VECTOR VALUED INFORMATION MEASURES AND INTEGRATION WITH RESPECT TO FUZZY VECTOR CAPACITIES

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Abstract. Integration with respect to vector-valued fuzzy measures is used to define and study information measuring tools. Motivated by some current developments in Information Science, we apply the integration of scalar functions with respect to vector-valued fuzzy measures, also called vector capacities. Bartle-Dunford-Schwartz integration (for the additive case) and Choquet type integration (for the non-additive case) are considered, showing that these formalisms can be used to define and develop vector-valued impact measures. Examples related to existing bibliometric tools as well as to new measuring indices are given.

1. Introduction

Classical Lebesgue integration of scalar functions has provided some fundamental tools in several areas of the Information Science, including the definition of indices for measuring some aspects of information items. For instance, a great part of mathematical definitions of impact indices for scientific journals can be modeled by means of integrals. Some current research has also pointed out that a natural vector-valued integration of scalar functions with respect to vector measures —the so called Bartle-Dunford-Schwartz integration— may be used to generalize some scalar theoretical settings (scalar-valued impact measures) to vector-valued settings (multi-valued impact measures). Vector-valued integration theory has appeared in the context of pure mathematics and until now this theory has found a lot of applications in Mathematical Analysis and Operator Theory. However, it can be used as an adequate framework for the analysis of problems in other scientific disciplines, since it provides a natural way of representing multi-valued mean properties of scalar functions by the simple rule of "putting each value in a different direction of the space" —that is, vector-valued integration— [4, 12].

In this paper we are concerned with suitable applications of the non-additive extensions of this integration theory to some open investigations in Information Science. An exhaustive study of the spaces of integrable functions that are integrable with respect to a vector-valued Choquet type integration theory has recently been published in [7]. The present paper can be considered a continuation of this line of research. Our aim now —after establishing a general framework of theoretical results—, is to give examples and concrete definitions of new indices with detailed explanations of several models for information measures.

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There are two facts that must be taken into account in relation with the definition and application of new impact parameters for measuring the tandem quantity/quality of scientific publications. On the one hand, and in the context of the new non-standard measures of information that are called altmetrics [24], there is increasing interest in the design of multi-valued indices. Indeed, decision making on research assessment based on several indices is an outstanding challenge in Information Science (see [1, 10, 12, 32] and references therein). The scientific community agrees on the fact that several (scalar) indices must be jointly used for research evaluation: the suitable mathematical setting for representing this idea is to consider vector-valued impact indices.

On the other hand, impact measures that are not defined by additive functions appear in almost all aspects of measurement of research activity. A relevant example is the h-index which measures a rate among the number of publications with a certain number of citations, and is not given by the usual integral. Integration with respect to (scalar) fuzzy measures is being used nowadays as a standard measure in information science (see for instance [3, 13, 17, 29]). As we said, we will analyze a vector-valued version of these integrals in order to enrich our knowledge about the design of new information measures.

Several matters appear immediately. A non-linear integral of scalar functions with respect to non-finitely additive set functions is needed. In the scalar case, the Choquet integral provides such a tool, although much of the examples recently introduced *are not* Choquet integrals [13, 17, 29]. However, it seems to be the first natural generalization, and so vector-valued Choquet integrals and related spaces of integrable functions for vector-valued capacities has recently been studied from the formal point of view by several authors [16, 30, 31], including some of the authors of the present paper [7]. Applications of this theory to other fields are also being currently developed (see for example [5]). Thus, we are interested in analyzing vector-valued information measures —impact indices— using this theory, since this is the better-known integration with respect to non-additive measures in the scalar case. This may show the way to a general analysis of non additive vector-valued information measures. In a sense, the present paper must be understood as a continuation of the research of one of the authors of the present paper, who presented a complete theory of the spaces of Choquet integrable functions with respect to a fuzzy capacity in the article [7]. The main theoretical results and the framework of the present work can be found in this paper [7]. However, it must be said that in [7] the problem of integration with respect to a vector-valued capacity is studied in its full generality, and so we have changed notation and consider more restricted types of integrals in the present paper for the aim of simplicity. Roughly speaking, in the integration model for impact measures in Information Science, each integrable function provides such an index. This relation will become clear in the present article.

Let us present now a picture of the "state of the art" regarding the subjects involved in this article. Integration with respect to general set functions has become a very active current research topic, mainly due to its potential applications —not only in Information Science—. Concepts such as fuzzy (scalar) measures, pseudo-additive measures, null-additive set functions and non-monotonic measures cover different aspects of this nonlinear theory. The interested reader can find information about these integration theories in a lot of classical and current sources; see for example [6, 8, 19, 20, 22, 23, 28] and references therein. Several authors have also recently paid attention to the second aspect that we want to point out in the paper —the vector-valued generalizations—; we may mention here the papers by Kawabe (see for example [14, 15, 16] and references therein), as well as by some other authors [30, 31]). For this vector measure case, the relation between integrable functions and weakly integrable functions with the Bocher, Dunford and Pettis integrability of the corresponding distribution functions have recently been studied in the interesting paper [11] by Fernández, Mayoral and Naranjo. Though we will assume some strong requirements on the vector-valued capacity, countable additivity is not one of them. Of course, some assumptions must be made on the vector-valued fuzzy measure for assuring a reasonable behavior of the integrals (see [14, 16] and references therein). Essentially, integration with respect to a Riesz space valued monotone capacity is defined and analyzed in these papers and the ones to which they refer. Our aim is to use a Lebesgue type integral being free of the order structure of the Banach space in which the capacity takes its values. In [26], such a kind of Bartle-Dunford-Schwartz integral for vector measures on quasi-Banach spaces is considered, but in this case the order properties of quasi-Banach lattices are also strongly used. In the case of the present paper, we do not take into account any lattice order in the Banach space where the capacity takes its values for the construction of our Choquet-Lebesgue type integral. Of course, Banach-lattice-valued capacities will be considered in examples and applications.

2. Preliminaries

Through this section and in the rest of the paper, let (Ω, Σ, μ) be a σ-finite measure space and E a Banach space. Consider a measurable function $f : \Omega \to \mathbb{R}^+$. The *Choquet integral* of f with respect to a scalar capacity $c : \Sigma \to \mathbb{R}^+$ —i.e. a monotone set function that satisfies $c(\emptyset) = 0$— is given by

$$\int_\Omega f\, dc := \int_0^\infty c(\{f > t\})\, dt.$$

We use the standard notation $\{f > t\} = \{w \in \Omega : f(w) > t\}$. Since $c(\{f > t\})$ is decreasing, this integral is defined in the Lebesgue sense, although it may of course be ∞.

We will also use the notion of the *Pettis integral,* an integral of a vector-valued function with respect to a scalar measure, which is sometimes called a weak integral. Consider a function ϕ acting in a set Ω with values in the Banach space E and a countably-additive measure μ on a sigma algebra Σ of subsets of Ω. The function ϕ is called *weakly measurable* if for any $x^* \in E^*$ the scalar function $x^* \circ \phi(\cdot) = \langle \phi(\cdot), x^* \rangle$ is measurable. The function ϕ is Pettis integrable over a measurable subset A if for each $x^* \in E^*$ and $A \in \Sigma$ the function $x^* \circ \phi$ is integrable on A and there exists an element $(P) - \int_A \phi\, d\mu \in E$ such that

$$\Big\langle (P) - \int_A \phi\, d\mu, x^* \Big\rangle = \int_A x^* \circ \phi\, d\mu = \int_A \langle \phi(t), x^* \rangle\, d\mu(t).$$

$(P) - \int_A \phi\, d\mu$ is called the Pettis integral of ϕ with respect to μ. The Dunford integral, which is defined in a similar way, is given when the integral is not necessarily in E; notice that the requirement of integrability of all the functions $x^* \circ \phi$ is enough for ϕ to have an integral in E^{**}, that is the case of Dunford integrable functions.

Let us define now the Bartle-Dunford-Schwartz integral with respect to a Banach-space-valued countably additive vector measure $m : \Sigma \to E$ for scalar measurable functions. Let f be a scalar-valued measurable function. Such a function f is said to be integrable with respect to m if the following two requirements are satisfied. The first one is that $|f|$ is integrable with respect to all measures $|\langle m, x^* \rangle|$, where $\langle m, x^* \rangle := x^* \circ m$ is a (non necessarily positive) scalar measure and $| \cdot |$ is the variation. The second one is that a vector-valued integral must exist: for each $A \in \Sigma$ there exists an element $\int_A f dm \in E$ such that $\langle \int_A f dm, x^* \rangle = \int_A f d\langle m, x^* \rangle$, $x^* \in E^*$. The space of all equivalence classes of integrable functions is denoted by $L^1(m)$; it is a σ-order continuous Banach lattice with the natural order and the norm

$$\|f\|_{L^1(m)} := \sup_{x^* \in B_{E^*}} \int_\Omega |f| d|\langle m, x^* \rangle|, \quad f \in L^1(m).$$

For fundamental notions and results on vector measures, Banach lattices and Banach function spaces we refer to [2, 9, 18, 21, 27]. If $X(\mu)$ is a σ-order continuous Banach function space over a finite measure μ and $T : X(\mu) \to E$ is an operator, the expression $m_T(A) := T(\chi_A)$, $A \in \Sigma$, provides a canonical way of defining vector measures which was shown to be a powerful tool in functional analysis (see i.e. [21, Ch.3]).

Let us introduce the main concept used in this paper. A set function $C : \Sigma \to E$ is a *vector capacity* —also called a fuzzy capacity or a fuzzy measure— if $C(\emptyset) = 0$. Sometimes —as in the definition of the Choquet integral in the scalar case given above— the monotonicity property is also required, that is, $\|C(A)\| \leq \|C(B)\|$ whenever $A \subset B$, $A, B \in \Sigma$. This requirement is not needed in the general case but is useful when defining an integral with respect to a Riesz-space-valued capacity (see [14]).

Fundamental examples of vector-valued capacities in classical analysis are easy to find. For instance, let $m \colon \Sigma \to E$ be a vector measure and $\Phi \colon E \to E$ a norm bounded function (i.e. $\sup_{x \in B_E} \|\Phi(x)\| < \infty$), with the requirements as are needed in each special case. The composition of these functions $\mathcal{C}(A) := \Phi\big(m(A)\big)$, $A \in \Sigma$, gives an example of a class of vector-valued capacities. This is of particular interest if Φ is given by a classical non-linear positive operator, as the Hardy-Littlewood maximal operator. Thus, in the same way that linear operators acting in Banach function spaces provide vector measures, non-linear maps in these spaces give vector-valued capacities in a natural way.

2.1. **Basics on integration with respect to vector-valued capacities.** Let us give now the mathematical motivation for the definition of an integral with respect to a vector-valued capacity. It follows the lines of the characterization obtained in [11] in the case of vector measures using the distribution function, which makes sense in the general case of vector-valued capacities. It also coincides with the natural representation of simple functions and the corresponding integrals for the case of an order based integral having values in Riesz spaces (see [14, Proposition 2.14]). Consider a Σ-simple function $f \geq 0$ represented as follows: there are sets $A_1, ..., A_n \in \Sigma$ such that $A_n \subseteq A_{n-1} \cdots \subseteq A_1$ and positive real numbers $\alpha_1, ..., \alpha_n$ such that

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i}.$$

Notice that this representation always exists for a positive simple function and is unique (μ-a.e.). We call it the decreasing representation of f. We can define the integral of such a function f over $A \in \Sigma$ with respect to the capacity C by

$$\int_A f\,dC := (P) - \int_0^\infty C(\{f\chi_A > t\})\,dt = \sum_{i=1}^n \alpha_i C(A \cap A_i). \tag{2.1}$$

This Choquet-type definition and the particular formula of the integral for simple functions provides an interesting framework for the general definition of impact indices in bibliometry (see for example Definition 3 in [29]). The reader can find a complete explanation of this relevant application in [13] (see all the paper, but mainly §4.). Actually the ideas laid out there were the main motivation for our research, since vector-valued capacities allow us to define a broader class of aggregation functions including the ones appearing in multi-indices based research evaluation (see also e.g. [25]).

Notice that the function $C_f : [0, \infty) \to E$ given by

$$C_f(t) := C(\{f > t\}),$$

whenever f is a positive simple function as above, is always well defined and —since it takes only a finite set of different values on E— it is also Pettis integrable. In fact, notice that we can define the integrals with respect to the scalar components $\langle C, x^*\rangle$ of C as

$$\int_A f\,d\langle C, x^*\rangle := \int_0^\infty \langle C_{f\chi_A}(t), x^*\rangle\,dt = \int_0^\infty \langle C(\{f\chi_A > t\}), x^*\rangle\,dt$$

$$= \sum_{i=1}^n \alpha_i \langle C(A \cap A_i), x^*\rangle.$$

With these definitions, notice that for every positive simple function f, $A \in \Sigma$ and $x^* \in E^*$,

$$\Big\langle \int_A f\,dC, x^* \Big\rangle = \int_A f\,d\langle C, x^*\rangle.$$

This motivates the following general definition, which almost coincides with the characterization of Bartle-Dunford-Schwartz integrable functions in terms of distribution functions in the case when C is a vector measure (see Theorem 2.7 in [11]). Note that $C_{|f|}$ can be defined as above for every measurable function f: we will call it the *distribution function of f*.

Definition 2.1. Consider a measurable capacity $C : \Sigma \to E$, that is, such that its distribution function is strongly measurable. We say that a measurable function $0 \le f : \Omega \to E$ is integrable with respect to the capacity C if and only if for every $A \in \Sigma$, the function $C_{f\chi_A} : [0, \infty) \to E$ is Pettis integrable. In this case,

$$\int_A f\,dC = (P) - \int_0^\infty C_{f\chi_A}\,dt,$$

and, by the properties of the Pettis integral,

$$\Big\langle \int_A f\,dC, x^* \Big\rangle = \int_0^\infty \langle C_{f\chi_A}(t), x^*\rangle\,dt = \int_A f d\langle C, x^*\rangle, \quad x^* \in E^*.$$

Section 5, and mainly Section 7 in [7] provide the technical requirements which are needed for this integral to extend to a complete space of integrable functions. The reader can find there all the information required for a general measurable function to belong to the natural completion of this space. The integration map I_C associates each simple function to its integral in E, that is

$$I_C(f) = \int_\Omega f dC, \quad f \text{ simple},$$

where the integral is defined using formula (2.1). The main properties of this operator can be found in [7] (see Lemma 4, Proposition 8 and other related results in Section 6 of this paper).

Remark 2.2. Let us compare the different norms which can be given to a space of (classes of) integrable functions —using the vector measure integration as a reference— with the Bochner and Pettis norms of the associated distribution functions. The relevance of these formulas is that each of them provide a different way of defining an information impact measure, as will be shown in the remainder of the paper. A total description of the relations between these norms can be found in [11]. Let us look at the case of a simple function f. Consider an ordered representation $\sum_{i=1}^n a_i \chi_{A_i}$ of $|f|$, where $A_n \subset A_{n-1} \cdots \subset A_1$ are measurable sets and $a_1, ..., a_n$ are real numbers. Let $C : \Sigma \to E$ be a measurable capacity whose semivariation is equivalent to a measure μ in the sense that they have the same null sets. Then we can define the following norm for the space of (classes of μ-a.e.) simple functions using the Bochner norm of the distribution function $C_{|f|}$,

$$\|f\|_C := \int_0^\infty \|C_{|f|}(t)\| \, dt, \quad f \textit{ simple.}$$

In the case when C is a vector measure m, the computation gives

$$\|f\|_C = \|f\|_m = \int_0^\infty \|m_{|f|}(t)\| \, dt = \sum_{i=1}^n a_i \|m(A_i)\|.$$

This formula *does not* provide the norm of f in $L^1(m)$. In the general case of C being just a capacity, we will call this norm —which in fact can be extended to the function space that is the completion of the space of simple functions— the *strong norm* for the space of integrable functions with respect to the capacity C.

However, assume now that $C = m$, a positive vector measure, that is, a countably additive vector measure having values in the positive cone of a Banach lattice. If we consider the Pettis norm for the Pettis integrable function $C_{|f|} : [0, \infty) \to E$ given by

$$\|f\|_P := \sup_{x^* \in B_{E^*}} \int_0^\infty |\langle C_{|f|}(t), x^* \rangle| \, dt,$$

for C being the positive vector measure m, we obtain

$$\|f\|_P = \sup_{x^* \in B_{E^*}^+} \int_0^\infty |\langle m_{|f|}(t), x^* \rangle| \, dt = \sup_{x^* \in B_{E^*}^+} \int_0^\infty |\langle \sum_{i=1}^n a_i m(A_i), x^* \rangle| \, dt.$$

$$= \| \sum_{i=1}^n a_i m(A_i) \|_E = \| \int_\Omega |f| \, dm \|_E.$$

This coincides with the norm of f in $L^1(m)$ (see for example Lemma 3.13 in [21]) for any integrable function $f \in L^1(m)$. We have that

$$\|f\|_P = \sup_{x^* \in B_{E^*}} \int_0^\infty |\langle m_{|f|}(t), x^* \rangle| \, dt \leq \sup_{x^* \in B_{E^*}} \int_0^\infty \langle m_{|f|}(t), |x^*| \rangle \, dt$$

$$= \sup_{x^* \in B^+_{E^*}} \int_0^\infty \langle m_{|f|}(t), x^* \rangle \, dt \leq \sup_{x^* \in B_{E^*}} \int_0^\infty |\langle m_{|f|}(t), x^* \rangle| \, dt.$$

Then

$$\|f\|_P = \sup_{x^* \in B^+_{E^*}} \int_0^\infty \langle m_{|f|}(t), x^* \rangle \, dt = \sup_{x^* \in B^+_{E^*}} \langle \int_\Omega |f| \, dm, x^* \rangle,$$

and so

$$\|f\|_P = \sup_{x^* \in B^+_{E^*}} \langle \sum_{i=1}^n a_i m(A_i), x^* \rangle = \| \int_\Omega |f| dm \| = \|f\|_{L^1(m)}.$$

Summing up, we obtain that the definition of integrability with respect to a vector-valued capacity which fits in better with the notion of integrability with respect to a countably additive vector measure is the Pettis integral of the distribution function, at least in the case of positive vector measures. This is the natural framework of this paper, since all reasonable vector-valued impact measures are defined by positive capacities, as we will show later on.

2.2. **Semivariations of capacities and scalar variations.** Variations and semivariations for vector measures provide natural norms for vector spaces so that they can be made into Banach spaces. Thus, it seems reasonable to ask for convenient adaptations of these notions in the case of vector-valued capacities in order to obtain something that could be considered as the natural topological structure for these spaces. Due to our interest in integration, the main properties that we need are associated to the behavior of capacities regarding the uniform limits of their evaluations on sequences of disjoint sets. As the last remark of the previous section indicates, it seems natural to associate, on the one hand, variation/Bochner integrability/strong integration of scalar functions, and on the other hand, semivariation/Pettis integrability/Bartle-Dunford-Schwartz integration of scalar functions. This is the reason why our first step consists in analyzing the notions of variation and semivariation for vector-valued capacities and the relations between them. So, in this section we introduce some fundamental definitions and analyze some properties of vector-valued capacities. We also prove a "capacity version" of one of the main results concerning vector measures, which provides an equivalence between the weak convergence properties and the norm convergence: the Orlicz-Pettis Theorem for vector measures.

Let (Ω, Σ) be a measure space. Consider a vector-valued capacity $C : \Sigma \to E$ and an element $x^* \in E^*$. Consider the scalar (signed) capacity $\langle C, x^* \rangle$ given by

$$\langle C, x^* \rangle(A) := \langle C(A), x^* \rangle, \quad A \in \Sigma.$$

Notice that the expression $|\langle C(\cdot), x^* \rangle|$ also defines a scalar capacity, in this case positive. We define the *variation* $|\langle C, x^* \rangle|$ of this capacity in a set $A \in \Sigma$ by

$$|\langle C, x^* \rangle|(A) := \sup_{(A_i)_{i=1}^n \subset \Sigma \, partition \, of \, A} \sum_{i=1}^n |\langle C(A_i), x^* \rangle|;$$

it is also a positive capacity. We say that the capacity C is *scalar bounded* if

$$\sup_{x^*\in B_{E^*}} |\langle C, x^*\rangle|(B) < \infty$$

for every $B \in \Sigma$.

The *semivariation* of a vector capacity C on a set $A \in \Sigma$ is given by

$$\|C\|(A) := \sup_{x^*\in B_{E^*}} |\langle C, x^*\rangle|(A).$$

We say that the vector capacity C is of *bounded semivariation* if for every $A \in \Sigma$, $\|C\|(A) < \infty$.

Let us give now some definitions regarding convexity/concavity type inequalities for scalar-valued capacities. We say that a scalar capacity $c : \Sigma \to \mathbb{R}$ is superadditive if for every couple of disjoint sets $A, B \in \Sigma$, $c(A \cup B) \geq c(A) + c(B)$. We say that c is subadditive if for every couple of disjoint sets $A, B \in \Sigma$, $c(A \cup B) \leq c(A) + c(B)$. These notions will be generalized and extended for vector-valued capacities in different manners later on in the paper.

Almost all the properties of semivariations of a vector capacity are consequences of the behavior of its scalar-valued components. The main properties of the semivariation of scalar-valued capacities we need in the present paper can be found in Lemma 2 of [7]. For definitions of semivariation type set functions and quasi-variations for vector-valued capacities, see Section 5 in [7]. In what follow we give some related results that will be used in the following sections.

Lemma 2.3. *Let $C : \Sigma \to E$ be a Banach-space-valued capacity. Then*

1. *The variation of a scalar capacity is a superadditive (and so monotone) scalar capacity.*
2. *C is scalar bounded if and only if*
$$\|C\|(\Omega) := \sup_{x^*\in B_{E^*}} |\langle C, x^*\rangle|(\Omega) < \infty.$$
3. *For every $A \in \Sigma$,*
$$\|C\|(A)$$
$$= \sup\Big\{\Big\| \sum_{i=1}^n \epsilon_i C(A_i)\Big\| : |\epsilon_i| = 1, \, (A_i)_{i=1}^n \subset \Sigma \, partition \, of \, A\Big\}.$$
4. *For every $A \in \Sigma$,*
$$\sup_{B\subseteq A, \, (B_i)_{i=1}^n \subset\Sigma \, partition \, of \, B} \| \sum_{i=1}^n C(B_i)\|$$
$$\leq \|C\|(A) \leq 2 \sup_{B\subseteq A, \, (B_i)_{i=1}^n \subset\Sigma \, partition \, of \, B} \| \sum_{i=1}^n C(B_i)\|.$$
Consequently, a vector capacity C is of bounded semivariation if and only if the set $\Big\{ \sum_{i=1}^n C(B_i) : B_1, ..., B_n \in \Sigma \, disjoint\Big\}$ is norm bounded.

Proof. (1) Fix a functional $x^* \in B_{E^*}$, a couple of disjoint sets $A, B \in \Sigma$, and measurable partitions $(A_i)_{i=1}^n$ of A and $(B_i)_{i=1}^m$ of B in Σ. Then $(A_i)_{i=1}^n \cup (B_i)_{i=1}^m$ define a partition of $A \cup B$, and

$$\sum_{i=1}^n |\langle C(A_i), x^* \rangle| + \sum_{i=1}^m |\langle C(B_i), x^* \rangle| \leq |\langle C, x^* \rangle|(A \cup B).$$

This obviously implies the superadditivity of $|\langle C, x^* \rangle|$, and the monotonicity $|\langle C, x^* \rangle|$ since this variation is always nonnegative.

(2) is an obvious consequence of (1). For (3) just take a measurable partition $(A_i)_{i=1}^n$ of A and $x^* \in B_{E^*}$. Then there are $\epsilon_i \in \{+1, -1\}$ such that

$$\sum_{i=1}^n |\langle C(A_i), x^* \rangle| = \sum_{i=1}^n \epsilon_i \langle C(A_i), x^* \rangle \leq \| \sum_{i=1}^n \epsilon_i C(A_i) \|,$$

which implies (3). The converse inequality also holds using the same kind of straightforward calculation. To see (4), just take into account that for every $A \in \Sigma$, every partition $(A_i)_{i=1}^n \subset \Sigma$ of A and every choice of signs ϵ_i, $i = 1, ..., n$, we can reorder $(A_i)_{i=1}^n$ in a way that $\sum_{i=1}^n \epsilon_i C(A_i) = \sum_{i=1}^{n_0} C(A_i) - \sum_{i=n_0+1}^n C(A_i)$, $0 \leq n_0 \leq n$, and so by (3)

$$\| \sum_{i=1}^n \epsilon_i C(A_i) \| \leq \| \sum_{i=1}^{n_0} C(A_i) \| + \| \sum_{i=n_0+1}^n C(A_i) \|$$

$$\leq 2 \sup_{B \subseteq A, \, (B_i)_{i=1}^n \subset \Sigma \, partition \, of \, B} \| \sum_{i=1}^n C(B_i) \| = \|C\|(A).$$

For the first inequality in (4), just take a partition $(B_i)_{i=1}^n \subset \Sigma$ of $B \subseteq A$, a functional $x^* \in B_{E^*}$ and consider the inequalities

$$\langle \sum_{i=1}^n C(B_i), x^* \rangle \leq \sum_{i=1}^n |\langle C(B_i), x^* \rangle| \leq \|C\|(A),$$

where the last inequality is a consequence of the monotonicity of $|\langle C, x^* \rangle|$. Then the result holds just taking the supremum with respect to every $x^* \in B_{E^*}$, every measurable $B \subseteq A$ and every partition on the left-hand side. □

From now on we will write $\|C\|_0(A)$ for the function

$$\|C\|_0(A) := \sup_{B \subseteq A, \, (B_i)_{i=1}^n \subset \Sigma \, partition \, of \, B} \| \sum_{i=1}^n C(B_i) \|,$$

that is defined for every $A \in \Sigma$.

Throughout the rest of the paper, we assume that the capacity C is bounded, i.e. $\|C\|(\Omega) < \infty$. After Lemma 2.3, this is equivalent to $\|C\|_0(\Omega) < \infty$.

In order to get better properties of the associated integration, we require stronger properties of the vector-valued *positive* capacities, which will be the case in most of the examples and models presented in the paper. Recall that we say a Banach-lattice-valued capacity is positive if $C : \Sigma \to E^+$, and monotone if $A \subseteq B$ implies $C(A) \leq C(B)$ in the lattice order. Recall also that $C(\emptyset) = 0$. Notice that if C is superadditive and positive, then for each couple of disjoint sets $A, B \in \Sigma$, $0 \leq C(A) \leq C(A) + C(B) \leq C(A \cup B)$, and so C is monotone. In particular, positive vector measures are always monotone.

A set $N \subseteq B_{E^*}$ is norming for C if $\|x\| := \sup_{x^* \in N} |\langle x, x^* \rangle|$, $x \in Sum(\mathcal{R}(C))$, the set of finite sums of products of elements of $\mathcal{R}(C)$ and positive scalars. The same definition makes sense for families of vector measures: N is norming for a set $\{C_\tau : \tau \in T\}$ it is such for each C_τ.

Definition 2.4. We say that a vector-valued capacity C is scalar subadditive with respect to a norming set N for C if for all disjoint $A, B \in \Sigma$ and $x^* \in N$,

$$|\langle C(A \cup B), x^* \rangle| \leq |\langle C(A), x^* \rangle| + |\langle C(B), x^* \rangle|,$$

i.e. the scalar capacity $|\langle C(\cdot), x^* \rangle|$ is subadditive for every $x^* \in N$.

Example 2.5. (1) Let $([0,1], \mathcal{B}, \mu)$ the Lebesgue measure space and consider the vector capacity $C_{1/2} : \mathcal{B} \to L^1[0,1]$ given by

$$C_{1/2}(A) = \chi_{(A \cap [0,1/2]) \cup (t_{-1/2}(A \cap [1/2,1]))},$$

where $t_{-1/2}$ is the linear transformation $t_{-1/2}(r) = r - 1/2$, $r \in [1/2, 1]$; that is $t_{-1/2}(A \cap [1/2,1]) = A \cap [1/2,1]) - 1/2$. Clearly, $C_{1/2}$ is not a vector measure, since it is not additive: for example, $C_{1/2}([0,1]) = \chi_{[0,1/2]}$, but $C_{1/2}([0,1/2]) + C_{1/2}([1/2,1]) = 2\chi_{[0,1/2]}$. However, it is subadditive when considered as a function having values in $L^1[0,1]$, and so it is scalar subadditive with respect to the positively norming set $N = (B_{L^\infty[0,1]})^+ = (B_{E^*})^+$.

(2) Every positive vector measure is scalar subadditive with respect to the positive cone of E^*. In fact, it is enough that for each pair of disjoint measurable sets A and B, the vector capacity C having values in the Banach lattice E satisfy $0 \leq C(A \cup B) \leq C(A) + C(B)$ in the lattice order. Since all the elements are positive, we have that for each $x^* \in N = (E^*)^+$ and disjoint $A, B \in \Sigma$,

$$|\langle C(A \cup B), x^* \rangle| = \langle C(A \cup B), x^* \rangle$$
$$\leq \langle C(A), x^* \rangle + \langle C(B), x^* \rangle = |\langle C(A), x^* \rangle| + |\langle C(B), x^* \rangle|.$$

Lemma 2.6. *If C is a scalar subadditive vector capacity and N is a norming set for C, then the variations of the scalar capacities $\langle C, x^* \rangle$, $x^* \in N$, are finitely additive measures.*

It is just a consequence of the definition of a scalar subadditive capacity and Lemma 2.3. In what follows we shall often assume that each scalar capacity $|\langle C, x^* \rangle|$, $x^* \in N$, is a finite (finitely additive) measure. In particular the lemma above shows that this happens if C is scalar subadditive with respect to N.

Definition 2.7. A family of E-valued capacities $\mathcal{C} = \{C_\tau : \Sigma \to E \,|\, \tau \in T\}$ is uniformly countably additive if for any sequence $(A_n)_{n=1}^\infty$ of pairwise disjoint sets of Σ, $\lim_n \| \sum_{m=n}^\infty C_\tau(A_m)\| = 0$ uniformly in $\tau \in T$.

Notice that *we are not assuming that the capacities are additive.* That is, countable additivity of a capacity does not imply additivity, as is natural in the non-additive context we are working in. The next theorem is the main result on semivariations of capacities in this paper. It can be understood as an extension to vector-valued capacities of some classical fundamental results on summability in Banach spaces and vector measures: the Orlicz-Pettis Theorem ([9, Cor. I.4.4]) and the Vitali-Hahn-Sacks Theorem ([9, Cor. I.5.6]).

Theorem 2.8. *Let $\mathcal{C} = \{C_\tau : \tau \in T\}$ be a family of E-valued capacities and let N be a norming set for $\mathcal{C}$. Then the following are equivalent.*

(1) *$\mathcal{C}$ is uniformly countably additive.*
(2) *The set $\{\langle C_\tau, x^*\rangle : \tau \in T,\, x^* \in N\}$ is uniformly countably additive.*
(3) *If (A_n) is a sequence of pairwise disjoint elements of Σ and $(A_n^j)_{j=1}^{m_n}$ is a partition of A_n for each n, then*
$$\lim_n \|\sum_{j=1}^{m_n} C_\tau(A_n^j)\| = 0$$
uniformly in $\tau \in T$.
(4) *If (A_n) is a sequence of pairwise disjoint elements of Σ, then*
$$\lim_n \|C_\tau\|_0(A_n) = 0$$
uniformly in $\tau \in T$.
(5) *If (A_n) is a sequence of pairwise disjoint elements of Σ, then*
$$\lim_n \|C_\tau\|(A_n) = 0$$
uniformly in $\tau \in T$.
(6) *The set $\{|\langle C_\tau, x^*\rangle| : \tau \in T,\, x^* \in B_{E^*}\}$ is uniformly countably additive.*

Proof. Clearly, (1) implies (2). To see that (2) implies (3), take $\varepsilon > 0$ and consider a sequence (A_n) with the corresponding partitions (A_n^j), and define the sequence of disjoint sets ordered by n putting consecutively the elements of each partition of A_n, $(A_n^j)_{n=1,j=1}^{\infty,m_n}$. Then there is an n_0 such that for every $x^* \in B_{E^*}$ and $\tau \in T$, $|\langle \sum_{k=n_0,j=1}^{\infty,m_k} C_\tau(A_k^j), x^*\rangle| \leq \varepsilon$, and so $\| \sum_{k=n_0,j=1}^{\infty,m_k} C_\tau(A_k^j)\| \leq \varepsilon$. Since this holds for every ε, this gives (3).

For (3) implies (4), suppose that (4) does not hold. Then there is a sequence (A_n) of pairwise disjoint elements of Σ such that $\sup_{\tau\in T} \|C_\tau\|_0(A_n) \geq 2\delta > 0$ for every n. Then there is a sequence of disjoint measurable sets $(B_n^j)_{j=1}^{m_n}$ such that $\cup_{j=1}^{m_n} B_n^j \subset A_n$ and
$$\sup_{\tau\in T} \|C_\tau\|_0(A_n) \leq \sup_{\tau\in T} \|\sum_{j=1}^{m_n} C_\tau(B_n^j)\| + \delta.$$
This contradicts (3).

By Lemma 2.3, (4) and (5) are equivalent. To see (5) implies (6), suppose that $\{|\langle C_\tau, x^*\rangle| : \tau \in T,\ x^* \in N\}$ is not uniformly countably additive. Then we find a disjoint sequence (A_n) and a sequence of positive numbers m_j such that
$$(*) = \sup\{\sum_{n=m_j+1}^{m_{j+1}} |\langle C_\tau, x^*\rangle|(A_n) :\, \tau \in T,\, x^* \in N,\} > \delta > 0.$$
But by Lemma 2.3(1), we have
$$\sup\{\|C_\tau\|(\cup_{n=m_j+1}^{m_{j+1}} A_n) :\, \tau \in T\}$$
$$\geq \sup\{|\langle C_\tau, x^*\rangle|(\cup_{n=m_j+1}^{m_{j+1}} A_n) :\, \tau \in T,\, x^* \in N\} \geq (*) > \delta > 0.$$
This contradicts (5) and gives the result. Obviously (6) implies (1). □

In the case of a single vector capacity, this result gives the following equivalences.

Corollary 2.9. *Let C be an E-valued capacity and let N be a norming set for C. Then the following statements are equivalent.*

(1) *C is countably additive.*
(2) *The set $\{\langle C, x^* \rangle : x^* \in N\}$ is uniformly countably additive.*
(3) *If (A_n) is a sequence of pairwise disjoint elements of Σ and $(A_n^j)_{j=1}^{m_n}$ is a partition of A_n for each n, then*

$$\lim_n \| \sum_{j=1}^{m_n} C(A_n^j) \| = 0.$$

(4) *If (A_n) is a sequence of pairwise disjoint elements of Σ, then*

$$\lim_n \|C\|_0(A_n) = 0.$$

(5) *If (A_n) is a sequence of pairwise disjoint elements of Σ, then*

$$\lim_n \|C\|(A_n) = 0.$$

(6) *The set $\{|\langle C, x^* \rangle| : x^* \in N\}$ is uniformly countably additive.*

The following definitions will be used later on. They allow us to give easy conditions under which some of the requirements of the previous results are fulfilled.

Definition 2.10. A vector capacity is said to satisfy the Fatou property if

This is the natural extension of the notion of *Fatou capacity* [6, p. 2] to the vector-valued case.

Moreover, a vector capacity is said to satisfy the weak Fatou property if for every $x^* \in E^*$ and $A_n, A \in \Sigma$, if $A_n \uparrow A$, then $\langle C(A_n), x^* \rangle \to \langle C(A), x^* \rangle$.

Finally, a vector-valued capacity is absolutely continuous with respect to a measure λ if and only if $\|C(A)\| \to 0$ whenever $\lambda(A) \to 0$.

Notice that if C is scalar subadditive and absolutely continuous with respect to a finite measure λ, then it is Fatou, and then weakly Fatou.

3. The first model: C-integrable functions of Bochner type

In this section we will explain the basics of one of the types of integration with respect to capacities —the strongest one— which we propose as models for impact measuring tools. We will use the symbol $L^1_{\mathcal{B}}(C)$ for this space in the present paper. The $\mathcal{B}$ in this notation refers to "Bochner", since we use the vector-valued integration for the distribution function in the definition. The space $L^1_{\mathcal{B}}(C)$ can be identified with the space $L^1(\|\Lambda\|)$ studied in Section 5 of [7], and is also related to the spaces $S_{\|\Lambda\|}$ and $w - L^1_v(\Lambda)$ appearing there. Let Σ be a σ-algebra of subsets of an abstract set Ω, E a Banach space and $C \colon \Sigma \to E$ a set function satisfying $C(\emptyset) = 0$. Sometimes, monotonicity of the set function C is also required in the definition of scalar capacities, specially in the setting of Information Science (see for example [29]). The reason is that it is difficult to find a reasonable meaning for an impact measure which would not increase when the set of information items increases. The vector-valued version of monotonicity involves the evaluation of the norm of the corresponding vectors, and will be considered latter together with subadditivity.

Denote by $\mathcal{M}$ the space of all Σ-measurable functions $f\colon \Omega \to \mathbb{R}$. For every $0 \leq f \in \mathcal{M}$, we denote by C_f the map $C_f\colon [0,\infty) \to E$ defined as

$$C_f(t) = C\big(\{\omega \in \Omega :\ f(\omega) > t\}\big).$$

Consider a simple function $\varphi\colon \Omega \to [0,\infty)$. If $\varphi = 0$ then we have $C_\varphi = 0$. In other case, we always can write $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_j}$ with $(A_j)_{j=1}^n$ being a disjoint sequence of measurable sets and $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_n$, and so

$$C_\varphi = \sum_{k=1}^n C\Big(\bigcup_{j=k}^n A_j\Big)\chi_{[\alpha_{k-1},\alpha_k)}.$$

Hence, C_φ is an E-step function (see [2, p. 423]) considering the Lebesgue measure m on $[0,\infty)$, and its integral with respect to m is given by

$$\int_\Omega C_\varphi \, dm = \sum_{k=1}^n C\Big(\bigcup_{j=k}^n A_j\Big)(\alpha_k - \alpha_{k-1}).$$

Note that if C is finitely additive then $\int_\Omega C_\varphi \, dm = \sum_{j=1}^n \alpha_j \, C\big(A_j\big)$, i.e. the integral of φ with respect to C.

We define the integral of φ with respect to the set function C as

$$\int_\Omega \varphi \, dC := \int_\Omega C_\varphi \, dm.$$

Assume now that C is Fatou. Then C_f is strongly m-measurable for every $0 \leq f \in \mathcal{M}$ ([2, Definition 11.36]). Indeed, taking a sequence (φ_n) of simple functions such that $0 \leq \varphi_n \uparrow f$, for each fixed $t \in [0,\infty)$ we have that $\{\omega \in \Omega :\ \varphi_n(\omega) > t\} \uparrow \{\omega \in \Omega :\ f(\omega) > t\}$ and so by (2.10), $C_{\varphi_n}(t) \to C_f(t)$ in E and C_{φ_n} is an E-simple function.

Definition 3.1. We will say that $f \in \mathcal{M}$ is *(strongly) integrable with respect to C* —in symbols, $f \in L^1_{\mathcal{B}}(C)$— if $C_{|f|}\colon [0,\infty) \to E$ is Bochner integrable with respect to m ([2, Definition 11.42]).

Assume that

$$A, B \in \Sigma \ \text{ with } \ A \subset B \ \Rightarrow\ \|C(A)\|_E \leq \|C(B)\|_E. \tag{3.1}$$

This is the vector-valued version of the notion of the monotonicity of capacity (see [6, p.98]) in the vector-valued case.

This result can be found in Proposition 15 (c) in [7], taking into account that here we say that f is (strongly) integrable if it belongs to the Choquet space $L^1(\|C\|)$ of the scalar capacity $\|C\|$ ($L^1(\|\Lambda\|)$ in the notation of [7]).

From now on, we assume that the *vector-valued capacity has the Fatou property* (condition (2.10) in what follows); although not always needed, it provides suitable conditions for structures with good properties. Notice that it is not needed for the capacity C for a simple function to be (strongly) integrable with respect to C, since for these functions the next result always holds.

Lemma 3.2. *Let C be a vector capacity with the Fatou property. A measurable function $f \in \mathcal{M}$ is (strongly) integrable with respect to C —that is, $f \in L^1_{\mathcal{B}}(C)$— if and only if $\int_0^\infty \|C_{|f|}(t)\|_E \, dt < \infty$.*

Proof. If $f \in \mathcal{M}$ is integrable with respect to C then, $C_{|f|}\colon [0,\infty) \to E$ is Bochner integrable with respect to m (by definition). So, there exists a sequence (ψ_n) of E-step functions on $[0,\infty)$ such that $\int_0^\infty \|C_{|f|}(t) - \psi_n\|_E \, dt \to 0$ and thus,

$$\int_0^\infty \|C_{|f|}(t)\|_E \, dt \leq \int_0^\infty \|C_{|f|}(t) - \psi_n\|_E \, dt + \int_0^\infty \|\psi_n\|_E \, dt < \infty.$$

Conversely, suppose that $f \in \mathcal{M}$ satisfies $\int_0^\infty \|C_{|f|}(t)\|_E \, dt < \infty$. Taking a sequence of simple functions (φ_n) such that $0 \leq \varphi_n \uparrow |f|$ by (2.10) it follows that

$$\|C_{|f|}(t) - C_{\varphi_n}(t)\|_E \to 0 \;\text{ pointwise for } t \in [0,\infty).$$

Since, by (3.1),

$$\|C_{|f|}(t) - C_{\varphi_n}(t)\|_E \leq \|C_{|f|}(t)\|_E + \|C_{\varphi_n}(t)\|_E \leq 2\|C_{|f|}(t)\|_E$$

and the function $t \to \|C_{|f|}(t)\|_E$ is in $L^1[0,\infty)$, applying the Dominated Convergence Theorem,

$$\int_0^\infty \|C_{|f|}(t) - C_{\varphi_n}(t)\|_E \, dt \to 0.$$

Hence $C_{|f|}$ is Bochner integrable as C_{φ_n} are E-step functions. Therefore f belongs to $L^1_{\mathcal{B}}(C)$, the space of integrable functions with respect to C. □

Let $\|\cdot\|_{L^1_{\mathcal{B}}(C)}$ be the positive map defined for every function $f \in \mathcal{M}$ by

$$\|f\|_{L^1_{\mathcal{B}}(C)} = \int_0^\infty \|C_{|f|}(t)\|_E \, dt \leq \infty.$$

Lemma 3.3. *The following assertions hold.*

a) $\|f\|_{L^1_{\mathcal{B}}(C)} = 0$ *if and only if* $f = 0$ C*-a.e. (i.e. except on a set* Z *of null capacity in the natural sense: for every* $A \subset Z$ *we have* $C(A) = 0$*, or equivalently (by* (3.1)*),* $C(Z) = 0$*.)*

b) $\|\lambda f\|_{L^1_{\mathcal{B}}(C)} = |\lambda| \cdot \|f\|_{L^1_{\mathcal{B}}(C)}$ *for all* $\lambda \in \mathbb{R}$ *and all* $f \in \mathcal{M}$*.*

c) *If* $f, g \in \mathcal{M}$ *are such that* $|f| \leq |g|$ *pointwise, then* $\|f\|_{L^1_{\mathcal{B}}(C)} \leq \|g\|_{L^1_{\mathcal{B}}(C)}$*.*

Proof. a) Suppose $f = 0$ C-a.e., that is, there exists $Z \in \Sigma$ of null capacity such that $f = 0$ in $\Omega\backslash Z$. Then, for every $t \in [0,\infty)$, we have that $\{\omega \in \Omega : |f| > t\} \subset Z$ and so $C_{|f|}(t) = 0$. Hence $\|f\|_{L^1_{\mathcal{B}}(C)} = 0$. Conversely, suppose that $\|f\|_{L^1_{\mathcal{B}}(C)} = 0$. Since

$$\big\{\omega \in \Omega :\, |f| > \tfrac{1}{n}\big\} \uparrow Z := \{\omega \in \Omega :\, |f| > 0\},$$

by (2.10), we have $C_{|f|}(\frac{1}{n}) \to C(Z)$. On the other hand, by (3.1) and since

$$\big\{\omega \in \Omega :\, |f| > \tfrac{1}{n}\big\} \subset \{\omega \in \Omega :\, |f| > t\}$$

for all $t \leq \frac{1}{n}$, we have

$$0 = \|f\|_{L^1_{\mathcal{B}}(C)} = \int_0^\infty \|C_{|f|}(t)\|_E \, dt \geq \int_0^{\frac{1}{n}} \|C_{|f|}(t)\|_E \, dt \geq \tfrac{1}{n} \, \|C_{|f|}(\tfrac{1}{n})\|_E.$$

So $\|C_{|f|}(\frac{1}{n})\|_E = 0$ for all n, and thus $C(Z) = 0$, that is, $f = 0$ C-a.e.

b) Let $f \in \mathcal{M}$ and $\lambda \in \mathbb{R}\backslash\{0\}$. It follows that

$$\begin{aligned} \|\lambda f\|_{L^1_{\mathcal{B}}(C)} &= \int_0^\infty \|C_{|\lambda f|}(t)\|_E \, dt = \int_0^\infty \|C_{|f|}(\tfrac{t}{|\lambda|})\|_E \, dt \\ &= |\lambda| \int_0^\infty \|C_{|f|}(s)\|_E \, ds = |\lambda| \, \|f\|_{L^1_{\mathcal{B}}(C)}. \end{aligned}$$

c) If $f, g \in \mathcal{M}$ are such that $|f| \le |g|$ pointwise, then $\{\omega \in \Omega : \, |f| > t\} \subset \{\omega \in \Omega : \, |g| > t\}$ for every $t \in [0, \infty)$ and so, by (3.1), $\|C_{|f|}(t)\|_E \le \|C_{|g|}(t)\|_E$ for every $t \in [0, \infty)$. Then $\|f\|_{L^1_{\mathcal{B}}(C)} \le \|g\|_{L^1_{\mathcal{B}}(C)}$. □

In what follows we assume that C is norm subadditive. Let us explain this concept.

Definition 3.4. A vector-valued capacity $C : \Sigma \to E$ is said to be quasi-subadditive if there is a constant $Q \ge 1$ such that

$$\|C(A \cup B)\|_E \le \; Q \left(\|C(A)\|_E + \|C(B)\|_E\right) \quad \text{for all} \quad A, B \in \Sigma. \tag{3.2}$$

If $Q = 1$, we simply say that it is subadditive. This is a natural extension of quasi-subadditive capacity [6, p.2] to the vector-valued case; all the cases we will consider will in fact be subadditive.

Remark 3.5.

(1) Recall that a vector-valued capacity C is monotone if $\|C(A)\| \le \|C(B)\|$ whenever $A \subseteq B$, $A, B \in \Sigma$ (see Equation (3.1)). This assumption, together with the previous one, are usually adopted for the aim of obtaining reasonably good properties for Choquet integration. This is not a strong restriction for the construction of impact measures, since the usual ones can be modeled by means of Choquet integrals with respect to capacities that satisfy these properties.

(2) All the vector-valued impact measures that we are analyzing are positive —that is, take values in the positive cone of a Banach lattice, usually $\mathbb{R}^n$ with the canonical order.

(3) The corresponding capacities are also monotone, and this makes it easier to verify some of the results on subadditivity. Although we have written the definition of subadditivity considering all pairs of sets, for monotone capacities the definition for only disjoint sets also works. To see this, note that obviously if C is subadditive for any pair of sets $A, B \in \Sigma$, then it is so when only disjoint A and B are considered. Conversely, if C is monotone and A and B are *any* pair of subsets in Σ, then we have that

$$\|C(A \cup B)\| \le Q \left(\|C(A \setminus B)\| + \|\mathcal{C}(B)\|\right) \le Q \left(\|C(A)\| + \|\mathcal{C}(B)\|\right),$$

and so both definitions coincide.

Example 3.6.

(1) An easy example of vector-valued additive measure is the one provided by an aggregation of weighted measures. Take a set $\mathcal{A}$ of articles that were published in 2010 and the (σ)-algebra Σ of all its subsets. Consider the canonical basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ endowed with the Euclidean norm $\|\cdot\|_2$, and define the vector measure $\nu : \Sigma \to \mathbb{R}^2$ by

$$\nu(A) = W(A)\, e_1 + V(A)\, e_2, \quad A \subseteq \mathcal{A}.$$

Here $W(A)$ is the number of citations the papers of the set A had until 2015, where the sum is weighted by the 2015 Thomson-Reuters 2-year impact factor of the journal in which each citation appeared. That is,

$$W(A) = \sum_{a\in A} w(a) = \sum_{a \in A} \Big(\sum_{c \in c(a)} IF(c) \Big), \quad A \subseteq \mathcal{A},$$

where $c(a)$ is the set of citations of the paper a and $IF(c)$ is the 2-year impact factor of the journal where the paper which contains the citation c was published. The scalar measure V is defined in the same way but changing the 2-year impact factor $IF(c)$ appearing in the definition of W by the 5-year (Thomson-Reuters) impact factor of the year 2015.

(2) Let us write an easy non-additive scalar case. We start with an example which is inspired by the *number of citations (NC)-index* explained as a case of Choquet integral in [29, Def.5]. Consider a (finite) set of authors $\mathcal{R}$ and the algebra Σ of all its subsets. Define the scalar capacity $\eta(A)$, $A \subseteq \mathcal{R}$, given by the number of citations that all the papers of the authors in A have received. This function is not in general additive but it is subadditive. To see this, suppose first that authors a_1 and a_2 in $\mathcal{R}$ are not coauthors of any paper. In this case we have that

$$\eta(\{a_1, a_2\}) = \eta(\{a_1\}) + \eta(\{a_2\}).$$

However, assume now that there is a paper coauthored by both a_1 and a_2 with at least one citation. Then we have that

$$\eta(\{a_1, a_2\}) < \eta(\{a_1, a_2\}) + 1 \leq \eta(\{a_1\}) + \eta(\{a_2\}).$$

since the coauthored paper is considered both in $\eta(\{a_1\})$ and $\eta(\{a_2\})$. Actually, it can be easily seen that subadditivity is satisfied for any pair of disjoint subsets of $\mathcal{R}$. Therefore, the capacity η is subadditive but not additive.

(3) Finally, we can use the ideas in (1) and (2) to define a 2-dimensional example of a vector-valued index that is a subadditive capacity but not a vector measure. In the context of (2), define a new scalar index ρ given by $\rho(A) = \frac{\eta(A)}{|A|}$, where $|A|$ is the number of elements in the set $A \subseteq \mathcal{R}$. This provides the citation author-mean of the set of authors A, and is subadditive since

$$\rho(A_1 \cup A_2) = \frac{\eta(A_1 \cup A_2)}{|A_1 \cup A_2|} \leq \frac{\eta(A_1)}{|A_1 \cup A_2|} + \frac{\eta(A_2)}{|A_1 \cup A_2|}$$

$$\leq \frac{\eta(A_1)}{|A_1|} + \frac{\eta(A_2)}{|A_2|} = \rho(A_1) + \rho(A_2)$$

for disjoint A_1 and A_2.

Define now a vector-valued function $\alpha : \Sigma \to \mathbb{R}^2$ by $\alpha(A) = \eta(A)\, e_1 + \rho(A)\, e_2$. For every pair of disjoint sets $A_1, A_2 \subseteq \mathcal{A}$ we have

$$\alpha(A_1 \cup A_2) = \eta(A_1 \cup A_2)\, e_1 + \rho(A_1 \cup A_2)\, e_2$$
$$\leq \eta(A_1)\, e_1 + \rho(A_1)\, e_2 + \eta(A_2)\, e_1 + \rho(A_2)\, e_2 = \alpha(A_1) + \alpha(A_2),$$

where the inequalities are written considering the standard ordering in the lattice $\mathbb{R}^2$. Therefore, $\|\alpha(A_1 \cup A_2)\|_2 \leq \|\alpha(A_1)\|_2 + \|\alpha(A_2)\|_2$. This gives an example of a vector-valued impact measure that is defined by a subadditive —but not additive— capacity.

Example 3.7. *Level indices.* Let us show a method for measuring the influence of a set of papers using not only the number of citations they have, but also the number of citations of all the papers which cite the items of the first set. We present a recursive construction of such a vector-valued capacity. Consider a set of papers $\mathcal{P}$ and let $B \subset \mathcal{P}$.

- First we will define a sequence of n-th level indices LI_n of B, where all indices of this form are scalar capacities. Let us take the set X_1 of all the papers which cite the papers in $B_0 := B$. We define $LI_1(B)$ as $|B_1|$ where $B_1 := X_1 \backslash B_0$ as we want to exclude the impact of citations between members of B_0. In the second step, we take the set X_2 of all the papers which cite the papers in B_1. Then $LI_2(B) := |B_2|$ for $B_2 := X_2 \backslash (B_0 \cup B_1)$, excluding the citations which were counted in the previous step. For the n-th step, $n \in \mathbb{N}$, we take the set X_n of all the papers which cite the papers in B_{n-1}. We set $B_n := X_n \backslash \bigcup_{k=0}^{n-1} B_k$ and $LI_n(B) := |B_n|$. Each n-th *level index* LI_n is a scalar capacity. We define the *level index* LI of B by
$$LI(B) := \sum_{k=1}^{\infty} LI_n(B_k) e_k.$$
$LI(B)$ is a vector-valued capacity and $\|LI(B)\|_{\ell^1}$ represents the number of all unique recursive citations of the papers from B. The above motivates our calling of $\|LI(\,\cdot\,)\|_{\ell^1}$ *the total index* TI. Clearly, $LI(B)$ cannot be additive in general. To see this, suppose first that B consists of two papers say p_1, p_2 which are cited by only one paper p_3. Let assume for simplicity that p_3 has no citations. In this case $LI_1(B) = 1$ and $LI_n(B) = 0$, $n > 1$, and
$$LI(B) = e_1 \neq 2e_1 = LI(\{p_1\})e_1 + LI(\{p_2\})e_1.$$
- Having defined the level index LI, we present a scalar variant of the previous construction. Let us take a sequence $\alpha := (\alpha_n)$ from $[0, 1]$ (e.g. $\alpha_n = \frac{1}{2^{n-1}}$). We define the α-*combined level index* LI_α by
$$LI_\alpha(B) := \|LI(B)\|_{\ell^1(\alpha)} = \sum_{n=1}^{\infty} \alpha_n |B_n|.$$
By the above, LI_α is (merely) a subadditive set function.
- We finish by giving a countably additive version of LI, namely
$$ALI(B) := \sum_{p \in B} LI(p).$$

Example 3.8. *A non-additive subadditive impact index.* Consider a set of papers $\mathcal{P}$ that satisfies the requirement that each of them has at least one of the members of a group of authors $\mathcal{A}$ among its authors. Define an impact index for the elements of $\mathcal{P}$ related to the authorship following similar ideas as the ones that can be found in the definition of the *interaction index* (see [19]). In our case, we define the index IA as follows.

- Let n be the number of authors in the set $\mathcal{A}$. Consider a set $\{p_i : i = 1, ..., n\}$ of positive real numbers that will be the weight given to each paper which has $a_i \in \mathcal{A}$ as author as follows. Write A_i for the set of articles in $\mathcal{P}$ which have a_i among their authors.
- Let $B \subset \mathcal{P}$, and define the index IA for the subset $B \subseteq \mathcal{P}$ as

$$IA(B) := \sum_{i=1}^{n} p_i(B),$$

where $p_i(B) = 0$ if $B \cap A_i = \emptyset$, and $p_i(B) = p_i$ if $B \cap A_i \neq \emptyset$.

Let us show that this measure is subadditive. Consider a pair B_1 and B_2 of disjoint sets of $\mathcal{P}$. Fix $i = 1, ..., n$. Then

$$p_i(B_1 \cup B_2) = 0, \quad \textit{if } (B_1 \cup B_2) \cap A_i = \emptyset,$$

and

$$p_i(B_1 \cup B_2) = p_i \ \textit{ if } \ (B_1 \cup B_2) \cap A_i \neq \emptyset.$$

For the same i, if $(B_1 \cup B_2) \cap A_i = \emptyset$ we have $p_i(B_1) = p_i(B_2) = 0$. On the other hand, if $(B_1 \cup B_2) \cap A_i \neq \emptyset$ we have $\max\{p_i(B_1), p_i(B_2)\} = p_i$, but $p_i(B_1) + p_i(B_2)$ can be equal to $2p_i$ if both $B_1 \cap A_i \neq \emptyset$ and $B_2 \cap A_i \neq \emptyset$. Summing up, and taking into account that this is true for every i we get that the scalar capacity IA is subadditive, but not necessarily additive. This provides a proper example of a subadditive capacity.

Lemma 3.9. *For every $f, g \in \mathcal{M}$, we have that $\|f+g\|_{L^1_{\mathcal{B}}(C)} \leq 2K\big(\|f\|_{L^1_{\mathcal{B}}(C)} + \|g\|_{L^1_{\mathcal{B}}(C)}\big)$*

Proof. Let $f, g \in \mathcal{M}$. Since

$$\{\omega \in \Omega : \, |f+g| > t\} \subset \big\{\omega \in \Omega : \, |f| > \tfrac{t}{2}\big\} \cup \big\{\omega \in \Omega : \, |g| > \tfrac{t}{2}\big\},$$

by (3.1) and (3.2) we have

$$\begin{aligned}
\|f+g\|_{L^1_{\mathcal{B}}(C)} &= \int_0^\infty \|C_{|f+g|}(t)\|_E \, dt \\
&\leq \int_0^\infty \Big\|C\big(\big\{\omega \in \Omega : \, |f| > \tfrac{t}{2}\big\} \cup \big\{\omega \in \Omega : \, |g| > \tfrac{t}{2}\big\}\big)\Big\|_E \, dt \\
&\leq K\Big(\int_0^\infty \|C_{|f|}(\tfrac{t}{2})\|_E \, dt + \int_0^\infty \|C_{|g|}(\tfrac{t}{2})\|_E \, dt\Big) \\
&= 2K\Big(\int_0^\infty \|C_{|f|}(s)\|_E \, ds + \int_0^\infty \|C_{|g|}(s)\|_E \, ds\Big) \\
&= 2K\big(\|f\|_{L^1_{\mathcal{B}}(C)} + \|g\|_{L^1_{\mathcal{B}}(C)}\big).
\end{aligned}$$

□

Lemma 3.10. *Let $f, g \in \mathcal{M}$ be such that $|f| \leq |g|$ C-a.e. Then, $\|f\|_{L^1_{\mathcal{B}}(C)} \leq K\|g\|_{L^1_{\mathcal{B}}(C)}$.*

Proof. Take a C-null set Z such that $|f| \leq |g|$ in $\Omega\backslash Z$. Since,

$$\begin{aligned}\{\omega\in\Omega:\,|f|>t\} &= \big(\{\omega\in\Omega:\,|f|>t\}\cap\Omega\backslash Z\big)\cup\big(\{\omega\in\Omega:\,|f|>t\}\cap Z\big)\\ &\subset \big(\{\omega\in\Omega:\,|g|>t\}\cap\Omega\backslash Z\big)\cup\big(\{\omega\in\Omega:\,|f|>t\}\cap Z\big),\end{aligned}$$

by (3.1) and (3.2) we have that

$$\begin{aligned}\|f\|_{L^1_{\mathcal{B}}(C)} &= \int_0^\infty \|C_{|f|}(t)\|_E\,dt\\ &\leq \int_0^\infty \big\|C\big((\{\omega:\,|g|>t\}\cap\Omega\backslash Z)\cup(\{\omega:\,|f|>t\}\cap Z)\big)\big\|_E\,dt\\ &\leq K\int_0^\infty \big\|C(\{\omega\in\Omega:\,|g|>t\}\cap\Omega\backslash Z)\big\|_E\\ &+ K\int_0^\infty \big\|C(\{\omega\in\Omega:\,|f|>t\}\cap Z))\big\|_E\,dt\\ &= K\int_0^\infty \big\|C(\{\omega\in\Omega:\,|g|>t\}\cap\Omega\backslash Z)\big\|_E\\ &\leq K\int_0^\infty \|C_{|g|}(t)\|_E\,dt = K\,\|g\|_{L^1_{\mathcal{B}}(C)}.\end{aligned}$$

□

Let us denote by $L^1_{\mathcal{B}}(C)$ the set of functions in $\mathcal{M}$ which are integrable with respect to C (i.e. $f\in\mathcal{M}$ such that $\|f\|_{L^1_{\mathcal{B}}(C)}<\infty$), where functions which are equal C-a.e. are identified. Then $L^1_{\mathcal{B}}(C)$ is a vector space and $\|\cdot\|_{L^1_{\mathcal{B}}(C)}$ is a quasi-norm, as a consequence of the previous lemma. We call it the *space of C-integrable functions of Bochner type.* Moreover it is an ideal of $\mathcal{M}$.

Proposition 3.11. *The space $L^1_{\mathcal{B}}(C)$ has the Fatou property, i.e. if $(f_n)\subset L^1_{\mathcal{B}}(C)$ is such that $0\leq f_n\uparrow f$ C-a.e. and $\sup\|f_n\|_{L^1_{\mathcal{B}}(C)}<\infty$, then $f:=\sup f_n\in L^1_{\mathcal{B}}(C)$ and $\|f_n\|_{L^1_{\mathcal{B}}(C)}\uparrow\|f\|_{L^1_{\mathcal{B}}(C)}$.*

Proof. Let us suppose first that $(f_n)\subset L^1_{\mathcal{B}}(C)$ is such that $0\leq f_n\uparrow$ pointwise and $\sup\|f_n\|_{L^1_{\mathcal{B}}(C)}<\infty$. Note that for the measurable function $f=\sup f_n\colon\Omega\to[0,\infty]$ we can consider the map $C_f(t)=C\big(\{\omega\in\Omega:\,f(\omega)>t\}\big)\in E$ for $t\in[0,\infty)$, which is strongly m-measurable just as in the finite-valued function case. Since, for every $t\in[0,\infty)$,

$$\{\omega\in\Omega:\,f_n(\omega)>t\}\uparrow\{\omega\in\Omega:\,f(\omega)>t\},$$

by the previous arguments and (2.10) and (3.1), $\|C_{f_n}(t)\|_E\uparrow\|C_f(t)\|_E$. Then, applying the Monotone Convergence Theorem in $L^1[0,\infty)$, we obtain $\|f_n\|_{L^1_{\mathcal{B}}(C)}\uparrow\|f\|_{L^1_{\mathcal{B}}(C)}$. In particular, $\|f\|_{L^1_{\mathcal{B}}(C)}<\infty$, that is, $f\in L^1_{\mathcal{B}}(C)$. Note that $\int_0^\infty\|C_f(t)\|_E\,dt<\infty$ implies $f<\infty$ C-a.e., and since

$$\{\omega\in\Omega:\,f(\omega)=\infty\}\subset\{\omega\in\Omega:\,f(\omega)>t\}\quad\text{for all } t\in[0,\infty),$$

then, by (3.1),

$$\|C\big(\{\omega\in\Omega:\,f(\omega)=\infty\}\big)\|_E\leq\|C_f(t)\|_E\quad\text{for all } t\in[0,\infty),$$

and so it must be $C\big(\{\omega\in\Omega:\,f(\omega)=\infty\}\big)=0$.

When $0 \le f_n \uparrow C$-a.e., we only have to take a C-null set Z such that $0 \le f_n \chi_{\Omega\setminus Z} \uparrow$ pointwise and apply the previous result (have in mind that if $f = g$ C-a.e. then $\|f\|_{L^1_{\mathcal{B}}(C)} = \|g\|_{L^1_{\mathcal{B}}(C)}$). □

Remark 3.12. From (3.2) with $K = 1$ and (2.10), it follows that

$$\|C(\cup_n A_n)\|_E \le \sum_n \|C(A_n)\|_E.$$

Indeed, $\|C(\cup_{j=1}^n A_j)\|_E \le \sum_{j=1}^n \|C(A_j)\|_E \le \sum_{j\ge1} \|C(A_j)\|_E$. On the other hand, $B_n = \cup_{j=1}^n A_j \uparrow B = \cup_{j\ge1} A_j$, hence $C(B_n) \to C(B)$ in E and consequently $\|C(B_n)\|_E \to \|C(B)\|_E$.

Proposition 3.13. *The space $L^1_{\mathcal{B}}(C)$ endowed with the quasi-norm $\|\cdot\|_{L^1_{\mathcal{B}}(C)}$ is complete. Consequently, $L^1_{\mathcal{B}}(C)$ is a quasi-Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $L^1_{\mathcal{B}}(C)$. There exists a strictly increasing sequence (n_j) such that $\|f_{n_{j+1}} - f_{n_j}\|_{L^1_{\mathcal{B}}(C)} \le \frac{1}{2^{2j}}$. Taking $g_k = \sum_{j=1}^k |f_{n_{j+1}} - f_{n_j}| \in L^1_{\mathcal{B}}(C)$ and $g = \sum_{j\ge1} |f_{n_{j+1}} - f_{n_j}|$, we have that $0 \le g_k \uparrow g$ pointwise. Moreover, since for each $t \in [0,\infty)$, we have that

$$\{\omega \in \Omega : g_k(\omega) > t\} \subset \{\omega \in \Omega : g(\omega) > t\} \subset \bigcup_{j\ge1} \{\omega : |f_{n_{j+1}} - f_{n_j}|(\omega) > \tfrac{t}{2^j}\},$$

by (3.1) and Remark 3.12, it follows that

$$\begin{aligned}
\|g_k\|_{L^1_{\mathcal{B}}(C)} &= \int_0^\infty \|C_{g_k}(t)\|_E\, dt \le \sum_{j\ge1} \int_0^\infty \|C_{|f_{n_{j+1}} - f_{n_j}|}(\tfrac{t}{2^j})\|_E\, dt \\
&= \sum_{j\ge1} 2^j \int_0^\infty \|C_{|f_{n_{j+1}} - f_{n_j}|}(s)\|_E\, ds = \sum_{j\ge1} 2^j \|f_{n_{j+1}} - f_{n_j}\|_{L^1_{\mathcal{B}}(C)} \\
&\le \sum_{j\ge1} 2^j \frac{1}{2^{2j}} < 1.
\end{aligned}$$

Hence, $\sup \|g_k\|_{L^1_{\mathcal{B}}(C)} < \infty$. Then, by Proposition 3.11, $g \in L^1_{\mathcal{B}}(C)$. Consider now $h = \sum_{j\ge1} (f_{n_{j+1}} - f_{n_j})$. Since $|h| \le |g|$ and $L^1_{\mathcal{B}}(C)$ is an ideal, we have that $h \in L^1_{\mathcal{B}}(C)$. Note that

$$\begin{aligned}
\Big\{\omega \in \Omega : |\sum_{j\ge k} (f_{n_{j+1}} - f_{n_j})|(\omega) > t\Big\} &\subset \Big\{\omega \in \Omega : \sum_{j\ge k} |f_{n_{j+1}} - f_{n_j}|(\omega) > t\Big\} \\
&\subset \bigcup_{j\ge k} \{\omega \in \Omega : |f_{n_{j+1}} - f_{n_j}|(\omega) > \tfrac{t}{2^j}\},
\end{aligned}$$

by (3.1) and Remark 3.12, it follows

$$\begin{aligned}
\Big\|\sum_{j\ge k} (f_{n_{j+1}} - f_{n_j})\Big\|_{L^1_{\mathcal{B}}(C)} &= \int_0^\infty \|C_{|\sum_{j\ge k}(f_{n_{j+1}} - f_{n_j})|}(t)\|_E\, dt \\
&\le \sum_{j\ge k} \int_0^\infty \|C_{|f_{n_{j+1}} - f_{n_j}|}(\tfrac{t}{2^j})\|_E\, dt \\
&= \sum_{j\ge k} 2^j \|f_{n_{j+1}} - f_{n_j}\|_{L^1_{\mathcal{B}}(C)} \le \sum_{j\ge k} \frac{1}{2^j}.
\end{aligned}$$

Therefore, $f_{n_k} \to h + f_{n_1}$ in $L^1_{\mathcal{B}}(C)$ and so does (f_n) as it is a Cauchy sequence. □

Remark 3.14. Let φ be a positive simple function. Then,

$$\Big\| \int \varphi \, dC \Big\|_E \leq \|\varphi\|_{L^1_{\mathcal{B}}(C)}.$$

Indeed, as a direct consequence of the definition of the Bochner integral of the distribution function C_φ we have

$$\Big\| \int \varphi \, dC \Big\|_E = \Big\| \int_0^\infty C_\varphi(t) \, dt \Big\|_E \leq \int_0^\infty \|C_\varphi(t)\|_E \, dt = \|\varphi\|_{L^1_{\mathcal{B}}(C)}.$$

Consequently, the integration operator with respect to C is continuous from the set of positive simple functions endowed with $\| \cdot \|_{L^1_{\mathcal{B}}(C)}$ into E. These inequalities will be useful when comparing different norms of functions appearing in the models of impact measures, which provide evaluations of such indices.

We finish the section with a characterization of one of the most relevant lattice properties of the spaces of integrable functions. If the space is σ-order continuous, in Information Science modeling it means that the evaluation of an impact measure given by a function f can be successfully approximated by means of an increasing sequence of simple functions converging pointwise to f.

It must be pointed out that this is not necessarily the case: condition (3.3) appearing below does not hold for every vector capacity. For example, consider Lebesgue measure space $([0,1], \mathcal{B}([0,1]), \mu)$ and the vector capacity $C_0 : \mathcal{B}([0,1]) \to E$ given by $C_0(A) = x_0 \neq 0$ if $\mu(A) \neq 0$ and $C_0(A) = 0$ if $\mu(A) = 0$, where E is a nontrivial Banach space and x_0 is a fixed vector in it.

Proposition 3.15. *The space $L^1_{\mathcal{B}}(C)$ has σ-order continuous norm if and only if the condition*

$$(A_n) \subset \Sigma \ \text{ with } \ A_n \downarrow \emptyset \quad \Rightarrow \quad C(A_n) \to 0 \ \text{ in } E \tag{3.3}$$

holds.

Proof. Suppose that $L^1_{\mathcal{B}}(C)$ is σ-order continuous. If $(A_n) \subset \Sigma$ is such that $A_n \downarrow \emptyset$, then $\chi_{A_n} \downarrow 0$ and so $\chi_{A_1 \setminus A_n} \uparrow \chi_{A_1}$. By σ-order continuity and from Remark 3.14,

$$\|C(A_n)\|_E \leq \|\chi_{A_n}\|_{L^1_{\mathcal{B}}(C)} = \|\chi_{A_1} - \chi_{A_1 \setminus A_n}\|_{L^1_{\mathcal{B}}(C)} \to 0.$$

Conversely, suppose that condition (3.3) holds. Given $f, f_n \in L^1_{\mathcal{B}}(C)$ such that $0 \leq f_n \uparrow f$ pointwise, since $0 \leq f - f_n \downarrow 0$ pointwise, for every $t > 0$ we have that

$$\{\omega \in \Omega : (f - f_n)(\omega) > t\} \downarrow \bigcap_n \{\omega \in \Omega : (f - f_n)(\omega) > t\} = \emptyset.$$

Then, by (3.3), $\|C_{f-f_n}(t)\|_E \to 0$. On the other hand, $\|C_{f-f_n}(t)\|_E \leq \|C_f(t)\|_E \in L^1[0,\infty)$. So, by applying the Dominated Convergence Theorem in $L^1[0,\infty)$, we obtain

$$\|f - f_n\|_{L^1_{\mathcal{B}}(C)} = \int_0^\infty \|C_{f-f_n}(t)\|_E \, dt \to 0.$$

Note that if $0 \leq f_n \uparrow f$ C-a.e., we only have to take a C-null set Z such that $0 \leq f_n \chi_{\Omega \setminus Z} \uparrow f \chi_{\Omega \setminus Z}$ pointwise and apply the previous result (have in mind that if $f = g$ C-a.e. then $\|f\|_{L^1_{\mathcal{B}}(C)} = \|g\|_{L^1_{\mathcal{B}}(C)}$). □

Note that under condition (3.3) the simple functions are dense in $L^1_{\mathcal{B}}(C)$. Indeed, every $0 \leq f \in L^1_{\mathcal{B}}(C)$ is the pointwise limit of simple functions $0 \leq \varphi_n \uparrow f$ and by σ-order continuity of $L^1_{\mathcal{B}}(C)$ (Proposition 3.15) we have that $\varphi_n \to f$ in $L^1_{\mathcal{B}}(C)$. For a general $f \in L^1_{\mathcal{B}}(C)$, the result follows by taking positive and negative parts.

4. The second model: C-integrable functions of Pettis type

Although the integration defined in the previous section would be enough for applications to impact indices, the definition provided there does not coincide with the one of integrable functions in the case when C is a countably additive vector measure. Indeed, as we already explained, the definition is stronger than the one needed for a direct extension of Bartle-Dunford-Schwartz integrability, and it will provide a different norm for measuring information, as will be shown in Section 5. The corresponding space of integrable functions can be identified with the space $w - L^1_c(\Lambda)$ appearing in Section 5 of [7]. In this section we will show how this integration can be generalized to the case of adequate vector capacities. The results concerning semivariations and variations of capacities presented in Section 2.2 will be necessary.

In order to have a nice definition of this kind of integrability, we will require C to be scalar bounded. Note that in this case we can define the Choquet integral of a measurable function $f \geq 0$ with respect to the scalar capacity $|\langle C, x^* \rangle|$ in the usual way,

$$\int_\Omega f \, d|\langle C, x^* \rangle| = \int_0^\infty |\langle C, x^* \rangle|_f \, dt.$$

The natural formula for defining the integral of a non-negative function —at least for simple functions— is the Pettis integral of the distribution function C_f; that is, if $f \geq 0$ is a measurable function we can define its integral as in the case of Bochner integrable functions,

$$\int_\Omega f \, dC = \int_0^\infty C_f(t) \, dt.$$

Definition 4.1. Let $N \subseteq B_{E^*}$ be a norming set for the N-scalar bounded vector capacity $C : \Sigma \to E$. We say that a measurable function f is weakly C-integrable with respect to N if

1. $C_{|f|}$ is Pettis integrable, and
2. the functional $\|f\|_{L^1_{\mathcal{P},N}(C)}$ defined as $\sup_{x^* \in N} \int_\Omega |f| \, d|\langle C, x^* \rangle|$ is finite.

Remark 4.2. Let us show now that under reasonable requirements and for positive lattice-valued capacities, it is possible to compare the functional $\| \cdot \|_{L^1_{\mathcal{P},N}(C)}$ with the Pettis norm of $C_{|f|}$ and with the norm of the integral $\int_\Omega |f| \, dC$, if it can be defined in the correct way. Assume that C is a positive Fatou capacity on the Banach lattice E (see Definition 2.10 in Section 2.2) and consider the norming set $N = B^+_{E^*}$ for C. Suppose also that E is reflexive. Note that in the case $\| \cdot \|_{L^1_{\mathcal{P},N}(C)}$ is a norm, the Pettis norm for the functions C_f is smaller than this norm, as the following calculations show. Take a simple measurable function f with a decreasing representation of $|f|$ given by $\sum_{i=1}^n a_i \chi_{B_i}$, $B_{i+1} \subset B_i$. Then for every $x^* \in N$,

$$\|f\|_{L^1_{\mathcal{P},N}(C)} \geq \int_\Omega |f| \, d|\langle C, x^* \rangle| = \int_0^\infty |\langle C, x^* \rangle|_{|f|} \, dt$$

$$= \sum_{i=1}^{n} a_i |\langle C, x^* \rangle|(B_i) \geq \sum_{i=1}^{n} a_i |\langle C(B_i), x^* \rangle| \geq |\langle \sum_{i=1}^{n} a_i C(B_i), x^* \rangle|$$
$$= \int_0^\infty |\langle C_{|f|}, x^* \rangle| \, dt.$$

This proves that norm $\| \cdot \|_{L^1_{\mathcal{P},N}(C)}$ is stronger than the Pettis norm for simple functions. Take now a measurable function $0 \leq f$ that is integrable with respect to every $|\langle C, x^* \rangle|$. Take a sequence of simple functions (f_n) such that $0 \leq f_n \uparrow f$. Assume that we have that $C_{f_n} \to C_f$ pointwise, and so for every $x^* \in (E^*)^+$, $\langle C_{f_n}, x^* \rangle \to \langle C_f, x^* \rangle$ pointwise (use the Fatou property for C). Then by the Monotone Convergence Theorem all the functions $\langle C_f, x^* \rangle$ are integrable, and an easy argument proves that for all $x = x^+ - x^- \in E^*$, $\langle C_f, x^* \rangle$ are also integrable. Thus we have that C_f is Dunford integrable and so Pettis integrable, since E is reflexive. Therefore f is weakly C-integrable with respect to N.

Moreover,

$$\int_0^\infty |\langle C_{|f|}, x^* \rangle| \, dt \geq \Big| \int_0^\infty \langle C_{|f|}, x^* \rangle \, dt \Big| = \Big| \langle \int_0^\infty C_{|f|} \, dt, x^* \rangle \Big|,$$

and taking sup with respect to all $x^* \in B_{E^*}$ we obtain that

$$\|f\|_{L^1_{\mathcal{P},N}(C)} \geq \| \int_\Omega |f| \, dC \|.$$

Definition 4.3. Let $N \subseteq B_{E^*}$ be a norming set for the scalar subadditive (with respect to N) vector capacity $C : \Sigma \to E$. Suppose that C is equivalent to a finite measure λ — that is, for all $A \in \Sigma$, $\sup_{B \subset A, \, B \in \Sigma} \|C(B)\| = 0$ if and only if $\lambda(A) = 0$. Then we define the space $L^1_{\mathcal{P},N}(C)$ as the subset of $L^0(\lambda)$ of the classes of C-a.e. equal functions that are weakly C-integrable with respect to N. We will simply write $L^1_{\mathcal{P}}(C)$ if $N = B_{E^*}$.

Standard arguments as the ones given in the previous sections prove also the next result.

Lemma 4.4. *Let N be a norming set for C. Then $L^1_{\mathcal{P},N}(\mathcal{C})$ is a linear space.*

In order to get reasonable properties of the space $L^1_{\mathcal{P},N}(C)$, from now on we will assume that C is scalar subadditive with respect to a norming set N and equivalent to a finite measure λ.

Lemma 4.5. *Suppose that E is reflexive and $C : \Sigma \to E$ is scalar subadditive with respect to a norming set N and equivalent to a finite measure λ. Then $L^1_{\mathcal{P},N}(C)$ is an ideal in $L^0(\lambda)$.*

Proof. Consider a couple of (classes of λ-a.e.) measurable functions f and g such that $|f| \leq |g|$ and $g \in L^1_{\mathcal{P},N}(\mathcal{C})$. Then clearly $f \in L^1(|\langle C, x^* \rangle|)$ for every $x^* \in N$ and $\sup_{x^* \in N} \int_\Omega |f| \, d|\langle C, x^* \rangle| < \infty$.

On the other hand, since

$$\{\omega \in \Omega : \, |f|(\omega) > t\} \subset \{\omega \in \Omega : \, |g|(\omega) > t\},$$

we have that for every $x^* \in N$, $\int_0^\infty \langle C_{|f|}, x^* \rangle \, dt \leq \int_0^\infty \langle C_{|g|}, x^* \rangle \, dt$, so $\langle C_{|f|}, x^* \rangle$ is integrable with respect to each $x^* \in N$. This implies that the integrals are uniformly bounded (by Remark 4.2). In particular, they are Dunford integrable and since E is reflexive they are also Pettis integrable. □

Theorem 4.6. *Suppose that C is positive, scalar subadditive for positive functionals, Fatou and such that C is equivalent to a finite measure λ. Assume that E is reflexive. Then the space $L^1_{\mathcal{P}}(C)$ endowed with the quasi-norm $\|\cdot\|_{L^1_{\mathcal{P}}(C)}$ is complete. Consequently, $L^1_{\mathcal{P}}(C)$ is a quasi-Banach space.*

Proof. First note that we can consider the intersection of the unit ball and the positive cone as N, since C is a positive capacity. Let (f_n) be a Cauchy sequence in $L^1_{\mathcal{P}}(C)$. First, notice that the space of all functions that are integrable with respect to all the measures $|\langle C, x^*\rangle|$ is complete, so there is a measurable function f that is integrable with respect to every such measure and $\|f\| < \infty$. Let us prove now that the Pettis integral of C_f exists for such an f.

1) Note that there exists a strictly increasing sequence (n_j) such that $\|f_{n_{j+1}} - f_{n_j}\|_{L^1_{\mathcal{B}}(C)} \leq \frac{1}{2^{2j}}$. If we define $g_k = \sum_{j=1}^k |f_{n_{j+1}} - f_{n_j}| \in L^1_{\mathcal{B}}(C)$ and $g = \sum_{j\geq 1} |f_{n_{j+1}} - f_{n_j}|$, we obtain that $g_k \in L^1_{\mathcal{P}}(C)$ by Lemma 4.4 and $0 \leq g_k \uparrow g$ pointwise. Moreover, we have that

$$\{\omega \in \Omega : \, g_k(\omega) > t\} \subset \{\omega \in \Omega : \, g(\omega) > t\} \subset \bigcup_{j\geq 1} \{\omega \in \Omega : \, |f_{n_{j+1}} - f_{n_j}|(\omega) > \tfrac{t}{2^j}\},$$

for every $t \in [0, \infty)$.

2) Fix a positive element $x^* \in (E^*)^+$. Then, by the subadditivity of $|\langle C(\cdot), x^*\rangle|$, we obtain

$$\begin{aligned}\langle C(\{\omega \in \Omega : \, g_k(\omega) > t\}), x^*\rangle &\leq \langle C(\{\omega \in \Omega : \, g(\omega) > t\}), x^*\rangle \\ &\leq \sum_{j\geq 1} \langle C(\{\omega \in \Omega : \, |f_{n_{j+1}} - f_{n_j}|(\omega) > \tfrac{t}{2^j}\}), x^*\rangle,\end{aligned}$$

for every $t \in [0, \infty)$. By the positive monotonicity of C and Lemma 2.3(2), it follows that

$$\begin{aligned}\int_0^\infty \langle C_{g_k}(t), x^*\rangle \, dt &\leq \sum_{j\geq 1} \int_0^\infty \langle C_{|f_{n_{j+1}} - f_{n_j}|}(\tfrac{t}{2^j}), x^*\rangle, dt \\ &= \sum_{j\geq 1} 2^j \int_0^\infty \langle C_{|f_{n_{j+1}} - f_{n_j}|}(s), x^*\rangle \, ds \leq \sum_{j\geq 1} 2^j \frac{1}{2^{2j}} < 1.\end{aligned}$$

Hence, $\sup_{x^* \in B_{E^*}, k\in\mathbb{N}} \int_0^\infty \langle C_{g_k}(t), x^*\rangle \, dt < \infty$. So the functions $\langle C_{g_k}(t), x^*\rangle$ define an increasing sequence that converges pointwise to $\langle C_g(t), x^*\rangle$ and by the Monotone Convergence Theorem, $\langle C_g(t), x^*\rangle \in L^1(|\langle C, x^*\rangle|)$ and

$$\int_0^\infty \langle C_{g_k}(t), x^*\rangle \, dt \to \int_0^\infty \langle C_g(t), x^*\rangle \, dt.$$

This implies that C_g is Pettis integrable. Indeed, it is easy to see that it is Dunford integrable, so by the reflexivity of E we have that it is Pettis integrable.

Therefore, $g \in L^1_{\mathcal{P}}(C)$. Since $f = \sum_{j\geq 1}(f_{n_{j+1}} - f_{n_j})$, $|f| \leq |g|$ and $L^1_{\mathcal{P}}(C)$ is an ideal, we have that $f \in L^1_{\mathcal{P}}(C)$.

□

To finish this section, let us note that under the assumptions on C given above if $f \in L^1_{\mathcal{P}}(C)$ there exists an integral of f that is $\int_\Omega f \, dC = \int_\Omega C_{f^+} \, dC - \int_\Omega C_{f^-} \, dC$, satisfying, for every $x^* \in E^*$, the equality

$$\Big\langle \int_\Omega f \, dC, x^* \Big\rangle = \int_\Omega f \, d\langle C, x^*\rangle.$$

This formula is the "vector capacity version" of the equality which is satisfied by integrable functions with respect to vector measures in the Bartle-Dunford-Schwartz sense.

Example 4.7. Consider the vector-valued capacity given in Example 3.6(3). Suppose that we want to use this index for evaluating which is the better combination of 2 teams of 3 researchers for developing a particular research program. Write $\mathcal{R}_3$ for all the subsets of 3 elements of $\mathcal{R}$. Each team has an initial mark reflecting its "quality" —for example, evaluating the previous common experience of the members of the group— which is given by a function $q : \mathcal{R}_3 \to \mathbb{R}^+$. Thus, in the model, each option Θ of two teams $\Theta = \{R_1, R_2\}$ can be represented by an integrable function $f_\Theta = q(R_1)\chi_{R_1} + q(R_2)\chi_{R_2}$. A measure of "how good" a given option is would be given by the norm of the integral of the function, that is

$$\Big\| \int_{\mathcal{R}} f_\Theta \, d\alpha \Big\|.$$

Comparing the norm of the integrals of all functions defined in this way would give the solution of the problem.

5. An application: an impact measure for databases

Suppose that there is a —potentially infinite— set of companies $S = \{c_i : i \in \mathbb{N}\}$, which provide databases for business purposes to a given company CO. This company offers them to individual customers. Consider the set Ω defined as the union of all the sets of information items Ω_i of each databases d_i provided by each company c_i. Assume that the information items are organized in all of the database d_i in a collection of subsets that define σ-algebras Σ_i; for example, if each set Ω_i is finite, we can consider the σ-algebra Σ_i of all the subsets of Ω_i for each $i \in \mathbb{N}$. Consider now the σ-algebra Σ on Ω generated by the countable unions of the elements of the σ-algebras Σ_i, $i \in \mathbb{N}$. We will assume that all the data in Ω are divided in a countable class of (disjoint) categories $\{D_k : k \in \mathbb{N}\}$ defined as thematic areas.

We will show two suitable constructions of impact measures for the database service Ω a company in S offers. Consider an information set A that offers; note that the following definition works for any subset $A \in \Sigma$ if we assume that the total number of searches for any set $A \in \Sigma$ in a year is finite and all of them refer to information items A that are distributed in a finite set of thematic areas. Each one of the following definitions would be useful depending on the use the company CO wants to make of them; note that both of them are naturally defined for use in the context.

(1) Consider a function $N : \Sigma \to \mathbb{R}^+$ defined as

$$N(A) = \textit{the number of queries about subsets } B \in \Sigma \textit{ of } A$$

$$\textit{in the preceding year}, \quad A \in \Sigma.$$

Notice that this function is not subadditive, but it is increasing. It measures how relevant a subset A is in view of the queries about the information items contained in its subsets. Consequently, let us define the vector capacity $C_1 : \Sigma \to \ell^2$ by

$$C_1(A) := \sum_{k=1}^{\infty} N(A \cap D_k)\, e_k \in \ell^2, \quad A \in \Sigma.$$

(2) Suppose now that we are interested in giving an impact index defined by a subadditive measure based on the numbers of queries about a given information set. We are interested in producing a measure which rewards the fact that the size of the information items provided by the companies is as small as possible but with a big impact. For this aim, define a new function $n : \Sigma \to \mathbb{R}^+$ by

$$n(A) = \textit{the number of queries about sets } B \in \Sigma \textit{ containing } A$$

$$\textit{in the preceding year}, \quad A \in \Sigma.$$

It is easy to see that this function is subadditive, but not increasing. In this case, we define the vector capacity $C_2 : \Sigma \to \ell^2$ by

$$C_2(A) := \sum_{k=1}^{\infty} n(A \cap D_k)\, e_k \in \ell^2, \quad A \in \Sigma.$$

Because of the assumption on the finite number of searches and thematic areas, both sums are in fact finite and so both capacities are well defined.

Suppose now that CO is planning a business policy for the next year and it wants to measure how to what extent the contracts with the companies of S should be preserved. They need to rank them by measuring their usefulness, taking into account that they have to reduce expenses both in contracts and in computers for storing data. Both measures explained above give information about the use of the elements of the databases, but the second is focused on rewarding frequently used small data sets. Therefore, we center our attention on C_2.

With the aim of ranking the set S we consider an "evaluation function" which will be computed for all the companies. To define it, all the information in Ω will be classified in three categories depending on its interest (measured in number of queries, for example): Cat_1 for the most relevant information, Cat_2 for the second-order data and Cat_3 for the —still interesting but— only occasionally searched information sets. We have then that $\Omega = \mathrm{Cat}_1 \cup \mathrm{Cat}_2 \cup \mathrm{Cat}_3$. Since we are interested in saving resources by rewarding important information as much as possible, we will consider the following testing function

$$f_{c_i}(w) := 3\,\chi_{\mathrm{Cat}_1 \cap \Omega_i}(w) + 2\,\chi_{\mathrm{Cat}_2 \cap \Omega_i}(w) + \chi_{\mathrm{Cat}_3 \cap \Omega_i}(w), \quad w \in \Omega,$$

for each company c_i, $i \in \mathbb{N}$. The decreasing rearrangement of the function f_{c_i} is given by

$$f_{c_i} = \chi_{\mathrm{Cat}_1 \cap \Omega_i} + \chi_{(\mathrm{Cat}_1 \cup \mathrm{Cat}_2) \cap \Omega_i} + \chi_{(\mathrm{Cat}_1 \cup \mathrm{Cat}_2 \cup \mathrm{Cat}_3) \cap \Omega_i}.$$

Integration with respect to C_2 can be used for ranking the companies. Using the notions introduced in the paper, two definitions are possible. By the computations explained at the beginning of the paper, we obtain the following indices.

(a) **A ranking index based on Bochner integration.** Define the ranking index $I_{\mathcal{B}}(c_i)$, $i \in \mathbb{N}$, as

$$I_{\mathcal{B}}(c_i) := \big\| f_{c_i} \big\|_{L^1_{\mathcal{B}}(C_2)} = \| C_2(\mathrm{Cat}_1 \cap \Omega_i) \|_E$$

$$+ \| C_2\big((\mathrm{Cat}_1 \cup \mathrm{Cat}_2) \cap \Omega_i \big) \|_E + \| C_2\big((\mathrm{Cat}_1 \cup \mathrm{Cat}_2 \cup \mathrm{Cat}_3) \cap \Omega_i \big) \|_E$$

$$= \Big(\sum_{k=1}^{\infty} n(\mathrm{Cat}_1 \cap \Omega_i \cap D_k)^2\Big)^{1/2} + \Big(\sum_{k=1}^{\infty} n((\mathrm{Cat}_1 \cup \mathrm{Cat}_2) \cap \Omega_i \cap D_k)^2\Big)^{1/2}$$

$$+\Big(\sum_{k=1}^{\infty} n((\mathrm{Cat}_1 \cup \mathrm{Cat}_2 \cup \mathrm{Cat}_3) \cap \Omega_i \cap D_k)^2\Big)^{1/2}.$$

(b) **A ranking index based on Pettis integration.** Other index can be defined with the aim of comparing the different companies using the (quasi) norm of the space $L^1_{\mathcal{P}}(C_2)$. It is more difficult to compute, but it also gives a ranking tool. Define the ranking index $I_{\mathcal{P}}(c_i)$, $i \in \mathbb{N}$, as

$$I_{\mathcal{P}}(c_i) := \|f_{c_i}\|_{L^1_{\mathcal{P}}(C_2)} =$$

$$= \sup_{x^* \in B_{\ell^2}} \int_{\Omega} |f|\, d|\langle C_2, x^*\rangle| = \sup_{x^* \in B_{\ell^2}} \int_0^{\infty} |\langle C_2, x^*\rangle|_{|f|}\, dt$$

$$= \sup_{x^* \in B_{\ell^2}} \Big(|\langle C_2, x^*\rangle|(\mathrm{Cat}_1 \cap \Omega_i) + |\langle C_2, x^*\rangle|(\mathrm{Cat}_1 \cap \mathrm{Cat}_2 \cap \Omega_i)$$

$$+ |\langle C_2, x^*\rangle|(\mathrm{Cat}_1 \cap \mathrm{Cat}_2 \cap \mathrm{Cat}_3 \cap \Omega_i)\Big),$$

where the variation of the scalar components of C_2 is given by

$$|\langle C_2, x^*\rangle|(A) = \sup_{\left[(A_i)_{i=1}^n \text{ partition of } A\right]} \sum_{i=1}^{n} \sum_{j=1}^{\infty} |\lambda_j|\, |\langle C_2(A_i), e_j\rangle|$$

$$= \sup_{\left[(A_i)_{i=1}^n \text{ partition of } A\right]} \sum_{i=1}^{n} \sum_{j=1}^{\infty} |\lambda_j|\, n(A_i \cap D_j),$$

for $A \in \Sigma$, where $x^* = \sum_{j=1}^{\infty} \lambda_j\, e_j \in B_{\ell^2}$.

(c) **A ranking index based on the integral of simple functions.** In this case, the index is just given by the norm of the integral, that is

$$I_{C_2}(c_i) := \Big\| \int_{\Omega} f_{c_i}\, dC_2 \Big\|$$

$$= \Big(\sum_{k=1}^{\infty} \Big(n(\mathrm{Cat}_1 \cap \Omega_i \cap D_k) + n((\mathrm{Cat}_1 \cup \mathrm{Cat}_2) \cap \Omega_i \cap D_k)$$

$$+ n((\mathrm{Cat}_1 \cup \mathrm{Cat}_2 \cup \mathrm{Cat}_3) \cap \Omega_i \cap D_k)\Big)^2\Big)^{1/2}.$$

The reader can notice that the indices given in (a) and (c) are easy to compute and have a clear meaning. They give measures of how relevant the databases provided by the different companies c_i are regarding the number of consultations of information sets containing their products. More precisely, they measure how many times a searched subset in a specific thematic area D_k contains one of the three sets of information of different categories $\mathrm{Cat}_1 \cap \Omega_i \cap D_k \subseteq (\mathrm{Cat}_1 \cup \mathrm{Cat}_2) \cap \Omega_i \cap D_k \subseteq (\mathrm{Cat}_1 \cup \mathrm{Cat}_2 \cup \mathrm{Cat}_3) \cap \Omega_i \cap D_k$ offered by a fixed company c_i. Both indices may then be used for ranking the relevance of the companies concerning the searching of information by the users of the global database provided by the company CO. And both are constructed using our technique of integration with respect to vector capacities, once the selection tool —the vector capacity C_2— and the test functions —the functions f_{c_i}— are fixed by the managing team of CO following their strategic criteria.

6. Conclusions

The increase in the number of new information measures that have appeared in recent developments in Information Science suggests the need to clearly establish the mathematical framework in which these measures should be included. Indeed, meaningful information measures should satisfy certain mathematical properties, but there are other classical requirements that are not needed and should be removed. Information measures should be vector valued functions, and in general do not need to be additive on disjoint sets. In this way, the vector valued nature of the measures allows information from different scalar indices to be combined in a single mathematical object. In addition, the lack of additivity of some of the most important impact measures, such as the h-index, must be accepted in fact.

These basic requirements justify the work shown in the paper, which proposes Choquet integrable functions with respect to vector capacities —both Bochner and Pettis type— as models for information measures. Once the model —an integrable function f— is set for a given information measure, there are three ways to define the index that evaluates it when it acts in information subsets: 1) the Bochner type norm of f, 2) the Pettis type norm of f, and 3) the norm of the vector valued integral of f. In addition to the specific calculation formulas, this procedure also provides a way of classifying the information measures, depending on the type of formula used in their definition. This was shown in Section 5 on a particular impact index, and opens up new perspectives in the development of these theoretical and applied aspects of Information Science.

While this opens the door to a systematic way of introducing new indices in Information Sciences, it should be noted that Choquet integration is not the only method of defining non-additive information measures. Once the additivity requirement is removed, many different integrals appear on the scene (fuzzy integrals, universal integrals,...). Some of them have already proved useful in some contexts, although they are mainly defined for scalar capacities. Therefore, it would be interesting to investigate also vector-valued versions of the most important non-additive scalar integrals, in order to facilitate the creation of new specific mathematical indices for measuring information sets.

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