}$ is a σ-algebra contained in Σ. A subfamily $\mathcal{H}$ of Σ is called Σ-*hereditary* if $\mathcal{H} = \cup\{\Sigma_A : A \in \mathcal{H}\}$. Unless otherwise stated we shall always work with an underlying measurable space (Ω, Σ).

Definition 2.1. *A subfamily $\mathcal{M}$ of a ring $\mathcal{R}$ of subsets of Ω is $\mathcal{R}$-singular if for each sequence $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{R}$ there is $\{M_n : n \in \mathbb{N}\} \subseteq \mathcal{M}$ with $\bigcup_{n=1}^{\infty}(A_n \setminus M_n) \in \mathcal{R}$.*

Example 2.2. *$\mathcal{M} = \{\emptyset\}$ is an $\mathcal{R}$-singular subfamily of every σ-ring $\mathcal{R}$ of subsets of Ω. If $\mathcal{Z}$ stands for the $2^{\mathbb{N}}$-hereditary ring of subsets of density zero of $\mathbb{N}$, it is easy to prove that the ring $\mathcal{M}$ of finite subsets of $\mathbb{N}$ is $\mathcal{Z}$-singular. Obviously, the countable union $\bigcup\{M : M \in \mathcal{M}\} = \mathbb{N}$ does not belong to $\mathcal{Z}$.*

Theorem 2.3. *Let $\mathcal{R}$ be a Σ-hereditary subring of Σ and $\mathcal{M}$ a Σ-hereditary and $\mathcal{R}$-singular subfamily of $\mathcal{R}$. If each subset T of $ba(\mathcal{R})$ which is pointwise bounded on $\mathcal{R}$ is uniformly bounded on $\mathcal{M}$, then $\mathcal{R}$ has property (N).*

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a sequence in $\mathcal{R}$. It suffices to show that T is uniformly bounded on $\{A_n : n \in \mathbb{N}\}$, i.e., that there exists $k > 0$ such that $\sup_{n\in\mathbb{N}} |\mu(A_n)| < k$ for every $\mu \in T$. By the hypotheses on $\mathcal{M}$ there is a sequence $\{M_n : n \in \mathbb{N}\}$ in $\mathcal{M}$ satisfying that $A := \bigcup_{n=1}^{\infty}(A_n \setminus M_n) \in \mathcal{R}$ with $M_n \subseteq A_n$ for each $n \in \mathbb{N}$. Since $\mathcal{R}$ is Σ-hereditary, the σ-algebra Σ_A is contained in $\mathcal{R}$. So, by the Nikodým-Grothendieck boundedness theorem, T is uniformly bounded on Σ_A. By hypothesis T is uniformly bounded on $\mathcal{M}$, hence T is uniformly bounded on $\{A_n : n \in \mathbb{N}\}$. □

Definition 2.4. *Let (Ω, Σ) be a measurable space. If we have a sequence $\{\mu_n\}_{n=1}^{\infty}$ of $[0,1]$-valued finitely additive measures that are countably subadditive and a pairwise disjoint sequence $\{E_n : n \in \mathbb{N}\}$ in Σ such that $\mu_n(E_n) = 1$ for each $n \in \mathbb{N}$, we shall call $\mathcal{R} = \{A \in \Sigma : \mu_n(A) \to 0\}$ the Σ-subring dominated by the sequence $\{(\mu_n, E_n) : n \in \mathbb{N}\}$.*

We shall also say that $\mathcal{R}$ is a *dominated Σ-subring*. Clearly, each dominated Σ-subring is Σ-hereditary and it does not have property (G).

Example 2.5. *The ring $\mathcal{Z}$ of subsets of $\mathbb{N}$ of density zero is a dominated $2^{\mathbb{N}}$-subring.*

Proof. For each natural number n let $E_n := \left\{2^{n-1}+1, 2^{n-1}+2, \ldots, 2^n\right\}$ and let μ_n be the $[0,1]$-valued positive measure defined on $2^{\mathbb{N}}$ by

$$\mu_n(A) = \frac{|A \cap E_n|}{2^{n-1}}.$$

The pairwise disjoint sets E_n verify that $\mu_n(E_n) = 1$ and, since finite sets have density zero, we have that $E_n \in \mathcal{Z}$ for every $n \in \mathbb{N}$. Let's prove that $\mathcal{Z}$ is exactly the $2^{\mathbb{N}}$-subring dominated by the sequence $\{(\mu_n, E_n) : n \in \mathbb{N}\}$. In fact, if $A \in \mathcal{Z}$ one has

$$\lim_{n\to\infty} \mu_n(A) = \lim_{n\to\infty} \frac{\left|A \cap \left(2^{n-1}, 2^n\right]\right|}{2^{n-1}} = 2 \times \lim_{n\to\infty} \frac{|A \cap (0, 2^n]|}{2^n} - \lim_{n\to\infty} \frac{\left|A \cap \left(0, 2^{n-1}\right]\right|}{2^{n-1}} = 0,$$

and, conversely, if $A \subseteq \mathbb{N}$ verifies that $\mu_n(A) \to 0$, then A is a set of density zero as a consequence of the Stolz convergence test. $\square$

Consequently, the ring $\mathcal{Z}$ does not have property (G) (a fact already observed in [2]).

Theorem 2.6. *If $\mathcal{R}$ is the Σ-subring dominated by a sequence $\{(\mu_n, E_n) : n \in \mathbb{N}\}$, the countable family $\mathcal{M} := \left\{\bigcup_{p=1}^{n} E_p : n \in \mathbb{N}\right\}$ is $\mathcal{R}$-singular.*

Proof. If $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{R}$ there exists a strictly increasing sequence $\{n_s\}_{s=1}^{\infty}$ in $\mathbb{N}$ such that for $k \geqslant n_s$

$$0 \leq \mu_k(A_1) + \cdots + \mu_k(A_s) < s^{-1}.$$

Since $\mu_k(E_p) = \delta_{kp}$, for $k < n_{s+1}$ we have that

$$0 \leq \mu_k\left(\bigcup_{i=1}^{\infty}\left(A_i \setminus \bigcup_{p=1}^{n_i} E_p\right)\right) \leq \sum_{i=1}^{s} \mu_k\left(A_i \setminus \bigcup_{p=1}^{n_i} E_p\right) \leq \sum_{i=1}^{s} \mu_k(A_i).$$

Consequently, if $n_s \leq k < n_{s+1}$ then

$$0 \leq \mu_k\left(\bigcup_{i=1}^{\infty}\left(A_i \setminus \bigcup_{p=1}^{n_i} E_p\right)\right) < s^{-1},$$

so that $\lim_{k\to\infty} \mu_k\left(\bigcup_{i=1}^{\infty}\left(A_i \setminus \bigcup_{p=1}^{n_i} E_p\right)\right) = 0$. Hence $\bigcup_{i=1}^{\infty}\left(A_i \setminus \bigcup_{p=1}^{n_i} E_p\right) \in \mathcal{R}$. $\square$

Remark 2.7. *Let $\mathcal{Z}$ be the ring of subsets of $\mathbb{N}$ of density zero. By the previous theorem and Example 2.5, the family $\mathcal{M} := \{[1, 2^n] : n \in \mathbb{N}\}$ is $\mathcal{Z}$-singular. Hence the family of finite subsets of $\mathbb{N}$ is $2^{\mathbb{N}}$-hereditary and $\mathcal{Z}$-singular.*

Theorem 2.8. *Assume that $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of atomless probability measures on Σ and $\{E_n : n \in \mathbb{N}\}$ a pairwise disjoint sequence in Σ with $\mu_n(E_m) = \delta_{n,m}$ for $n, m \in \mathbb{N}$. Then the Σ-subring $\mathcal{R}$ dominated by $\{(\mu_n, E_n) : n \in \mathbb{N}\}$ has property (N).*

Proof. For each $s \in \mathbb{N}$ let $D_s := \bigcup_{p=1}^{s} E_p$. By Theorem 2.6 the family $\{D_s : s \in \mathbb{N}\}$ is $\mathcal{R}$-singular and hence $\mathcal{M} = \{\Sigma_{D_s} : s \in \mathbb{N}\}$ is Σ-hereditary and $\mathcal{R}$-singular. According to Theorem 2.3, it suffices to prove that each subset H of $ba(\mathcal{R})$ pointwise bounded on $\mathcal{R}$ is uniformly bounded on $\mathcal{M}$.

Let us proceed by contradiction by supposing that H is a subset of $ba(\mathcal{R})$ which it is pointwise bounded on $\mathcal{R}$ but not uniformly bounded on $\mathcal{M}$. Fix $n \in \mathbb{N}$ and for each $p \in \mathbb{N}$ let $\{E_{p,j}^{n} : 1 \leq j \leq n\}$ denote a partition of E_p consisting of subsets of Σ such that $\mu_p(E_{p,j}^{n}) = n^{-1}$ for $1 \leq j \leq n$. Then, for $s \in \mathbb{N}$ and $1 \leq j \leq n$ set $D_{s,j}^{n} := \bigcup_{p=1}^{s} E_{p,j}^{n}$ and

$$\mathcal{M}_j^n := \{\Sigma_{D_{s,j}^n} : s \in \mathbb{N}\}.$$

Since H is not uniformly bounded on $\mathcal{M}$, for each $n \in \mathbb{N}$ there is j_n with $1 \leq j_n \leq n$ such that H is not uniformly bounded on $\mathcal{M}_{j_n}^n$. By the Nikodým-Grothendieck boundedness theorem we get that for each natural number m_n the set H is uniformly bounded on the σ-algebra $\Sigma_{D_{m_n}}$, hence for each pair of natural numbers n and m_n the set H is not uniformly bounded on $\mathcal{M}_{j_n}^n \setminus \Sigma_{D_{m_n}}$. So, for each pair of natural numbers n and m_n there exist $v_n \in H$, $m_{n+1} > m_n$ and $A_n \subseteq \bigcup\{E_{p,j_n}^n : m_n < p \leq m_{n+1}\}$ with

$$|v_n(A_n)| > n, \text{ for each } n \in \mathbb{N}. \tag{1}$$

Let $A := \bigcup\{A_n : n \in \mathbb{N}\} \in \Sigma$. If $m_n < p \leq m_{n+1}$ we obtain by construction that $\mu_p(A) = \mu_p(A_n) \leq \mu_p(E_{p,j_n}^n) = n^{-1}$. Hence, $\lim_{p\to\infty} \mu_p(A) = 0$ and consequently $A \in \mathcal{R}$. Since the σ-algebra Σ_A is contained in the Σ-hereditary ring $\mathcal{R}$, it turns out that H must be uniformly bounded in Σ_A, which contradicts the inequalities (1). $\square$

Example 2.9. *Dominated Σ-subrings with property (N). Let* $\Omega = [0,1]$ *and* Σ *be the* σ*-algebra of Lebesgue measurable subsets of the interval* $[0,1]$*. Define the atomless measures*

$$\mu_n(A) = \int_A f_n(t)\, d\lambda(t)$$

on Σ*, where* $f_n : [0,1] \to \mathbb{R}$ *is the function whose graph consists of a flat peak of height* 2^n *over the segment* $\left(2^{-n}, 2^{-n+1}\right]$ *along with the segments* $\{(x,0) : x \in [0,2^{-n}] \cup \left(2^{-n+1},1\right]\}$ *and* λ *stands for the Lebesgue probability measure of* $[0,1]$*. Set* $E_n := \left(2^{-n}, 2^{-n+1}\right]$ *for each* $n \in \mathbb{N}$*. The* Σ*-subring* $\mathcal{R}$ *of subsets of* $[0,1]$ *dominated by* $\{(\mu_n, E_n) : n \in \mathbb{N}\}$ *has property (N) by virtue of the previous theorem and* $\mathcal{R}$*, as every dominated* Σ*-subring, does not have property (G). Each Lebesgue measurable set that meets only finitely many sets* E_n *belongs to* $\mathcal{R}$*. Moreover* $M = \bigcup_{n=1}^{\infty} \left(\frac{2^{n+1}-1}{4^n}, \frac{1}{2^{n-1}}\right] \in \mathcal{R}$ *since* $\mu_n(M) = 2^{-n} \to 0$*, and* M *meets each* E_n*.*

3. Rings for which (N) ⇔ (wN)

We exhibit a class of rings for which properties (N) and (wN) are equivalent. This provides a partial positive solution of the still open problem for algebras of sets concerning whether $(N) \Rightarrow (sN)$ [27, Problem 1].

If a subfamily Δ of a ring $\mathcal{R}$ is not a *Nikodým set* for $ba(\mathcal{R})$ there exists an unbounded sequence $\{\mu_n\}_{n=1}^{\infty}$ in $ba(\mathcal{R})$ which is pointwise bounded on Δ. Consequently Δ is the union of the sets $\Delta_m := \cup\{A \in \Delta : \left|\mu_n(\chi_A)\right| \leq m, \forall n \in \mathbb{N}\}$ for $m \in \mathbb{N}$. Since $\{m^{-1}\mu_n : n \in \mathbb{N}\} \subseteq \{\chi_A : A \in \Delta_m\}^0$, it follows that $\{\chi_A : A \in \Delta_m\}^0$ is an unbounded subset of $ba(\mathcal{R})$ for every $m \in \mathbb{N}$. Conversely, if Δ is the union of an increasing sequence $\{\Delta_m\}_{m=1}^{\infty}$ and each $\{\chi_A : A \in \Delta_m\}^0$ is unbounded, there is $\mu_m \in \{\chi_A : A \in \Delta_m\}^0$ with $\left\|\mu_m\right\| > m$ for each $m \in \mathbb{N}$. Since $\{\mu_n : n \in \mathbb{N}\}$ is Δ-pointwise bounded, Δ is not a Nikodým set.

Therefore a subfamily Δ of a ring $\mathcal{R}$ is a *Nikodým set* for $ba(\mathcal{R})$ if and only if for each increasing covering $\{\Delta_m\}_{m=1}^{\infty}$ of Δ there exists Δ_n such that $\{\chi_A : A \in \Delta_n\}^0$ is a bounded subset of $ba(\mathcal{R})$ or, equivalently, if the closed absolutely convex hull of $\{\chi_A : A \in \Delta_n\}$ is a neighborhood of zero in $\ell_0^{\infty}(\mathcal{R})$. This result also follows from the Amemiya-Kōmura property (see [21]).

If a subfamily Δ of a ring $\mathcal{R}$ is a Nikodým set for $ba(\mathcal{R})$ then $F := \mathrm{span}\{\chi_A : A \in \Delta\}$ is a subspace of $\ell_0^{\infty}(\mathcal{R})$ dense and barrelled (i.e., each subset $\{\mu_\alpha : \alpha \in \Lambda\}$ of $ba(\mathcal{R})$ which is pointwise bounded on F verifies that $\sup_{\alpha\in\Lambda} \left\|\mu_\alpha|_F\right\| < \infty$). The converse is obvious because a subset $\{\mu_\alpha : \alpha \in \Lambda\}$ of $ba(\mathcal{R})$ is pointwise bounded on F if and only if is pointwise bounded in Δ and, by density $\left\|\mu_\alpha|_F\right\| = \left\|\mu_\alpha\right\|$. Therefore Δ is a Nikodým set if and only if $\mathrm{span}\{\chi_A : A \in \Delta\}$ is a subspace of $\ell_0^{\infty}(\mathcal{R})$ dense and barrelled. In particular, the barrelledness of $\ell_0^{\infty}(\mathcal{R})$ is equivalent to the fact that $\mathcal{R}$ has property (N).

Lemma 3.1. *Let* $\mathcal{R}$ *be a ring. If* $\mathcal{N}$ *is a Nikodým set for* $ba(\mathcal{R})$ *and* $\{\mathcal{N}_n : n \in \mathbb{N}\}$ *is an increasing covering of* $\mathcal{N}$*, there exists* $m \in \mathbb{N}$ *such that* $\mathrm{span}\{\chi_A : A \in \mathcal{N}_m\}$ *is dense in* $\ell_0^{\infty}(\mathcal{R})$*. If* $\mathcal{N}$ *is not a Nikodým set for* $ba(\mathcal{R})$ *and* $\mathrm{span}\{\chi_A : A \in \mathcal{N}\}$ *is dense in* $\ell_0^{\infty}(\mathcal{R})$*, for each countable subfamily* $\mathcal{M}$ *of* $\mathcal{R}$ *it holds that* $\mathcal{N} \cup \mathcal{M}$ *is not a Nikodým set for* $ba(\mathcal{R})$*.*

Proof. If $\mathcal{N}$ is a Nikodým set then there exists $\mathcal{N}_m$ such that the closed absolutely convex hull of $\{\chi_A : A \in \mathcal{N}_m\}$ is a neighborhood of 0 in $\ell_0^{\infty}(\mathcal{R})$. When $\mathcal{N}$ is not a Nikodým set for $ba(\mathcal{R})$ and $\mathrm{span}\{\chi_A : A \in \mathcal{N}\}$ is dense in $\ell_0^{\infty}(\mathcal{R})$, it turns out that $\mathrm{span}\{\chi_A : A \in \mathcal{N}\}$ is a non barrelled subspace of $\ell_0^{\infty}(\mathcal{R})$. So, if $\mathcal{M}$ is countable, the countable dimension of $\ell_0^{\infty}(\mathcal{M})$ implies that $\mathrm{span}\{\chi_A : A \in \mathcal{N} \cup \mathcal{M}\}$ is also non barrelled (*cf.* [19, Theorem 4.3.6]). Hence $\mathcal{N} \cup \mathcal{M}$ is not a Nikodým set for $ba(\mathcal{R})$. □

Lemma 3.2. *Let* $\mathcal{R}$ *be a ring with property (N) which fails to have property (wN). Then there exists an increasing web* $\{\mathcal{R}_t : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ *in* $\mathcal{R}$ *such that for each countable subfamily* $\mathcal{M}$ *of* $\mathcal{R}$ *the increasing web* $\{\mathcal{R}_t \cup \mathcal{M} : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ *does not contain any strand consisting entirely of Nikodým sets for* $ba(\mathcal{R})$*.*

Proof. Let $\{\mathcal{R}'_t : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ be an increasing web in $\mathcal{R}$ without any strand consisting of Nikodým sets and let J be the subset of $\bigcup_{s\in\mathbb{N}} \mathbb{N}^s$ such that $t \in J$ whenever both $\mathcal{R}'_t$ is a Nikodým set and $\operatorname{span}\{\chi_A : A \in \mathcal{R}'_t\}$ is dense in $\ell_0^\infty(\mathcal{R})$. Since $\mathcal{R}$ has property (N), by Lemma 3.1 there exists $m \in \mathbb{N}_0$ such that $\operatorname{span}\{\chi_A : A \in \mathcal{R}'_{t_1}\}$ is dense in $\ell_0^\infty(\mathcal{R})$ for each $t_1 \geqslant m$. If J were the empty set, no $\mathcal{R}'_{t_1}$ would be a Nikodým set for $ba(\mathcal{R})$. Hence, due to Lemma 3.1 and the increasing web condition, no $\mathcal{R}'_{t_1} \cup \mathcal{M}$ is a Nikodým set for each countable subset $\mathcal{M}$ of $\mathcal{R}$ and each $t_1 \in \mathbb{N}$. So, the web formed by the sets $\mathcal{R}_t = \mathcal{R}'_{t_1}$ for $t = (t_1, t_2, \cdots, t_p) \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s$ verifies that $\mathcal{R}_t \cup \mathcal{M}$ is not a Nikodým set for each countable subset $\mathcal{M}$ of $\mathcal{R}$.

If $t = (t_1, t_2, \cdots, t_p) \in J$ and $(t_1, t_2, \cdots, t_p, t_{p+1}) \notin J$, for each $t_{p+1} \in \mathbb{N}$, then it is obvious that

$$\{(t'_1) : t'_1 \geqslant t_1\} \cup \{(t_1, t'_2) : t'_2 \geqslant t_2\} \cup \cdots \cup \{(t_1, t_2, \cdots, t'_p) : t'_p \geqslant t_p\} \subseteq J$$

and $\mathcal{R}'_{t_1,t_2,\cdots,t_p}$ is a Nikodým set. Applying Lemma 3.1 with $\mathcal{N} = \mathcal{R}'_{t_1,t_2,\cdots,t_p}$ and $\mathcal{N}_n = \mathcal{R}'_{t_1,t_2,\cdots,t_p,n}$ we get $m = m_{t_1,t_2,\cdots,t_p} \in \mathbb{N}_0$ such that $\operatorname{span}\{\chi_A : A \in \mathcal{R}'_{t_1,t_2,\cdots,t_p,t_{p+1}}\}$ is dense in $\ell_0^\infty(\mathcal{R})$ for each $t_{p+1} \geqslant m$. So, $\mathcal{R}'_{t_1,t_2,\cdots,t_p,t_{p+1}}$ is not a Nikodým set for every $t_{p+1} \geqslant m$. Consequently, Lemma 3.1 implies that $\mathcal{R}'_{t_1,t_2,\cdots,t_p,t_{p+1}} \cup \mathcal{M}$ is not a Nikodým set for $ba(\mathcal{R})$ for each $t_{p+1} \in \mathbb{N}$ and each countable subset $\mathcal{M}$ of $\mathcal{R}$. We establish the lemma by means of the increasing web determined by the sets $\mathcal{R}'_t$ with $t \in J$ together with the sets $\mathcal{R}_{t_1,t_2,\cdots,t_p,t_{p+1},\cdots t_{p+s}} = \mathcal{R}'_{t_1,t_2,\cdots,t_p,t_{p+1}}$ when $(t_1, t_2, \cdots, t_p) \in J$, $(t_1, t_2, \cdots, t_p, t_{p+1}) \notin J$ for each $t_{p+1} \in \mathbb{N}$, and $(t_{p+1}, \cdots t_{p+s}) \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s$. □

Theorem 3.3. *Let $\mathcal{M}$ be a Σ-hereditary, countable and singular subfamily of the Σ-hereditary ring $\mathcal{R}$. If $\mathcal{R}$ has property (N), then $\mathcal{R}$ has property (wN).*

Proof. Assume by way of contradiction that $\mathcal{R}$ has property (N) but does not have property (wN). By Lemma 3.2 there exists an increasing web $\{\mathcal{R}_t : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ in $\mathcal{R}$ such that the increasing web $\{\mathcal{R}_t \cup \mathcal{M} : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ does not contain any strand formed entirely by Nikodým sets for $ba(\mathcal{R})$. Let

$$J := \{t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s : \mathcal{R}_t \cup \mathcal{M} \text{ is not a Nikodým set for } ba(\mathcal{R})\}.$$

Then for each $t \in J$ there exists in $ba(\mathcal{R})$ a subset T_t which is pointwise bounded on $\mathcal{R}_t \cup \mathcal{M}$ but is not uniformly bounded on $\mathcal{R}$. Since $\mathcal{R}$ is a Nikodým set, T_t cannot be pointwise bounded on $\mathcal{R}$ so that there exists $A_t \in \mathcal{R}$ such that T_t is unbounded in A_t for each $t \in J$. On the other hand, since $\mathcal{M}$ is a Σ-hereditary and singular, for each $t \in J$ the set A_t contains a subset $M_t \in \mathcal{M}$ such that $A := \bigcup\{A_t \setminus M_t : t \in J\} \in \mathcal{R}$. As in addition the Σ-hereditary ring $\mathcal{R}$ contains the σ-algebra Σ_A, which has property (wN), there exists a sequence $\left\{m_p\right\}_{p=1}^\infty \subseteq \mathbb{N}$ such that, for each $p \in \mathbb{N}$ one has that

$$\{\mathcal{R}_{m_1 m_2 \cdots m_p} \cup \mathcal{M}\} \cap \Sigma_A \text{ is a Nikodým set for } ba(\Sigma_A). \tag{2}$$

Given that $\{\mathcal{R}_t \cup \mathcal{M} : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ does not contain any strand formed entirely by Nikodým sets for $ba(\mathcal{R})$, there exists $q \in \mathbb{N}$ such that $t_q := (m_1, m_2, \cdots, m_q) \in J$ and then T_{t_q} is unbounded in A_{t_q}. By construction T_{t_q} is pointwise bounded on $\mathcal{R}_{t_q} \cup \mathcal{M}$, in particular T_{t_q} is bounded in M_{t_q}, and by (2) with $p = q$ it follows that T_{t_q} is also uniformly bounded in Σ_A. In particular, since $A_{t_q} \setminus M_{t_q} \in \Sigma_A$, it turns out that T_{t_q} is bounded in $M_{t_q} \cup \left(A_{t_q} \setminus M_{t_q}\right) = A_{t_q}$, a contradiction. □

Remark 3.4. *By [2] (see also [3] and [9]), the ring $\mathcal{Z}$ of subsets of density zero of $\mathbb{N}$ has property (N), hence Remark 2.7 and Theorem 3.3 imply that $\mathcal{Z}$ has property (wN).*

4. Rainwater sets for *ba* $(\mathcal{R})$

As mentioned in the preliminaries a subset X of the closed dual unit ball B_{E^*} of a Banach space E is called a *Rainwater set* for E if every *bounded* sequence $\{x_n\}_{n=1}^\infty$ of E that converges pointwise on X, i.e., such that $x^* x_n \to x^* x$ for each $x^* \in X$, converges weakly in E. By [23, Corollary 11] each James boundary J for B_{E^*} is a Rainwater set for E (the converse is not true). In particular $\operatorname{Ext} B_{E^*}$ is a Rainwater set for E, [20]. This latter fact also follows from Choquet's integral representation theorem, and implies that for each compact space K the set of evaluations $\{\delta_a : a \in K\}$ is a Rainwater set for $C(K)$. Recently, Rainwater sets for the Banach space $C^b(X)$ of continuous and bounded real-valued functions defined on a completely regular space X have been investigated in [6]. The next proposition provides a relation between Rainwater sets and property (G).

Proposition 4.1. *Let $\mathcal{R}$ be a ring of subsets of a set Ω. The following are equivalent*

1. *$\mathcal{R}$ has property (G).*
2. *The set of evaluations $\{\delta_A : A \in \mathcal{R}\}$ is a Rainwater set for $ba(\mathcal{R})$.*

Proof. First we suppose that $\mathcal{R}$ is an algebra. Assume that $\mathcal{R}$ has property (G) and that $\{\mu_n\}_{n=1}^{\infty}$ is a bounded sequence in $ba(\mathcal{R})$ such that $\langle \mu_n, \delta_A \rangle \to \langle \mu, \delta_A \rangle$ for every $A \in \mathcal{R}$, i.e., such that $\mu_n(A) \to \mu(A)$ for each $A \in \mathcal{R}$. Since $\{\mu_n\}_{n=1}^{\infty}$ is a bounded sequence in $ba(\mathcal{R})$ that converges pointwise on $\mathcal{R}$, according to [22, condition G_1 in Definition 2.3] the sequence $M = \{\mu_n : n \in \mathbb{N}\}$ is uniformly exhaustive on $\mathcal{R}$. Given that M is uniformly exhaustive and bounded on the members of $\mathcal{R}$, [22, Proposition 1.2] ensures that M is a relatively weakly compact set of $ba(\mathcal{R})$. Thus, by Eberlein's theorem, M is weakly sequentially compact. Then, as $\mu_n(A) \to \mu(A)$ for each $A \in \mathcal{R}$, we get that μ is the only possible weakly adherent point of the sequence $\{\mu_n\}_{n=1}^{\infty}$. So, $\mu_n \to \mu$ weakly in $ba(\mathcal{R})$ and $\{\delta_A : A \in \mathcal{R}\}$ is a Rainwater set for $ba(\mathcal{R})$.

Assume conversely that the set of evaluations $\{\delta_A : A \in \mathcal{R}\}$ is a Rainwater set for $ba(\mathcal{R})$. Let $\{\mu_n\}_{n=1}^{\infty}$ be any bounded sequence in $ba(\mathcal{R})$ that converges pointwise on $\mathcal{R}$ to some $\mu \in ba(\mathcal{R})$. The latter means that $\mu_n(A) \to \mu(A)$ for each $A \in \mathcal{R}$, so that

$$\langle \mu_n, \delta_A \rangle \to \langle \mu, \delta_A \rangle$$

for all $A \in \mathcal{R}$. Hence $\mu_n \to \mu$ weakly in $ba(\mathcal{R})$, so that $\{\mu_n : n \in \mathbb{N}\}$ is a relatively weakly compact subset of $ba(\mathcal{R})$. Again by [22, Proposition 1.2] we have that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is uniformly exhaustive, which according to [22, equivalence $(G_1) \Leftrightarrow (G_2)$ of Definition 2.3] means that $\mathcal{R}$ has property (G).

To get the proof for a ring $\mathcal{R}$, notice that as the algebra $\mathcal{F}$ generated by $\mathcal{R}$ and $\{\Omega\}$ verifies that the codimension of $l_0^\infty(\mathcal{R})$ in $l_0^\infty(\mathcal{F})$ is 1, then Proposition 1.2. in [22] as well as the equivalence $(G_1) \Leftrightarrow (G_2)$ of Definiton 2.3. in [22] hold for the ring $\mathcal{R}$. □

Corollary 4.2. *In the (wN)-ring $\mathcal{Z}$ of subsets of density zero of $\mathbb{N}$ the set of evaluations $\{\delta_A : A \in \mathcal{Z}\}$ is not a Rainwater set for $ba(\mathcal{Z})$.*

Proof. Since no dominated subring has property (G), this is consequence of Example 2.5 and Proposition 4.1. □

Corollary 4.3. *Let $\mathcal{N}$ be a Nikodým set for $ba(\mathcal{R})$ such that $\{\delta_A : A \in \mathcal{N}\}$ is a Rainwater set for $ba(\mathcal{R})$. Then each sequence $\{\mu_n : n \in \mathbb{N}\}$ in $ba(\mathcal{R})$ pointwise convergent on $\mathcal{N}$ is weakly convergent in $ba(\mathcal{R})$.*

Proof. Since $\mu_n(A) \to \mu(A)$ for every $A \in \mathcal{N}$, the sequence $\{\mu_n : n \in \mathbb{N}\}$ is pointwise bounded on $\mathcal{N}$, hence norm bounded in $ba(\mathcal{R})$ due to $\mathcal{N}$ is a Nikodým set. As in addition $\{\delta_A : A \in \mathcal{N}\}$ is a Rainwater set for $ba(\mathcal{R})$, then $\mu_n \to \mu$ weakly in $ba(\mathcal{R})$. □

Corollary 4.4. *If a ring $\mathcal{R}$ of subsets of Ω has both properties (N) and (G), i.e., $\mathcal{R}$ is a so-called ring with the Vitali-Hahn-Saks property, or property (VHS), then each sequence in $ba(\mathcal{R})$ pointwise convergent on $\mathcal{R}$ is weakly convergent in $ba(\mathcal{R})$.*

Remark 4.5. *There have been several attempts of introducing boundedness properties stronger than property (wN) defined in terms of increasing webs, as properties $(w\text{-}sN)$ or $\left(w^2N\right)$ (see [12] and [15]), but all them have shown to be equivalent to property (wN) (this follows from [15, Proposition 1]). It is easy to prove that a ring $\mathcal{R}$ has (wN)-property if and only $\ell_0^\infty(\mathcal{R})$ is baireled, i.e. if each increasing web $\left\{E_{n_1,n_2,\ldots,n_p} : p, n_1, n_2, \ldots, n_p \in \mathbb{N}\right\}$ on $\ell_0^\infty(\mathcal{R})$ formed by linear subspaces contains a strand $\{E_{m_1,m_2,\ldots,m_p} : p \in \mathbb{N}\}$ formed by subspaces both dense and barrelled [8]. Other classic barrelledness properties stronger than baireledness fail for the space $\ell_0^\infty(\mathcal{R})$ even if $\mathcal{R}$ is a σ-algebra (see [8, 9] for details).*

Summarizing, if (Ω, Σ) is a measurable space and $\mathcal{R}$ is a Σ-hereditary ring of subsets of Ω that contains a Σ-hereditary, countable and singular subfamily $\mathcal{M}$, then $\mathcal{R}$ has property (wN) if and only if it has property (N), which provides a partial solution to Valdivia's question. We have also shown that a ring of sets $\mathcal{R}$ has property (G) if and only if the family of evaluations $\{\delta_A : A \in \mathcal{R}\}$ is a Rainwater set for $ba(\mathcal{R})$.

Problem 4.6. *Characterize those rings $\mathcal{R}$ of subsets of a set Ω for which $(N) \Leftrightarrow (wN)$.*

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