Departamento de Matemática Aplicada
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Dynamic Modeling and Stability Analysis of Stochastic Multi-Physical Systems Applied to Electric Power Systems

TESIS DE DOCTORADO

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Abbreviations

AE	algebraic equation
AVR	automatic voltage regulator
a.s.	almost sure
CIGRE	Conseil International des Grands Réseaux Électriques
CIR	Cox–Ingersoll–Ross
CPU	central processing unit
DAE	differential-algebraic equation
DG	distributed generation
d-index	differentiation index
DSA	dynamic stability analysis
DSM	demand-side management
e.g.	for example (*“exempli gratia”*)
EM	Euler-Maruyama
i.e.	that is (*“id est”*)
IEEE	Institute of Electrical and Electronics Engineers
IVP	initial value problem
LE	Lyapunov exponent
LLE	largest Lyapunov exponent
MDS	metric dynamical system
MET	multiplicative ergodic theorem
m.s.	mean square
NERC	North American Electric Reliability Council
ODE	ordinary differential equation
OU	Ornstein-Uhlenbeck
PSS	power system stabilizer
RDE	random differential equation
RDS	random dynamical system
RK	Runge-Kutta
SDAE	stochastic differential-algebraic equation
SDE	stochastic differential equation
s-index	strangeness index
SMIB	single-machine infinite-bus
SSSA	small-signal stability assessment

SVD	singular value decomposition
t-index	tractability index
TSA	transient stability analysis
uSDE	underlying stochastic differential equation
VPP	virtual power plant

Notation

$\mathbb{C}$	field of complex numbers
$\mathbb{D}$	domain of definition of a differential equations system
$\mathbb{N}$	field of natural numbers
$\mathbb{R}$	field of real numbers
$\mathbb{R}^+$	set of real nonnegative numbers
X	defined euclidean d-dimentional space $\mathbb{R}^d$
$\mathbb{I} = [t_0, t_f]$	closed time interval of a differential equations system
$\mathbb{L}_\nu$	set of solutions of the derivative array F^ν of order ν
$\mathcal{C}(\mathbb{I}, \mathbb{R}^n)$	set of continuous functions
$\mathcal{C}^k(\mathbb{I}, \mathbb{R}^n)$	set of k-times continuously differentiable functions
$\mathcal{C}_b^{k,\delta}$	Banach space of $\mathcal{C}^k$ functions on $\mathbb{R}^d$ where the k-th derivative is δ-Hölder continuous
Ω	sample space
$\mathcal{F}$	collection of events, σ-algebra
$(\mathcal{F}_t)_{t \geq t_0}$	filtration
$\mathbb{P}$	probability space
$\mathcal{B}(\Omega)$	Borel sets in Ω
$(\Omega, \mathcal{F}, \mathbb{P})$	complete probability space
$(\Omega, \mathcal{F}, \mathbb{P}, (\theta)_{t \in \mathbb{I}})$	or simply θ, metric dynamical system
$\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$	Banach space of functions $f : \Omega \to \mathbb{R}$ wich are $\mathcal{F}$- and $\mathcal{B}$-measurable on $\mathbb{R}$, and $\mathbb{P}$-integrable
$\boldsymbol{E}$	leading matrix of a quasi-linear systems of differential equations
F	function of a nonlinear DAE
F^l	derivative array of order l
$\boldsymbol{M} \in \mathbb{R}^{m,n}$	matrix with m rows and n columns
$\det \boldsymbol{M}$	determinant of $\boldsymbol{M}$
M^{ij}	entry (i, j) of a matrix $\boldsymbol{M}$
$\operatorname{diag}(\boldsymbol{M})$	diagonal matrix of $\boldsymbol{M}$
$\dim(S)$	dimension of a (sub)space $S \in \mathbb{R}^n$
$\ker \boldsymbol{M}$	kernel of a matrix $\boldsymbol{M}$
$\operatorname{range} \boldsymbol{M}$	range of a matrix $\boldsymbol{M}$
$\operatorname{rank} \boldsymbol{M}$	rank of a matrix $\boldsymbol{M}$

$\mathrm{tr}(\boldsymbol{M})$	trace of a matrix $\boldsymbol{M}$
$\boldsymbol{M}^T$	transpose of a matrix $\boldsymbol{M}$
$\boldsymbol{I}$, $\boldsymbol{I}_d$	identity matrix (of dimension d)
x_t	state variable of a system of differential equations on t
$\dot{x}_t, \ddot{x}_t, \dddot{x}_t, x_t^{(i)}$	total derivatives of x_t with respect to t, i.e., $\dot{x}_t = \frac{d}{dt}x_t$, $\ddot{x}_t = \frac{d^2}{dt^2}x_t$, $\dddot{x}_t = \frac{d^3}{dt^3}x_t$, $x_t^{(i)} = \frac{d^i}{dt^i}x_t$
$\frac{\partial \cdot}{\partial \cdot}$	partial derivative
v_t	state variable of the variational equation system
$\boldsymbol{J}$	Jacobian matrix
t	independent variable of differential equations
T	defined time interval
$\mathcal{V}$	Lyapunov function
h	stepsize
t_0	initial time
t_f	final time
ξ_t	Brownian motion
w_t	Wiener process
θ_t	driving dynamical system
Θ	random dynamical system
φ	nonlinear cocycle
$\boldsymbol{\mathcal{G}}(\cdot)$	Lie algebra or Lie group
$\boldsymbol{\Phi}$	matrix cocycle
λ	Lyapunov exponent
$\boldsymbol{V}$	fundamental solution matrix
$\boldsymbol{Q}$	orthogonal matrix
$\boldsymbol{R}$	upper triangular matrix
$\mathbb{E}[\cdot]$	expectation, mean, or average
$\mathbb{V}[\cdot] = \mathbb{E}[\cdot]^2$	variance
$\exp(\cdot)$	(natural) exponential function
$\lim$	limit
$\log(\cdot)$	(natural) logarithm function
$\log^+(\cdot)$	positive part of log
$\min(\cdot)$	minimum
$\sup(\cdot)$	supremum
μ	mean
σ	standar deviation
$\mathcal{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$\oplus$	direct sum
$\lvert\cdot\rvert$	vector and matrix norms in $\mathbb{R}^n$
$\mathbf{0}$	zero matrix

$\|\cdot\|$	function norm
ν_d	differentiation index
ν_s	strangeness index
$\succeq$	positive semidefinite
$\preceq$	negative semidefinite
$\forall$	for all
$\circ$	composition
$\mathfrak{Re}(\cdot)$	real part
$\mathfrak{Im}(\cdot)$	imaginary part

Resumen

La naturaleza aleatoria que caracteriza algunos fenómenos en sistemas físicos reales (e.g., ingeniería, biología, economía, finanzas, epidemiología y otros) nos ha planteado el desafío de un cambio de paradigma del modelado matemático y el análisis de sistemas dinámicos, y a tratar los fenómenos aleatorios como variables aleatorias o procesos estocásticos. Este enfoque novedoso ha traído como consecuencia nuevas especificidades que la teoría clásica del modelado y análisis de sistemas dinámicos deterministas no ha podido cubrir. Afortunadamente, maravillosas contribuciones, realizadas sobre todo en el último siglo, desde el campo de las matemáticas por científicos como Kolmogorov, Langevin, Lévy, Itô, Stratonovich, sólo por nombrar algunos; han abierto las puertas para un estudio bien fundamentado de la dinámica de sistemas físicos perturbados por ruido.

En la presente tesis se discute el uso de ecuaciones diferenciales algebraicas estocásticas (EDAEs) para el modelado de sistemas multifísicos en red afectados por perturbaciones estocásticas, así como la evaluación de su estabilidad asintótica a través de exponentes de Lyapunov (ELs). El estudio está enfocado en EDAEs d-index-1 y su reformulación como ecuaciones diferenciales estocásticas ordinarias (EDEs). Fundamentados en la teoría ergódica, es factible analizar los ELs a través de sistemas dinámicos aleatorios (SDAs) generados por EDEs subyacentes. Una vez garantizada la existencia de ELs bien definidas, hemos procedido al uso de técnicas de simulación numérica para determinar los ELs numéricamente. Hemos implementado métodos numéricos basados en descomposición QR discreta y continua para el cómputo de la matriz de solución fundamental y su uso en el cálculo de los ELs. Las características numéricas y computacionales más relevantes de ambos métodos se ilustran mediante pruebas numéricas. Toda esta investigación sobre el modelado de sistemas con EDAEs y evaluación de su estabilidad a través de ELs calculados numéricamente, tiene una interesante aplicación en ingeniería. Esta es la evaluación de la estabilidad dinámica de sistemas eléctricos de potencia. En el presente trabajo de investigación, implementamos nuestros métodos numéricos basados en

descomposición QR para el test de estabilidad dinámica en dos modelos de sistemas eléctricos de potencia de una-máquina bus-infinito (UMBI) afectados por diferentes perturbaciones ruidosas. El análisis en pequeña-señal evidencia el potencial de las técnicas propuestas en aplicaciones de ingeniería.

Abstract

The random nature that characterizes some phenomena in the real-world physical systems (e.g., engineering, biology, economics, finance, epidemiology, and others) has posed the challenge of changing the modeling and analysis paradigm and treat these phenomena as random variables or stochastic processes. Consequently, this novel approach has brought new specificities that the classical theory of modeling and analysis for deterministic dynamical systems cannot cover. Fortunately, stunning contributions made overall in the last century from the mathematics field by scientists such as Kolmogorov, Langevin, Lévy, Itô, Stratonovich, to name a few; have opened avenues for a well-founded study of the dynamics in physical systems perturbed by noise.

In the present thesis, we discuss stochastic differential-algebraic equations (SDAEs) for modeling multi-physical network systems under stochastic disturbances, and their asymptotic stability assessment via Lyapunov exponents (LEs). We focus on d-index-1 SDAEs and their reformulation as ordinary stochastic differential equations (SDEs). Supported by the ergodic theory, it is feasible to analyze the LEs via the random dynamical system (RDSs) generated by the underlying SDEs. Once the existence of well-defined LEs is guaranteed, we proceed to the use of numerical simulation techniques to determine the LEs numerically. Discrete and continuous QR decomposition-based numerical methods are implemented to compute the fundamental solution matrix and use it in the computation of the LEs. Important numerical and computational features of both methods are illustrated through numerical tests. All this investigation concerning systems modeling through SDAEs and their stability assessment via computed LEs finds an appealing engineering application in the dynamic stability assessment of power systems. In this research work, we implement our QR-based numerical methods for testing the dynamic stability in two types of single-machine infinite-bus (SMIB) power system models perturbed by different noisy disturbances. The analysis in small-signal evidences the potential of the proposed techniques in engineering applications.

Resum

La naturalesa aleatòria que caracteritza alguns fenòmens en sistemes físics reals (e.g., enginyeria, biologia, economia, finances, epidemiologia i uns altres) ens ha plantejat el desafiament d'un canvi de paradigma del modelatge matemàtic i l'anàlisi de sistemes dinàmics, i a tractar els fenòmens aleatoris com a variables aleatòries o processos estocàstics. Aquest enfocament nou ha portat com a conseqüència noves especificitats que la teoria clàssica del modelatge i anàlisi de sistemes dinàmics deterministes no ha pogut cobrir. Afortunadament, meravelloses contribucions, realitzades sobretot en l'últim segle, des del camp de les matemàtiques per científics com Kolmogorov, Langevin, Lévy, Itô, Stratonovich, només per nomenar alguns; han obert les portes per a un estudi ben fonamentat de la dinàmica de sistemes físics pertorbats per soroll.

En la present tesi es discuteix l'ús d'equacions diferencials algebraiques estocàstiques (EDAEs) per al modelatge de sistemes multifísicos en xarxa afectats per pertorbacions estocàstiques, així com l'avaluació de la seua estabilitat asimptòtica a través d'exponents de Lyapunov (ELs). L'estudi està enfocat en EDAEs d-index-1 i la seua reformulació com a equacions diferencials estocàstiques ordinàries (EDEs). Fonamentats en la teoria ergòdica, és factible analitzar els ELs a través de sistemes dinàmics aleatoris (SDAs) generats per EDEs subjacents. Una vegada garantida l'existència d'ELs ben definides, hem procedit a l'ús de tècniques de simulació numèrica per a determinar els ELs numèricament. Hem implementat mètodes numèrics basats en descomposició QR discreta i contínua per al còmput de la matriu de solució fonamental i el seu ús en el càlcul dels ELs. Les característiques numèriques i computacionals més rellevants de tots dos mètodes s'illustren mitjançant proves numèriques. Tota aquesta investigació sobre el modelatge de sistemes amb EDAEs i avaluació de la seua estabilitat a través d'ELs calculats numèricament, té una interessant aplicació en enginyeria. Aquesta és l'avaluació de l'estabilitat dinàmica de sistemes elèctrics de potència. En el present treball de recerca, implementem els nostres mètodes numèrics basats en descomposició QR per al test d'estabilitat

dinàmica en dos models de sistemes elèctrics de potència d'una-màquina bus-infinit (UMBI) afectats per diferents pertorbacions sorolloses. L'anàlisi en xicotet-senyal evidencia el potencial de les tècniques proposades en aplicacions d'enginyeria.

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Este trabajo está dedicado a las tres mujeres más importantes en mi vida, mi madre Noemí, mi esposa Diana, y mi hija Aitana.

CHAPTER 1

Introduction and Objectives

1.1. Introduction

1.1.1. Modeling Multi-Physical Systems

Mathematical modeling is defined as the representation of the behavior of systems, devices, objects, etc. by using concepts expressed in a mathematical language. Mathematical models are extensively applied in the scientific and engineering area. By manipulating and interpreting mathematical terms, scientists and engineers are able to understand the phenomena under study. In engineering, mathematical modeling is crucial to understanding the behavior of complex systems that are generally a combination of mechanical, electrical, chemical, or other physical components, which are known as *multi-physical systems*.

Nevertheless, to create a mathematical model of a complex multi-physical system is an increasingly challenging exercise. First of all, it is essential to decide what we will do with the model, whether it is a robustness analysis, numerical simulation, optimization, or control process of the network. Depending on the task, different modeling details may be needed. In some cases, highly simplified (reduced) models may be enough, and in other cases, a highly detailed model could be necessary. With these differing requirements in mind, model hierarchies are usually developed for multi-physical systems. A model hierarchy allows to choose the most simple model for a desired task, preserving as much as possible the physical properties need for that task, being as accurate as needed, and being its implementation as efficiently as possible.

An appealing class of multi-physical systems in the electrical engineering area, where suitable model hierarchies have been built, are the *power network systems*. In these systems, a task such as the *power flow analysis*, where just the steady-state of the system is need to be taken into account, a set of algebraic equations that models the constraints of the system (i.e., topological, geometric, electrical, mechanical, etc. constraints) is enough. Nevertheless, when it is necessary to study the system's transient-state behavior, then a dynamical model through differential equations is needed in addition to the model of algebraic equations. Furthermore, if some parameters are treated as spatially distributed, a more complex model based on partial differential equations should be considered. Other considerations, e.g. including (or neglecting) specific physical phenomena, provoke the creation of new models with different shapes, which are also part of the power systems model hierarchy.

Let us focus on multi-physical systems assuming their parameters are lumped. This kind of systems' dynamical behavior is generally modeled employing ordinary differential equations (ODEs) coupled to algebraic equations (AEs) representing such systems' constraints. This combined set of equations is called *Differential-Algebraic Equations* (DAEs), (DAEs), also known in the control context as *descriptor systems*. They are a natural and widely requested mathematical representation paradigm in many areas of science and engineering. In the most general case, a DAE model can be expressed in the fully implicit form

$$F(\dot{x}, x, u, t) = 0, \tag{1.1}$$

typically accompanied by an initial condition $x_{t_0} = x_0$ and possibly combined with an output equation $y = G(x, u, t)$. Besides the term of descriptor systems, multi-physical systems formulated in this fashion have other denominations are implicit, generalized, singular, constrained, degenerate, and semi-state systems.

1.1.2. Incorporating Uncertainties

On the other hand, it is important to mention that real-world dynamical systems evolve in the presence of noisy disturbances either on their parameters, inputs, or initial conditions. The noise source has diverse origins such as measurement errors, earthquake motion, epidemics, solar radiation, wind speed, thermal noise, population fluctuation, and many others. Regardless of its source, the presence of noise (also referred to as randomness or stochasticity) can lead to non-trivial effects over the dynamical system. Possible uncertain variations in the states' trajectories of the system or changes in their stability properties are some of the consequences registered, turning the deterministic-based studies inadequate for this new situation. Hence, it is essential to propose a way to incorporate the noise into our dynamic models. A suitable choice is to couple the noise as an additional term to the deterministic dynamic equations as follows

$$\frac{dx}{dt} = f(x,t,u) + \text{“}\textit{noise}\text{”}. \tag{1.2}$$

These noise-coupled expressions are well known as *“stochastic differential equations”* (SDEs), and the uncertain term is the white noise process [68]. However, the implementation of SDEs is not trivial. Working with them requires the use of new rules of calculus. Along the time, two versions of stochastic calculus have been dominant; these are Itô and Stratonovich calculi.

Dynamical systems perturbed by uncertain disturbances have appropriately been modeled and analyzed in many disciplines through SDEs (Itô or Stratonovich -type). During the last decades, there has been a broad and fruitful development either in theory and applications. In like manner, DAEs have been the choice to treat constrained dynamical systems on a deterministic framework, as mentioned before, see [5], [48], [61], [68] and [15], [16], [52], [55] respectively. However, modeling and analyzing the dynamic behavior of constrained systems subject to white noise disturbances has stated the need for a generalized concept that covers both SDEs and DAEs. Such a requirement has led to the appearance of *“stochastic differential-algebraic equations”* (SDAEs). This is a relatively new framework, the study of its concepts, mathematical properties, and numerical treatment has been limited to the date. Some relevant contributions can be found in [22], [80], [95].

1.1.3. Power Systems Modeled as SDAEs

A kind of constrained multi-physical systems of particular interest, because of their importance in the energy context, are the power network systems. Here, the dynamical behavior of the different components into a power system (i.e., synchronous machines, controllers, power converters, transmission lines, loads, etc.) together with its constraints (given by the network topology, physical laws and restrictions, etc.) have been traditionally represented by DAEs. Nonetheless, the need of incorporating perturbing phenomena of random/stochastic nature (such as wind or solar generation, rotor vibrations in synchronous machines, stochastic variations of loads, electromagnetic transients, measurement errors in control devices, etc.) in the form of random variables or stochastic processes makes it necessary the use of SDAEs. However, the scarce development experienced with SDAE's theory has had repercussions in the applications, the studies of multi-physical systems based in SDAEs have been modest and sparse. An exemplary case of this exiguous implementation is the power systems area. Some relevant engineering oriented works which implement SDAEs in the study to power systems are [65], [94]. Most of the engineering studies have instead made use of the SDE forms. Nonetheless, it is significant to point out that right modeling of constrained systems necessarily entails an explicit representation of the system's constraints via algebraic equations, or even more, the possible implicit presence of constraints into the differential equations system, which it is the case of the system modeled as d-index greater than 1 SDAE system. With this scenario, the SDAE setting is the right choice to avoid issues in subsequent analysis and numerical treatment.

1.1.4. Lyapunov Exponents in SDAEs

The presence of noise perturbing a multi-physical system can drastically modify its dynamical behavior compared to the one in deterministic condition. Among others, qualitative changes linked to the dynamic stability of the system could come out. For example, noise can shift the bifurcations stabilizing the system, or induce new stable states that did not exist in the deterministic counterpart, or exciting internal oscillation modes. In this context, it is worth having techniques that enable us to assess the dynamic stability conditions with noisy disturbances. An interesting tool is the one inspired in the *"characteristic exponents"* introduced by A.M. Lyapunov in his remarkable Ph.D. thesis [59] entitled *"The general problem of the stability of motion"*. The nowadays called Lyapunov exponents (LEs), are a powerful instrument in the asymptotic stability evaluation of dynamical systems. Although the theoretical background of LEs is not new, this experimented with a crucial development since the contribution given by Oseledets (1968). The Multiplicative Ergodic Theorem (MET) of Oseledets [69] ensures the regularity and the existence of the LEs belonging to a linear cocycle over a metric dynamical system under some assumptions and conditions. A fundamental merit of the ergodic theory framework is that it leads us to a whole new field, the theory of random dynamical systems (RDS). Moreover, this enables us to derive a theory of the LEs for RDSs, which can be generated by random and stochastic, ordinary and partial differential equations. The interested reader is referred to [7] for a rigorous presentation of this theory.

1.2. Thesis Objectives

Based on the information stated in the previous section, it is noticeable the still not mature enough development of theory, and numerical treatment techniques around SDAE systems. This added to the scarcity of applications of the stability analysis strategies in real-world study-cases such as power network systems. In order to perform a contribution in these fields of studies, we state as *main objective* the study of concepts and methods of modeling and stability analysis of constrained dynamical systems under uncertainty based on stochastic differential-algebraic equations, with an application to power network systems. Based on this postulate, the specific objectives are identified as follows:

- Acquire a deep understanding of the fundamental concepts related to stochastic differential-algebraic equations and the generation of random dynamical systems.
- Verify through the theory the use of Lyapunov exponents as asymptotic stability index of dynamical systems perturbed by noise modeled via SDAEs.
- Develop suitable numerical techniques to compute the Lyapunov exponents from an SDAE setting.
- Evaluate the methods through their implementation into some nu-

merical examples in order to evaluate its precision and computational cost.

- Working out selected conventional and validated study-cases of power systems modeled through SDAEs using the proposed numerical methods for computing Lyapunov exponents.

1.2.1. The Road Towards the Objectives' Achievement

The starting point of the present research work is the review of concepts that permit the correct asymptotic stability assessment of differential-algebraic equations driven by Gaussian white noise through the method of LEs. We start by surveying the theory of SDAEs. In particular, we are interested in SDAEs systems of the form

$$\boldsymbol{E}d\boldsymbol{x}_t = \boldsymbol{f}_0(\boldsymbol{x}_t)dt + \sum_{j=1}^{m} \boldsymbol{f}_j(\boldsymbol{x}_t)dw_t^j, \qquad t \in \mathbb{R}^+,$$

together with consistent initial value $\boldsymbol{x}_{t_0} = \boldsymbol{x}_0$. Following the ideas from [22], [23], [54], [80], [95], properties such as strong solutions' existence and uniqueness are reviewed. Analogously to the DAE case, We also define the class of index-1 SDAEs. Further, we show that SDAE systems with this structure can be reduced to an equivalent SDE system that preserves the inherent dynamics of the original SDAE. Additionally, we know from the RDSs theory that an autonomous Itô SDE generates an RDS (see [7]). Therefore, the Oseledets' MET can be applied to define the Lypaunov exponents and Lyapunov spectrum.

Once we have verified the theoretical framework that guarantees the existence of well-defined LEs, we study the numerical methods based on the QR factorization of the fundamental solution matrix that allow the numerical computation of spectral values associated to the Lyapunov spectra. The first technique requires computing the fundamental solution matrix and forming an orthogonal factorization; the second one involves performing a continuous QR decomposition of the fundamental solution matrix. Both techniques have been extensively studied in works with applications to deterministic ODE and DAE systems, see [12], [13], [28], [29], [57], [58]. This research essentially follows the ideas exposed in [18], who extended the QR methods to a stochastic version.

Finally, the whole concepts and computational methodologies are applied to assess the asymptotic stability of power systems affected by stochastic disturbances using the LEs associated with their SDAE models. The stability under stochastic perturbations of didactic models such as the single machine infinite bus system is tested. This is an attractive attempt to evidence the usefulness of the LEs as an index of stability for dynamical systems working under uncertainty and opening the doors to its potential application on large-scale systems stability analysis.

The research studies and results presented in this thesis document has been carried out within the *Research Group INTERTECH* of *Universitat Politècnica de València* (UPV), led by Prof. Dr. Pedro Fernández de Córdoba. Additionally, the research activities have been partially performed in the *Research Group Numerical Mathematics* of *Technische Universität Berlin* (Germany) under the supervision of Prof. Dr. Volker Mehrmann.

1.3. Outlines of the Thesis

The present thesis document, which contains a compilation of author's doctoral researh work, is organized in the following manner:

We start with **Chapter 2** (*Modeling Dynamics of Constrained Systems under Uncertainty*). This chapter is a summary of the theory related to constrained ODE systems, which are called DAEs. The important concept of *index* is tackled and explained, with emphasis on the differential and strange indexes, which are of interest to our studies. We continue with the theory of SDEs, as the mathematical expression of a dynamical system perturbed by noise. The SDAE, understood as the constrained version of the SDE systems is finally studied. Specifically, we refer to the d-index-1 (or strangeness-free) quasi-linear SDAE form, which can be reduced to an underlying system. Some methods for the numerical integration of SDEs are digested in the last section.

Chapter 3 (*Random Dynamical Systems and Lyapunov Stability*), is devoted to briefly recall some basic concepts about Random Dynamical Systems generated by SDAEs and their properties. The fundamentals of this topic were mainly taken from L. Arnold's work published in his book [7]. We revisit the main concepts about stability in the sense of Lyapunov, as well as the two methods to assess the stability of a given solution. Afterward, we focus on the first method for testing stability based in *characteristic numbers* better known as *Lyapunov exponents*, whose value is a suitable index to determine the asymptotic growth rate of the system solutions. This section summarizes explanations about the existence and well-posedness of the Lyapunov exponents.

With base on the theory reviewed about the existence of well-defined Lyapunov Exponents in the RDS generated by SDEs, along with **Chapter 4** (*Numerical Methods for computing LEs*), we provide numerical techniques for the numerical computation of Lyapunov Exponents for given SDE systems. The methods are based on QR decomposition of the fundamental solution matrix. Two different methods, namely continuous and discrete, are studied and implemented along with the use of Euler-Maruyama and Milstein discretization schemes for numerical integration. The numerical accuracy, computational cost, and robustness of the QR-based methods are tested numerically through two examples.

In **Chapter 5** (*Application to Power Systems*), we present a survey of the main concepts regarding power network systems in dynamic regime, emphasizing topics such as technical definitions and classification of stability (according to IEEE/CIGRE), dynamical modeling considering uncertainties with SDAEs, and dynamic stability analysis. Here, we show the benefits of using the computed Lyapunov Exponents calculated by QR-based methods. In the two single-machine infinite-bus (SMIB) power system test cases proposed in the last section of this chapter, it has been possible to assess the system's asympotic stability condition for different stochastic perturbation levels and the maximum stochastic volatility allowed by the system before losing stability.

Lastly, **Chapter 6** (*Final Discussion*) contains the main concluding ideas, relevant considerations and remarks, future works of interest in line with the thesis, and an itemized list of the publications made along the present doctoral research.

CHAPTER 2

Modeling Dynamics of Constrained Systems under Uncertainty

2.1. Differential-Algebraic Equations

The exercise of modeling the dynamics of multi-physical systems leads typically to a formulation based on differential equations. Nevertheless, in real-wold systems in general, the states are usually constrained by conservation laws, geometric, topological, geometric, electrical, mechanical, etc. restrictions. They must then be included in the model through *Algebraic Equations* (AEs), which model such restrictions. This indicates that constrained dynamic models arise naturally from the real system. It is a common operation to comprise such constraints of the system, transforming them algebraically (vanishing the constraints) to a set of the known *ordinary differential equations* (ODEs) in minimal coordinates. However, in practice, such transformation could not be feasible in some cases, and even not recommendable due to the numerical and analytical issues this conversion could cause. As far as possible, the algebraic constraints must be kept, and the system be treated analytically and numerically within this form.

The combined group of ODEs accompanied by AEs are called *differential-algebraic equations* (DAEs), also named descriptor systems. The subject of DAEs has been extensively studied from the analytic, numerical and the application point of view. Abundant literature is available on this topic, see e.g. [15], [16], [52], [55] for further reading. The most general formulation of DAE, known as *standard form*, which is used in the general mathematical analysis and general numerical methods, has the following fully implicit form

$$\boldsymbol{F}(\dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t) = 0, \tag{2.1}$$

where we denote by $\mathcal{C}^0(\mathbb{I}, \mathbb{R}^n)$ the set of continuous functions, $\mathbb{I} \subseteq \mathbb{R}$ is a compact time interval, and $\boldsymbol{x}_t \in \mathbb{R}^n$ represents the system's unknown variables. We assume that $\boldsymbol{F} \in \mathcal{C}^0(\mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}}, \mathbb{R}^n)$ is sufficiently smooth and that $\mathbb{D}_x, \mathbb{D}_{\dot{x}} \subseteq \mathbb{R}^n$ are open sets [52]. If an initial condition

$$\boldsymbol{x}_{t_0} = \boldsymbol{x}_0, \qquad \boldsymbol{x}_0 \in \mathbb{R}^n, \tag{2.2}$$

on the domain $\mathbb{I} := [t_0, t_f]$, accompanies the DAE (2.1), it is known as an *initial value problem* (IVP) DAE. Commonly, to analyze the nonlinear form (2.1) requires its linearization along with its solutions. The result of the linearization process is the linear DAE form

$$\boldsymbol{E}_t \dot{\boldsymbol{x}}_t = \boldsymbol{A}_t \boldsymbol{x}_t, \qquad t \in \mathbb{I}, \tag{2.3}$$

with the matrix pair $\boldsymbol{E}_t$, $\boldsymbol{A}_t \in \mathcal{C}^0(\mathbb{I}, \mathbb{R}^{n\times n})$, which is also known as *variational equation.*

Concerning a DAE system's solutions, a function can be consedered a solution of an specific DAE if it satisfies such DAE pointwise and satisfies the initial conditions. Furthermore, a whole IVP DAE, like (2.1) with the initial value (2.2), is called *consistent* if there exist at least one solution that satisfies such initial value problem. It is interesting to see a DAE solution, from a geometrical point of view, as an ODE whose solutions are restricted to manifolds due to the constraints.

The validation of a model given by the IVP DAE, in order to guarantee its consistency with the system that represents, requires the study of the existence and uniqueness of the solutions. This is not an easy task in the DAEs' field. Firstly, there is an important property to accomplish; this is the regularity. A DAE system is called *regular* if its number of equations is equal to the number of unknowns. If the DAE is not regular, a regularization process is needed [15]. Also, the analysis and numerical treatment of the solutions for the different DAE forms require remodeling procedures that involve transformations of the equations, as well as differentiations to filter out the hidden constraints. The characterization of the DAEs, in order to perform these transformations and differentiations, is discussed in the following section.

2.1.1. The Index of DAEs

The analysis and numerical treatment of ODEs, which can be considered as an special type of DAEs, has been extrensively studied and is well understood. In fact, the most of the techniques of simulation, stability analysis, control, and optimization are available for ODEs. On the contrary, several difficulties arise when the DAEs are anlyzed and treated. Identifying and measuring those issues has led to the birth and the development of different approaches called *index* concepts. An index of a DAE is a nonnegative integer that provides useful information about the mathematical structure and potential difficulties in its analysis and the numerical solution. The height of an index in the DAE is directly proportional to the specific difficulties one can expect. A wide variety of indexes have been proposed and developed, based on the kind of difficulty to be measured. Extensive bibliography can be found in [52], [55], [63], [84] about index concepts such as geometric index, global index, uniform index, index of nilpotency, perturbation index, or structural index. However, their individual characteristics and the relations between them are beyond this study. Since our research work needs are mainly oriented to the equivalent transformations and differentiations of the DAEs, in order to apply classical numerical methods, we briefly get focus on three important indexes *tractability index*, *differentiation index*, and the *strangeness index*.

2.1.1.1. Tractability Index

Shortly identified in the literature as *t-index*, the tractability index is adequate for the class of DAEs with properly stated leading term such as

$$\boldsymbol{E}_t \frac{d}{dt}(\boldsymbol{D}_t \boldsymbol{x_t}) = \boldsymbol{b}(\boldsymbol{x}_t, t), \qquad t \in \mathbb{I}, \tag{2.4}$$

where $\boldsymbol{E}_t \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{n,n})$, $\boldsymbol{D} \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^{n,n})$, $\boldsymbol{b}(\boldsymbol{x}_t, t) \in \mathbb{R}^n$, $\boldsymbol{x}_t \in \mathbb{R}^n$, $\ker(\boldsymbol{E}_t) \oplus$ $\text{range}(\boldsymbol{D}_t) = \mathbb{R}^n$ for all $t \in \mathbb{I}$, and there exists a projector $\boldsymbol{\mathcal{R}}_t \in$

$\mathcal{C}^1(\mathbb{I}, \mathbb{R}^{n,n})$ such that

$$\text{range}(\boldsymbol{\mathcal{R}}_t) = \text{range}(\boldsymbol{D}_t), \quad \ker(\boldsymbol{\mathcal{R}}_t) = \ker(\boldsymbol{E}_t) \quad \text{for all} \quad t \in \mathbb{I}.$$

The t-index proposes the construction of a chain of matrices for decoupling the DAE into characteristic components. Here, the t-index value is given by the length of the sequence of matrices need to accomplish the decoupling. The proposal of the t-index is not developed in view of numerical methods, but it is suited either for discretizations or for the analysis of the DAE in aspects such as the determination of consistent initial values, the determination of the smoothness requirements to the solution, and system matrices and vectors, see [62], [84].

2.1.1.2. Differentiation Index

Either the differentiation index as well as the strangeness index presented below, are oriented to the differentiation process of the DAE. They are based on the derivative array approach developed by Campbell [16]. Let us consider the general nonlinear DAE (2.1). If we successive differentiate it with respect to t up to order l and summarize all the derivatives together with the original form, we obtain the so-called *derivative array*. This inflated system has the form

$$\boldsymbol{F}^l(\boldsymbol{x}_t^{(l+1)}, \ldots, \dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t) = \begin{bmatrix} \boldsymbol{F}(\dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t) \\ \frac{d}{dt}\boldsymbol{F}(\dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t) \\ \vdots \\ \left(\frac{d}{dt}\right)^l \boldsymbol{F}(\dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t) \end{bmatrix}. \tag{2.5}$$

We require the solvability of (2.5) in an open set. Furthermore, we consider every smooth solution of (2.1) solves (2.5) as well, and vice-versa. Are also defined the Jacobians

$$\boldsymbol{\mathcal{M}}^l(\boldsymbol{x}_t^{(l+1)}, \ldots, \dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t) = \boldsymbol{F}^{l;\dot{\boldsymbol{x}}_t,\ldots,\boldsymbol{x}_t^{(l+1)}}(\boldsymbol{x}_t^{(l+1)}, \ldots, \dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t), \tag{2.6}$$

$$\boldsymbol{\mathcal{N}}^l(\boldsymbol{x}_t^{(l+1)}, \ldots, \dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t) = -\boldsymbol{F}^{l;\boldsymbol{x}}(\boldsymbol{x}_t^{(l+1)}, \ldots, \dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t), \tag{2.7}$$

which corresponds to the derivative array in the linear form (2.3) [84]. Then, we are ready to define the differentiation index.

Definition 2.1.1 (Differentiation index)**.** *Suppose that the DAE (2.1) is a solvable system on an open set. If ν_d is the smallest integer l (if it exists) such that the solution $\boldsymbol{x}$ is uniquely defined by (2.5) for all consistent initial values, then we call ν_d the differentiation index (d-index) of the DAE (2.1).*

The d-index, roughly speaking, tells us the number of differentiations that are required to convert a DAE system into an ODE. The d-index is the most known and the most used approach for DAEs. But, it has an important drawback, it is not suitable for general DAE forms since it is based on a solvability concept that requires unique solvability. Therefore, another concept has been developed to generalize this approach.

2.1.1.3. Strangeness Index

The concept of the strangeness index was developed by Kunkel and Mehrmann. It is a generalization of the differentiation index to over- and underdetermined DAE systems. The concept is based on the following Hypothesis

Hypothesis 2.1.2 ([63])**.** *Consider the general nonlinear DAE (2.1) and suppose that there exist integers ν, a, and d such that the set*

$$\mathbb{L}_\nu = \{\boldsymbol{z}_t \in \mathbb{R}^{(\nu+2)n+1} | \boldsymbol{F}^\nu(\boldsymbol{z}_t) = 0\}$$

associated with $\boldsymbol{F}$ is nonempty and such that for every point

$$\boldsymbol{z}_0 = (\boldsymbol{x}_0^{(\nu+1)}, \dots, \dot{\boldsymbol{x}}_0, \boldsymbol{x}_0, t_0) \in \mathbb{L}_\nu$$

there exists a sufficiently small neighborhood in which the following properties are satisfied:

1. *We have* $\operatorname{rank} \boldsymbol{\mathcal{M}}^\nu(\boldsymbol{z}_t) = (\nu+1)n - a$ *on* $\mathbb{L}_\nu$ *such that there exists a smooth matrix function* $\boldsymbol{Z}_2$ *of size* $(\nu+1)n \times a$ *and pointwise maximal rank, satisfying* $\boldsymbol{Z}_2^T \boldsymbol{\mathcal{M}}^\nu = 0$ *on* $\mathbb{L}_\nu$.
2. *We have* $\operatorname{rank} \hat{\boldsymbol{A}}_2(\boldsymbol{z}_t) = a$, *where* $\hat{\boldsymbol{A}}_2(\boldsymbol{z}_t) = \boldsymbol{Z}_2^T \boldsymbol{\mathcal{N}}^\nu [\boldsymbol{I}_n \ 0 \ \cdots \ 0]^T$ *such that there exists a smooth matrix function* $\boldsymbol{T}_2$ *of size* $n \times d$, $d = n - a$, *and pointwise maximal rank, satisfying* $\hat{\boldsymbol{A}}_2 \boldsymbol{T}_2 = 0$.
3. *We have* $\operatorname{rank} \boldsymbol{F}_{\dot{\boldsymbol{x}}_t}(\dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t) \boldsymbol{T}_2(\boldsymbol{z}_t) = d$ *such that there exists a smooth matrix function* $\boldsymbol{Z}_1$ *of size* $n \times d$ *and pointwise maximal rank, satisfying* $\operatorname{rank} \hat{\boldsymbol{E}}_1 \boldsymbol{T}_2 = d$, *where* $\hat{\boldsymbol{E}}_1 = \boldsymbol{Z}_1^T \boldsymbol{F}_{\dot{\boldsymbol{x}}_t}$.

With the Hypothesis 2.1.2 as base, we define the strangeness index concept as follows.

Definition 2.1.3 (Strangeness index). *Given the nonlinear DAE in the general form (2.1), the smallest value of ν, denoted by ν_s, such that the right-hand side $\boldsymbol{F} = (\dot{\boldsymbol{x}}_t, \boldsymbol{x}_t, t)$ of (2.1) satisfies the Hypothesis 2.1.2, is called the strangeness index (s-index) of (2.1). Furthermore, if (2.1) has $\nu = 0$ (vanishing s-index), then it is called strangeness-free (s-free).*

The s-index owns the same simple interpretation as d-index, which is the minimum number of differentiations needed to transform the DAE into an ODE. But, the value of s-index is one below the d-index, considering that both indexes exist. Notice that not all the systems can have this generalized differentiation index, but many of those from applications do. In fact, s-free DAEs are commonly found in transmission power systems modeled as DAEs.

Under very mild assumptions, DAE systems for which there exists an s-index, can be reformulated as a s-free system with the same solution, in which the algebraic and differential parts of the system are easily separated.

2.1.2. Some Classes of DAEs

The DAE in standard form (2.1) is a beneficial general representation that permit its suitable analysis and numerical treatment. Nevertheless, in practice, the structure of a DAE strongly depends on the application where they come from. Based on this, there is an extensive classification of DAEs arising from different origins and with different structures such as DAEs with linear derivative term, nonlinear derivative term; fully implicit, semi-implicit, semi-explicit, quasi-linear, linear; with variable coefficients (nonautonomous), or constant coefficients (autonomous), among others; see [16], [52], [55] for details. The classification of DAEs by their structure is fundamental since most of the analysis concepts and numerical methods have been developed for each specific form. In this section, we define three forms of DAEs that are of interest to our research. The first one is the *autonomous quasi-linear DAE* and has the form

$$\boldsymbol{E}\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t), \qquad t \in \mathbb{I}, \tag{2.8}$$

where $\boldsymbol{E} \in \mathbb{R}^{n\times n}$ is known as *"leading matrix"*, the vector-valued function $\boldsymbol{f} \in \mathcal{C}^k(\mathbb{D}_{\boldsymbol{x}}, \mathbb{R}^n)$ (for some $k \geq 1$) is called *right-hand side function*, and $\boldsymbol{x}_t \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^n)$ are the *unknown variables* of the quasi-linear DAE.

$\mathbb{D}_{\boldsymbol{x}} \subseteq \mathbb{R}^n$ is an open set. Moreover, an special subclass of quasi-linear DAE is the *semi-implicit DAE*

$$\boldsymbol{E}_1\dot{\boldsymbol{x}}_t = \boldsymbol{f}_1(\boldsymbol{x}_t), \tag{2.9a}$$

$$0 = \boldsymbol{f}_2(\boldsymbol{x}_t), \tag{2.9b}$$

with $t \in \mathbb{I}$. Where $\boldsymbol{f}_1 \in \mathcal{C}^k(\mathbb{D}_{\boldsymbol{x}}, \mathbb{R}^d)$, $\boldsymbol{f}_2 \in \mathcal{C}^k(\mathbb{D}_{\boldsymbol{x}}, \mathbb{R}^a)$, and $\boldsymbol{x}_t \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^n)$. Here the leading matrix of (2.9) is a singular matrix and has the structure

$$\boldsymbol{E} := \begin{bmatrix} \boldsymbol{E}_1 \\ \boldsymbol{0} \end{bmatrix},$$

with $\boldsymbol{E}_1 \in \mathbb{R}^{d\times n}$. Here we consider $d = n-a$, with d denoting the number of differential equations and a the number of algebraic equations. Furthermore, if the block matrix of (2.9a) satisfies $\boldsymbol{E}_1 := [\,\boldsymbol{I}_d\ \boldsymbol{0}\,]$ (where $\boldsymbol{I}_d$ is the identity matrix), then the form (2.9) corresponds to a *semi-explicit DAE*. As an interesting remark, there an interesting further subclass of semi-explicit DAE whose special structure arises naturally in many applications; this is the *Hessenberg DAE* form, see [16], [55]. When a nonlinear DAE form is linearized with respect to a (constant) solution, we have a linear DAE. In this section, we present the *autonomous linear DAE*

$$\boldsymbol{E}\dot{\boldsymbol{x}}_t = \boldsymbol{A}\boldsymbol{x}_t, \qquad t \in \mathbb{I}, \tag{2.10}$$

where the leading matrix $\boldsymbol{E} \in \mathbb{R}^{n\times n}$ and the *right-hand side matrix* $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ are constant matrices. The vector $\boldsymbol{x}_t \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^n)$ represents the unknown variables.

2.2. Stochastic Differential Equations

The development of theory and numerical treatment for dynamical systems expressed in ODEs and DAEs, and their applications to the study of multi-physical systems, has been the most common activity in the scientific and engineering area. However, as discussed previously in Chapter 1, the dynamical systems studies' orientation towards a deterministic viewpoint is an incomplete exercise. The strong presence of randomness or stochasticity, so-called uncertainty in general, shaped like interference, noise, or other uncertain physical phenomena in the real-world systems makes unavoidable neglecting their incorporation into dynamical models. It is possible to categorize the presence of uncertainties into systems in three groups:

- initial conditions,
- inputs or stimulations,
- physical parameters.

In the last two categories, one can incorporate the uncertainties in the form of additive or multiplicative uncertain disturbances, respectively. So, in line with the equation (1.2) presented in the introductory chapter, we can mathematically define a noisy differential equation by coupling the uncertain disturbance in the form of a noise process ξ_t into a general ODE form as follows

$$\frac{dx}{dt} = f_0(x_t, t) + f_1(x_t, t)\xi_t. \tag{2.11}$$

Then, the original ODE's deterministic evolution becomes a *stochastic differential equation* (SDE) with the incorporation of the noisy forcing. Here, the form of the expression $f_1(x_t, t)$ determines whether the noise is additive or multiplicative in the SDE (if $f_1 \equiv$ cte. the noise is additive).

Although the presence of noise usually does nothing more than blurring the deterministic trajectories of the system variables. In some dynamical systems, overall in the ones characterized by a high nonlinearity where stochastics acts as a driving force, the noise's action can radically change their dynamics compared to their behavior in a deterministic regime. In this context, a formal study on the noise effects in the system dynamics is feasible in terms of SDEs.

There are some options of stochastic processes to describe the noise ξ_t in (2.11) such as the Brownian motion, fractional Brownian motion, Lévy noise, Poisson noise, etc. The Brownian motion, better known in mathematics as *Wiener process* (in fact, Brownian motion is a concept from physics, but mathematically defined with the name of Wiener process), is widely used to model uncertainty in many applications in engineering, biology, economics and finance [72]. The Wiener process is characterized by being a continuous process with stationary independent increments. It is the most basic stochastic process that allows us to model continuous uncertainty.

Definition 2.2.1. *(Wiener process) A stochastic process $w_t(\omega) = \omega(t)$ with $t \in \mathbb{R}^+$, where $\omega \in \Omega$, defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \geq t_0}$, where $\mathcal{F} = \mathcal{B}(\Omega)$ is the σ-algebra of Borel sets in Ω, is called Wiener process if the following conditions hold:*

1) $w_0 = 0$ *a.s.,*

2) sample paths of the function w_t is a.s. continuous,
3) the function w_t is of unbounded variation in every interval,
4) w_t has independent increments with $w_{t+h} - w_t \sim \mathcal{N}(0, h)\ \forall\, t, h > 0$, where $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2.

Furthermore, Each j-th Wiener process is understood as a process with independent increments such that $(w_t - w_s) \sim \mathcal{N}(0, t-s)$, i.e., is a Gaussian random variable for all $0 \leq s < t$, such that

$$\mathbb{E}[w_t - w_s] = 0, \quad \mathbb{E}[w_t - w_s]^2 = t - s, \quad \mathbb{E}[w_t w_s] = \min(s, t).$$

By using the differential equation given in the form (2.11), and the Brownian motion expressed as the time derivative of the Wiener process

$$\frac{dw_t}{dt} = \xi_t, \tag{2.12}$$

also called *white noise* process, we obtain the SDE of the form

$$dx = f_0(x_t, t)dt + f_1(x_t, t)dw_t, \quad t \in \mathbb{I}, \tag{2.13}$$

where the function $f_0 \in \mathcal{C}^k(\mathbb{D}_x, \mathbb{R})$ (for some $k \geq 1$) is known as *"drift"*, the function $f_1 \in \mathcal{C}^{k+1}(\mathbb{D}_x, \mathbb{R})$ is known as the *"diffusion"*, with the Wiener process w_t. Since the Wiener process is nowhere differentiable, it is more convenient to represent equation (2.13) in its integral form as

$$x_t = x_{t_0} + \int_{t_0}^{t} f_0(x_s, s)ds + \int_{t_0}^{t} f_1(x_s, s)dw_s, \quad t \in \mathbb{I}. \tag{2.14}$$

Here, the first integral is a stochastic Riemann-Stieltjes integral, and the second one is a stochastic integral (see, e.g. [68]). The presence of a stochastic integral requires the use of special rules of calculus. In this regard, two approaches are known: the Itô and the Stratonovich scheme. In the case of SDEs with additive noise, the representations in Itô and Stratonovich are equivalent. In other cases, swapping between Itô and Stratonovich interpretations is feasible by converting the SDEs through the following formula that relates the two definitions

$$\bar{f}_0(x_t) = f_0(x_t) - \frac{1}{2} f_1 \frac{\partial f_1}{\partial x_t}(x_t). \tag{2.15}$$

Both the Itô and Stratonovich interpretations have their own rules of calculus. Each of the two with their advantages and drawbacks, see

[68] for further details. In the present research work, we will adopt the Itô's interpretation.

With base on the equation (2.13), we present the general formulation of a d-dimensional nonlinear SDE with multiple noise sources as follows

$$d\boldsymbol{x}_t = \boldsymbol{f}_0(\boldsymbol{x}_t)dt + \sum_{j=1}^{m} \boldsymbol{f}_j(\boldsymbol{x}_t)dw_t^j, \quad t \in \mathbb{I}, \tag{2.16}$$

with the drift $\boldsymbol{f}_0 \in \mathcal{C}^k(\mathbb{D}_{\boldsymbol{x}}, \mathbb{R}^d)$ (for some $k \geq 1$), and the diffusions $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m \in \mathcal{C}^{k+1}(\mathbb{D}_{\boldsymbol{x}}, \mathbb{R}^d)$. Here, w_t^j (for $j = 1, \ldots, m$) form an m-dimensional *Wiener process* defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \geq t_0}$, where $\mathcal{F} = \mathcal{B}(\Omega)$ is the σ-algebra (or collection of events) of Borel sets in Ω. Each j-th Wiener process w_t is understood as a process such that $w_t(\omega) = \omega(t)$, where $\omega \in \Omega$, i.e., the elements of Ω are identified with the paths.

In this case, the relationship between Itô and Stratonovich approaches is denoted by

$$\bar{f}_0^i(\boldsymbol{x}_t) = f_0^i(\boldsymbol{x}_t) - \frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{m} f_{jk} \frac{\partial \boldsymbol{f}_k}{\partial \boldsymbol{x}_t}(\boldsymbol{x}_t), \qquad i = 1, \ldots, d, \tag{2.17}$$

The constrained nature of real-world noisy dynamical systems makes it necessary to include the algebraic equations that model the restrictions to the systems of SDEs. This new class of differential equations on manifolds is discussed below.

2.3. Stochastic Differential-Algebraic Equations

Consider a system of quasi-linear stochastic differential-algebraic equations (SDAEs) of the form

$$\boldsymbol{E}d\boldsymbol{x}_t = \boldsymbol{f}_0(\boldsymbol{x}_t)dt + \sum_{j=1}^{m} \boldsymbol{f}_j(\boldsymbol{x}_t)dw_t^j, \quad t \in \mathbb{I}, \tag{2.18}$$

with a singular matrix $\boldsymbol{E} \in \mathbb{R}^{n \times n}$ of rank $d < n$. The function $\boldsymbol{f}_0 \in \mathcal{C}^k(\mathbb{D}_{\boldsymbol{x}}, \mathbb{R}^n)$ (for some $k \geq 1$) is the drift, and $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m \in \mathcal{C}^{k+1}(\mathbb{D}_{\boldsymbol{x}}, \mathbb{R}^n)$ are the diffusions. Here $\mathbb{I} := [t_0, t_f] \subseteq \mathbb{R}^+$ is a closed time interval and $\mathbb{D}_{\boldsymbol{x}} \subseteq \mathbb{R}^n$ is an open set. Furthermore, w_t^j (for $j = 1, \ldots, m$) form an m-dimensional *Wiener process* defined on the complete probability space

$(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \geq t_0}$. Each j-th Wiener process w_t is a process such that $w_t(\omega) = \omega(t)$, where $\omega \in \Omega$.

The representation of equation (2.18) in its integral form is

$$\boldsymbol{E}\boldsymbol{x}_t = \boldsymbol{E}\boldsymbol{x}_{t_0} + \int_{t_0}^{t} \boldsymbol{f}_0(\boldsymbol{x}_s) ds + \sum_{j=1}^{m} \int_{t_0}^{t} \boldsymbol{f}_j(\boldsymbol{x}_s) dw_s^j, \quad t \in \mathbb{I}. \tag{2.19}$$

We assume consistent initial values $\boldsymbol{x}_{t_0} = \boldsymbol{x}_0$, independent of the Wiener processes w_t^j and with finite second moments [68]. A solution $\boldsymbol{x}_t = \boldsymbol{x}(t, \omega)$ of (2.19) is an n-dimensional vector-valued Markovian stochastic process depending on $t \in \mathbb{I}$ and $\omega \in \Omega$ (the parameter ω is commonly omitted in the notation of $\boldsymbol{x}$). Such a solution can be defined as *strong solution* if it fulfills the following conditions, see e.g. [23], [95].

- $\boldsymbol{x}(\cdot)$ is adapted to the filtration $(\mathcal{F}_t)_{t \geq t_0}$,
- $\int_{t_0}^{t_f} |f_0^{\ell}(x_s)| ds < \infty$ almost sure (a.s.), for all $\ell = 1, \ldots, n$,
- $\int_{t_0}^{t_f} |f_j^{\ell}(x_s)|^2 dw_s^j < \infty$ a.s., for all $j = 1, \ldots, m$, and $\ell = 1, \ldots, n$,
- (2.19) holds for every $t \in \mathbb{I}$ a.s.

Because of the presence of the algebraic equations associated with the kernel of $\boldsymbol{E}$, the solution components associated with these equations would be directly affected by white noise and not integrated. To avoid this, a reasonable restriction is to ensure that the noise sources do not appear in the algebraic constraints. According to [80], [95], this assumption can be accomplished in SDAE systems whose deterministic part

$$\boldsymbol{E}\dot{\boldsymbol{x}}_t = \boldsymbol{f}_0(\boldsymbol{x}_t), \quad t \in \mathbb{I}, \tag{2.20}$$

which is an autonomous quasi-linear DAE as (2.8), has tractability index-1 [55], [95], whose constraints are regularly and globally uniquely solvable for parts of the solution vector. Based on these ideas, the following definition is stated:

Definition 2.3.1. *([95]) The SDAE system (2.18) is called an index-1 SDAE if*

- *the noise sources do not appear in the constraints, and*

- *the constraints are globally uniquely solvable for the algebraic variables.*

In this work, we slightly modify the previous assumptions and consider SDAE systems whose deterministic part (2.20) is a regular *strangeness-free* DAE [52] i.e., it has differentiation index-1. A system with these characteristics can be transformed into a semi-explicit form employing an appropriate kinematic equivalence transformation [15], [57]. Then, there exists a unique pair of projector matrices $\boldsymbol{\mathcal{P}}$

$$\boldsymbol{\mathcal{P}}\boldsymbol{E}\boldsymbol{\mathcal{Q}} = \begin{bmatrix} \boldsymbol{I}_d & 0 \\ 0 & 0 \end{bmatrix}.$$

Pre-multiplying (2.18) by $\boldsymbol{\mathcal{P}}$, and changing the variables $\boldsymbol{x}_t$ according to the transformation $\boldsymbol{x}_t = \boldsymbol{\mathcal{Q}}\hat{\boldsymbol{x}}_t$, with $\hat{\boldsymbol{x}}_t = \left[\hat{\boldsymbol{x}}_t^D, \hat{\boldsymbol{x}}_t^A\right]$, we obtain the decoupled semi-explicit form

$$d\hat{\boldsymbol{x}}_t^D = \hat{\boldsymbol{f}}_0^D(\hat{\boldsymbol{x}}_t^D, \hat{\boldsymbol{x}}_t^A)dt + \sum_{j=1}^{m} \hat{\boldsymbol{f}}_j^D(\hat{\boldsymbol{x}}_t^D, \hat{\boldsymbol{x}}_t^A)dw_t^j, \tag{2.21a}$$

$$0 = \hat{\boldsymbol{f}}_0^A(\hat{\boldsymbol{x}}_t^D, \hat{\boldsymbol{x}}_t^A)dt + \sum_{j=1}^{m} \hat{\boldsymbol{f}}_j^A(\hat{\boldsymbol{x}}_t^D, \hat{\boldsymbol{x}}_t^A)dw_t^j, \tag{2.21b}$$

where $\hat{\boldsymbol{x}}_t^D$ and $\hat{\boldsymbol{x}}_t^A$ is a separation of the transformed state into differential and algebraic variables, respectively, that is performed in such a way that the Jacobian of the function $\hat{\boldsymbol{f}}_0^A$ with respect to the algebraic variables is nonsingular, see [52] for details of the construction. The condition that the noise sources do not appear in the constraints, implies that $\sum_{j=1}^{m} \hat{\boldsymbol{f}}_j^A \equiv 0$, so that the algebraic equations in (2.21b) can be solved as

$$\hat{\boldsymbol{x}}_t^A = \boldsymbol{F}^A(\hat{\boldsymbol{x}}_t^D),$$

and inserted in the dynamic equations (2.21a) yielding an ordinary SDE

$$d\hat{\boldsymbol{x}}_t^D = \hat{\boldsymbol{f}}_0^D(\hat{\boldsymbol{x}}_t^D, \boldsymbol{F}^A(\hat{\boldsymbol{x}}_t^D))dt + \sum_{j=1}^{m} \hat{\boldsymbol{f}}_j^D(\hat{\boldsymbol{x}}_t^D, \boldsymbol{F}^A(\hat{\boldsymbol{x}}_t^D))dw_t^j. \tag{2.22}$$

The resulting equation is termed as *underlying sotchastic differential equation* (uSDE) of the strangeness-free SDAE. It acts in the lower-dimensional subspace $\mathbb{R}^d$, with $d = n - a$ (where a denotes the number of algebraic equations). The SDE system (2.22) preserves the inherent dynamics of a strangeness-free SDAE system [55]. In this way, the algebraic equations can be removed from the system. However, whenever a numerical method is used for the numerical integration, then one has to make sure that the algebraic equations are solved adequately at each time step so that the back-transformation to the original state variables can be performed.

2.4. Numerical Integration Methods

It has been seen that a d-index-1 (or strangeness-free) SDAE can be reduced to its uSDE by using suitable transformations. This makes it feasible to solve an IVP SDAE (2.18) by integrating the IVP uSDE (2.22). Most of the SDEs are non-integrable analytically. So, it is required to use numerical methods to approximate the solutions numerically. In this regard, the numerical schemes used for the solution of ODEs, have a very low performance when implemented in SDEs because of their poor numerical convergence. There are discretization schemes specifically developed to integrate SDE systems numerically. In literature such as [45], [48], [61], [68] and references therein, one can find diverse implicit and explicit methods. Some of these schemes more convenient for the Itô and other ones for the Stratonovich formulation (e.g., Usually only the Itô calculus allows us to exploit powerful martingale results for numerical analysis.) [73].

The different numerical schemes are identified through their order of convergence, which is a measure of efficiency. This plays a crucial role in the design of numerical algorithms. Depending on the problem, there are mainly two types of convergence. These are distinguished on base whether it is required

- having approximations to the process trajectories themselves, or
- having approximations to the corresponding distributions.

These types of convergence are shortly identified as the *strong* and the *weak* convergence criterion, respectively.

For the proposes of this research work, the numerical integration of our SDE systems will focus on the use of *Taylor schemes*. So, let us consider the IVP SDE system of the form

$$d\boldsymbol{x}_t = \boldsymbol{f}_0(\boldsymbol{x}_t)dt + \sum_{j=1}^{m} \boldsymbol{f}_j(\boldsymbol{x}_t)dw_t^j, \qquad \boldsymbol{x}_{t_0} = \boldsymbol{x}_0, \tag{2.23}$$

to be integrated numerically on the defined time interval $[t_0, T]$. It is chosen a time discretization with the grid points

$$t_0 < t_1 < \ldots < t_{N-1} < t_N = T,$$

with a step-size $h = T/N$. Then, we discuss in the following sections some widely-used first-order numerical schemes for Itô-type SDE systems. Here, we restrict ourselves to the particular case known as *diagonal-noise case*, where $m = d$ and no correlations between the diffusion terms in the SDE system are assumed, see [48].

2.4.1. Euler-Maruyama Method

The Euler-Maruyama (EM) scheme is the simplest method used to solve SDEs numerically. The method requires the system to be formulated exclusively in Itô form (actually, its equivalent scheme for the Stratonovich form is the Euler-Heun method). The method is the analog of the Euler method for ODEs generalized to SDEs.

Considering the diagonal-noise case. The EM approximation for the kth equation (with $k = 1, \ldots, d$) of the Itô-type SDE system (2.23) is a continuous-time stochastic process that satisfies the iterative scheme

$$Y_{n+1}^k = Y_n^k + h f_0^k(Y_n) + f_k^k(Y_n)\Delta W_n^k, \qquad Y_0 = x_0^k, \tag{2.24}$$

for $n = 0, 1, \ldots, (N-1)$, where $Y_n = Y_{t_n}$, the step-size $h = T/N = t_{n+1} - t_n$, $\Delta W_n = [W_{t_{n+1}} - W_{t_n}] \sim \mathcal{N}(0, h)$ with $W_{t_0} = 0$. The random variables ΔW_n are independent and identically distributed normal random variables with expected value zero and variance h. The EM scheme has a strong order of convergence 0.5, and weak order of convergence 1. The method is considered simple and crude, somewhat inefficient, and inaccurate unless a small step-size h is used. Also, it often exhibits poor stability properties.

2.4.2. Milstein Method

If we consider one additional term from the stochastic Taylor expansion, a strong order of convergence 1.0 is obtained. This is known as Milstein scheme. In the same way than section 2.4.2, for a d-dimensional Itô-type SDE system (2.23), the Milstein scheme for the kth component (with $k = 1, \ldots, d$) is given by

$$\begin{aligned} Y_{n+1}^k &= Y_n^k + h f_0^k(Y_n) + f_k^k(Y_n)\Delta W_n^k \\ &\quad + \frac{1}{2} f_k^k(Y_n) \frac{\partial f_k^k}{\partial x_t^k}(Y_n) \left[(\Delta W_n^k)^2 - h \right], \quad Y_0 = x_0^k, \quad (2.25) \end{aligned}$$

for $n = 0, 1, \ldots, (N-1)$, where $Y_n = Y_{t_n}$, the step-size $h = T/N = t_{n+1} - t_n$, $\Delta W_n = [W_{t_{n+1}} - W_{t_n}] \sim \mathcal{N}(0, h)$ with $W_{t_0} = 0$. The Milstein scheme converges strongly with order 1 and weakly with order 1. Note that the additional term considered implies the calculation of the first derivative of f_k^k. The interested reader can review in [48] a derivative-free version of the Milstein scheme and its properties.

2.4.3. Runge-Kutta Method

Similarly to Euler scheme, the Runge-Kutta (RK) method for SDEs is a generalization of the Runge-Kutta method for ODEs to SDEs. For the same multi-dimensional Itô-type SDE system (once again in the diagonal-noise case) we have the most basic one-step RK scheme for the kth equation (with $k = 1, \ldots, d$) as

$$\begin{aligned} Y_{n+1}^k &= Y_n^k + h f_0^k(Y_n) + f_k^k(Y_n)\Delta W_n^k \\ &\quad + \frac{1}{2} \left[f_k^k(Z_n) - f_k^k(Y_n) \right] \left[(\Delta W_n)^2 - h \right] h^{-1/2}, \\ &\quad Y_0 = x_0^k, \qquad (2.26) \end{aligned}$$

for $n = 0, 1, \ldots, (N-1)$, where $Y_n = Y_{t_n}$, the step-size $h = T/N = t_{n+1} - t_n$, $\Delta W_n = [W_{t_{n+1}} - W_{t_n}] \sim \mathcal{N}(0, h)$ with $W_{t_0} = 0$. Here, the expression

$$Z_n = Y_n + f_0^k(Y_n)h + f_k^k(Y_n)h^{1/2}.$$

The RK scheme has strong and weak order of convergence, both equal to 1.

CHAPTER 3

Random Dynamical Systems and Lyapunov Stability

In Chapter 2, we have discussed the properties of an autonomous s-free SDAE, as well as its reduction to an uSDE, which preserves the dynamic characteristics of the original system. Using the back-transformation, the definitions and properties attributed to the uSDE, and its analysis can be extended to the original SDAE. For this reason, in the present Chapter we will talk directly about random dynamical systems generated by SDE systems and assume this dynamical system with noise also corresponds to the original SDAE within which the aforementioned SDE system underlies.

Next, along with the Chapter, we evoke the theory of stability in dynamical systems in Lyapunov's sense. Assuming the ergodicity of random dynamical systems and under some integrability conditions, we define the Lyapunov exponents in linear cocycles as a generalization of the Lyapunov exponents in the deterministic case. The theory discussed in this part is strongly inspired in the concepts of random dynamical systems and Lyapunov exponents developed by L. Arnold in [6], [7], and other remarkable contributions mainly made by his known as "Bremen group".

3.1. Random Dynamical Systems

A random dynamical system or (or simply named cocycle) is essentially, roughly speaking, a nonautonomous dynamical system consisting of two essential ingredients:

- a model of noise,
- a model of the system which is perturbed by the noise (cocycle).

This two-parameter stochastic flows can be generated by random and stochastic, ordinary and partial differential equations. The formal definition of a random dynamical system requires to start with the notion of a metric dynamical system

Definition 3.1.1 (Metric Dynamical System [7], [17], [82]). *A metric dynamical system (MDS) $\theta \equiv (\Omega, \mathcal{F}, \mathbb{P}, (\theta)_{t\in\mathbb{I}})$ with time interval $\mathbb{I}$, is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a flow on Ω defined as a family of transformations (mappings) $\theta_t : \Omega \to \Omega,\ t \in \mathbb{I}$ such that*

- *it is an one-parameter group, i.e.*

$$\theta_0 = \mathrm{id}_\Omega, \quad \theta_t \circ \theta_s = \theta_{t+s} \quad \forall\ t, s \in \mathbb{I},$$

  where d_θ is the identical map on Ω,
- *The mapping $(t, \omega) \mapsto \theta_t\omega$ is $(\mathcal{B}(\mathbb{I}) \times \mathcal{F}, \mathcal{F})$-measurable,*
- *$\theta_t\mathbb{P} = \mathbb{P} \quad \forall t \in \mathbb{I}$, i.e. $\mathbb{P}(\theta_t B) = \mathbb{P}(B)$, $\forall\ B \in \mathcal{F}$ and $\forall\ t \in \mathbb{I}$.*

A set $B \in \mathcal{F}$ is called θ-invariant (for short invariant) if $\theta_t B = B$ for all $t \in \mathbb{I}$. A MDS is called ergodic under $\mathbb{P}$ if for any invariant set $B \in \mathcal{F}$ we have either $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$. An ergodic MDS, denoted by $\theta \equiv (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ with the filtration $(\mathcal{F}_t)_{t\geq t_0}$, is also defined by the *Wiener shift*

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t), \quad t \in \mathbb{I}, \quad \forall\ \omega \in \Omega,$$

which means that a shift transformation given by θ is measure-preserving and ergodic [17]. Once a MDS has been defined, we are ready to state a random dynamical system's formal definition.

Definition 3.1.2 (Random Dynamical System [7], [17]). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = \mathbb{R}^d$. A measurable random dynamical system $\Theta = (\theta, \varphi)$ (henceforth abbreviated as RDS) on the measurable* *space X over (or covering, or extending) an MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{I}})$ is a mapping*

$$\boldsymbol{\varphi} : \mathbb{I} \times \Omega \times X \to X, \quad (t, \omega, \boldsymbol{x}) \mapsto \varphi(t, \omega, X),$$

with the following properties:

- *Measurability:* $(t, \omega, \boldsymbol{x}) \mapsto \boldsymbol{\varphi}(t, \omega, \boldsymbol{x})$ *is measurable,*
- *Cocycle property: The mappings* $\boldsymbol{\varphi}(t, \omega) := \boldsymbol{\varphi}(t, \omega, \cdot) : X \to X$ *form a cocycle over* $\theta(\cdot)$*, i.e., they satisfy*

$$\boldsymbol{\varphi}(0, \omega) = \mathrm{id}_X \quad \forall\, \omega \in \Omega \quad (\mathit{if} \quad 0 \in \mathbb{I}),$$

$$\boldsymbol{\varphi}(t + s, \omega) = \boldsymbol{\varphi}(t, \theta_s\omega) \circ \boldsymbol{\varphi}(s, \omega) \quad \forall\, s, t \in \mathbb{I}, \ \mathit{and}\ \omega \in \Omega,$$

where id_X *is the identical map on* X*.*

- *continuity:* $\boldsymbol{x} \mapsto \boldsymbol{\varphi}(t, \omega, \boldsymbol{x})$ *is continuous for all* $(t, \omega) \in \mathbb{I} \times \Omega$*.*

Here "$\circ$" means composition, which canonically defines an action on the left of the semigroup of self-mappings of $\mathbb{R}^d$ on the space $\mathbb{R}^d$, i.e. $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$.

Figure 3.1 helps us to imagine an RDS as fiber maps on the (trivial) bundle $\Omega \times X$. The figure can be explained as follows: according as ω is shifted by the dynamical system θ in time s towards $\theta_s\omega$ on the base space Ω, the cocycle $\varphi(s, \omega)$ moves the point x in the fiber $\{\omega\} \times X$ over ω to $\varphi(s, \omega)x$ in the fiber $\{\theta_s\omega\} \times X$ over the point $\theta_s\omega$. The cocycle property can be clearly displayed on this bundle.

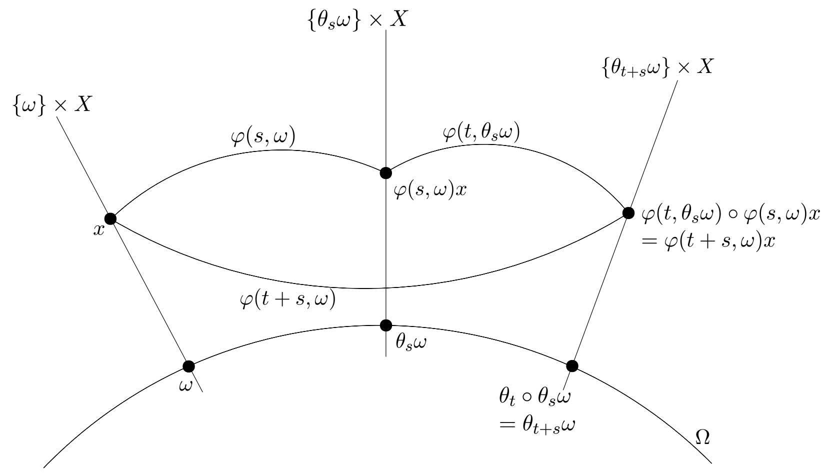


Figure 3.1: A RDS as an action on the bundle $\Omega \times X$

3.1.1. RDSs generated by SDEs

Now, let us take back the Itô SDE (2.22) that was already stated in the previous Chapter.

$$d\boldsymbol{x}_t = \boldsymbol{f}_0(\boldsymbol{x}_t)dt + \sum_{j=1}^{m} \boldsymbol{f}_j(\boldsymbol{x}_t)dw_t^j = \sum_{j=0}^{m} \boldsymbol{f}_j(\boldsymbol{x}_t)dw_t^j, \quad t \in \mathbb{I}. \tag{3.1}$$

For simplicity, we have modified the Itô SDE, with the drift and diffusion terms combined into one term, and with the convention $dw_t^0 \equiv dt$ to obtain this compacted version. Additionally, we assume that the differential operator $L := \boldsymbol{f}_0 + \frac{1}{2}\sum_{j=1}^{m}(\boldsymbol{f}_j)^2$ is strong hypoelliptic in the sense that the Lie algebra $\boldsymbol{\mathcal{G}}(\boldsymbol{f}_0, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_m)$ generated by the vector fields $\boldsymbol{f}_j$ (with $j = 0, \ldots, m$) has dimension d for all $\boldsymbol{x}_t \in \mathbb{R}^d$ [7]. Once again, w_t^j (for $j = 0, \ldots, m$) is a m-dimensional Wiener process.

For a given initial value $\boldsymbol{x}_{t_0} = \boldsymbol{x}_0$, the solution process generates a *Markovian stochastic process*, and the SDE (3.1) generates a RDS $\Theta = (\theta, \boldsymbol{\varphi})$. The details about how a $\mathcal{C}^k$ RDS is generated from an Itô SDE as (3.1) are presented in the following Theorem.

Theorem 3.1.3 (RDS from an Itô SDE)**.** *Let $\boldsymbol{f}_0 \in \mathcal{C}_b^{k,\delta}$, $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m \in \mathcal{C}_b^{k+1,\delta}$ and $\sum_{j=1}^{m}\sum_{i=1}^{d} f_j^i \frac{\partial}{\partial x_i} f_j \in \mathcal{C}_b^{k,\delta}$ for some $k \geq 1$ and $\delta > 0$. Here $\mathcal{C}_b^{k,\delta}$ is the Banach space of $\mathcal{C}^k$ vector fields on $\mathbb{R}^d$ with linear growth and bounded derivatives up to order k and the k-th derivative is δ-Hölder continuous. Then:*

1. *The quasi-linear Itô SDE*

$$d\boldsymbol{x}_t = \sum_{j=0}^{m} \boldsymbol{f}_j(\boldsymbol{x}_t)dw_t^j, \quad t \in \mathbb{I}, \tag{3.2}$$

   with $dw_t^0 \equiv dt$, generates a unique (up to indistinguishability) $\mathcal{C}^k$ RDS $\boldsymbol{\varphi}$ over the filtered dynamical system describing Brownian motion. For any $\epsilon \in (0, \delta)$, $\boldsymbol{\varphi}$ is a $\mathcal{C}^{k,\epsilon}$-semimartingale cocycle and $(t, \boldsymbol{x}) \mapsto \boldsymbol{\varphi}(t, \omega)\boldsymbol{x}$ belongs to $\mathcal{C}^{0,\beta;k,\epsilon}$ for all $\beta < \frac{1}{2}$ and $\epsilon < \delta$.
2. *The RDS $\boldsymbol{\varphi}$ has stationary independent (multiplicative) increments, i.e., for all $t_0 \leq t_1 \leq \cdots \leq t_n$ the random variables*

$$\boldsymbol{\varphi}(t_1) \circ \boldsymbol{\varphi}(t_0)^{-1},\ \boldsymbol{\varphi}(t_2) \circ \boldsymbol{\varphi}(t_1)^{-1},\ \ldots,\ \boldsymbol{\varphi}(t_n) \circ \boldsymbol{\varphi}(t_{n-1})^{-1}$$

   (here, "$\circ$" means composition) are independent, and the law of $\boldsymbol{\varphi}(t+h) \circ \boldsymbol{\varphi}(t)^{-1}$ is independent of t (homogeneous Brownian motion in the group $\mathrm{Diff}^k(\mathbb{R}^d)$).

3. *If $\boldsymbol{\Phi}(t, \omega, \boldsymbol{x})$ denotes the Jacobian of $\boldsymbol{\varphi}(t, \omega)$ at $\boldsymbol{x}_t$, then $(\boldsymbol{\varphi}, \boldsymbol{\Phi})$ is a $\mathcal{C}^{k-1}$ RDS uniquely generated by (3.2) together with the variational equation*

$$d\boldsymbol{v}_t = \sum_{j=0}^{m} \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{v}_t \; dw_t^j, \quad t \in \mathbb{I}, \tag{3.3}$$

where $\boldsymbol{J}_j(\boldsymbol{x}_t) := \left(\frac{\partial \boldsymbol{f}_j}{\partial \boldsymbol{x}}\right)\Big|_{\boldsymbol{x}(t;t_0)}$ is obtained after linearizing (3.2) along a solution. Hence $\boldsymbol{\Phi}$ uniquely solves (3.3) on $\mathbb{I}$,

$$\boldsymbol{\Phi}(t, \omega, \boldsymbol{x}) = \boldsymbol{I}_d + \sum_{j=0}^{m} \int_0^t \boldsymbol{J}_j(\boldsymbol{\varphi}(s)\boldsymbol{x})\boldsymbol{\Phi}(s, \boldsymbol{x}) dw_s^j, \quad t \in \mathbb{I}, \tag{3.4}$$

where $\boldsymbol{I}_d$ denotes the identity matrix of size d. Therefore, $\boldsymbol{\Phi}$ is a matrix cocycle over $\Theta = (\theta, \boldsymbol{\varphi})$.

4. *The determinant $\det \boldsymbol{\Phi}(t, \omega, \boldsymbol{x})$ satisfies Liouville's equation on $\mathbb{I}$,*

$$\det \boldsymbol{\Phi} = \exp\left(\sum_{j=0}^{m} \int_0^t \mathrm{tr}(\boldsymbol{J_j}(\boldsymbol{\varphi}(s)\boldsymbol{x}) dw_s^j\right), \quad t \in \mathbb{I}, \tag{3.5}$$

being then a scalar cocycle over Θ.

See Theorems 2.3.32 and 2.3.39-40 in [7] for the background theory and proof of this Theorem.

As it was pointed out previously, a RDS Θ can be generated by an IVP SDE system, but also by pathwise ODE that contain a real noise stochastic process, i.e. with a wider range of probability distributions, in their vector field functions. The so-called random differential equations (RDEs). An autonomous RDE has the form

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{\eta}_t(\omega)), \qquad \omega \in \Omega, \quad \text{and} \quad t \in \mathbb{I}, \tag{3.6}$$

where $\boldsymbol{\eta}_t \in \mathcal{C}^1(\mathbb{I} \times \Omega, \mathbb{R}^m)$ valued stochastic process with continuous sample paths defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \geq t_0}$, and $\boldsymbol{f} \in \mathcal{C}^k(\mathbb{D}_{\boldsymbol{x}} \times \mathbb{D}_{\boldsymbol{\eta}}, \mathbb{R}^d)$ are the vector field (or right-hand side) functions. If we consider an IVP, the RDE (3.6) comes accompanied by an initial condition, which could be considered to be affected by randomness as well, $\boldsymbol{x}_{t_0}(\omega) = \boldsymbol{x}_0(\omega)$, whose solutions are able to generate a cocycle $\Theta = (\theta, \boldsymbol{\varphi})$. Moreover, in cases where a model based on SDEs is inappropriate because of the special structure on the noisy disturbance, the SDE system could be transformed into a system of RDEs by suitable change of variables, see [17]. A detailed research about this type of RDS generator is beyond of our scope of study. The interested reader can find further information about RDEs in [7], [17], [42], [83].

3.2. Stability of Dynamical Systems

A crucial issue in the theory of dynamical systems, in general, is what in physics is described as the ability of such systems, given an initial condition, to regain a state of equilibrium after perturbations in the system states. The mentioned property of the system is the very known concept of *stability*. The rigorous mathematical theory of stability appeared, and has been developed and formalized in the last centuries, overall in studying mechanical motions (definitions given by Lagrange, Dirichlet, Poisson, Laplace, Poincaré, and others). Nevertheless, the most complete and mathematically rigorous contribution to the stability analysis was proposed by the Russian mathematician A. M. Lyapunov, whose results were published for the very first time in his remarkable Ph.D. thesis entitled *The general problem of the stability of motion* at Moscow University in 1892. The definitions of stability and the methods proposed for its assessment are nowadays a powerful framework for the stability analysis and control of nonlinear dynamical systems.

3.2.1. Lyapunov Stability Theory

In this Section, we present a survey of the classical concepts of stability in Lyapunov's sense, initially developed for ODEs. In that context, let us consider a dynamical system generated by the following general nonlinear ODE system

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, t), \qquad t \in \mathbb{I}, \tag{3.7}$$

with the initial condition $\boldsymbol{x}_{t_0} = \boldsymbol{x}_0$. Here $\boldsymbol{x}_t \in \mathbb{R}^d$ are the state (unknown) variables, and $\boldsymbol{f} \in \mathcal{C}(\mathbb{I} \times \mathbb{D}_{\boldsymbol{x}}, \mathbb{R}^d)$ are functions or vector fields In the following definition, we present different types of stability of equilibrium points of the system (3.7). An equilibrium point $\boldsymbol{x}^*$ (also known as steady state) is a constant particular solution of (3.7) that satisfies $\boldsymbol{f}(\boldsymbol{x}^*) = 0$, (for further details see [17], [41], [53]).

Definition 3.2.1 (Stability in the sense of Lyapunov)**.** *An equilibrium point $\boldsymbol{x}^*$ of the system (3.7) is said to be:*

1. *"Lyapunov stable", or simply "stable", if for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, if $\|\boldsymbol{x}_0 - \boldsymbol{x}^*\| < \delta(\varepsilon)$, then for every $t \geq 0$ we have $\|\boldsymbol{x}_t - \boldsymbol{x}^*\| < \varepsilon$.*

2. *"Asymptotically stable", if it is Lyapunov stable and there exists* $\delta(\varepsilon) > 0$ *such that if* $\|\boldsymbol{x}_0 - \boldsymbol{x}^*\| < \delta$*, then* $\lim_{t\to\infty} \|\boldsymbol{x}_t - \boldsymbol{x}^*\| = 0$.
3. *"Exponentially stable", if it is asymptotically stable and there exist* $\alpha > 0,\ \beta > 0,\ \delta(\varepsilon) > 0$ *such that if* $\|\boldsymbol{x}_0 - \boldsymbol{x}^*\| < \delta$*, then* $\|\boldsymbol{x}_t - \boldsymbol{x}^*\| \leq \alpha\|\boldsymbol{x}_0 - \boldsymbol{x}^*\|e^{-\beta t}$*, for all* $t \geq 0$.
4. *"Unstable" if is not Lyapunov stable.*

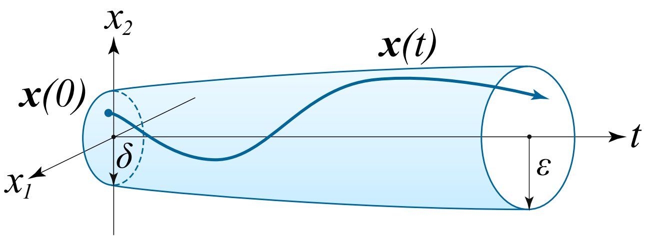
Figures 3.2a, 3.2b, and 3.2c describe graphically the notions of Lyapunov stability, asymptotic stability, and exponential stability respectively; for an equilibrium point. It is clear the existence of a hierarchy in these definitions. Exponential stability implies asymptotic stability, asymptotic stability implies stability, and the absence of at least Lyapunov stability implies instability.

Although the stability concepts in the sense of Lyapunov stated in Definition 3.2.1 have been proposed initially for ODEs, since their first appearance, such definitions have been extended to dynamical systems generated by other kinds of realizations as DAEs, SDEs, SDAEs, RDEs, and others (e.g., see [53] for the DAE case).

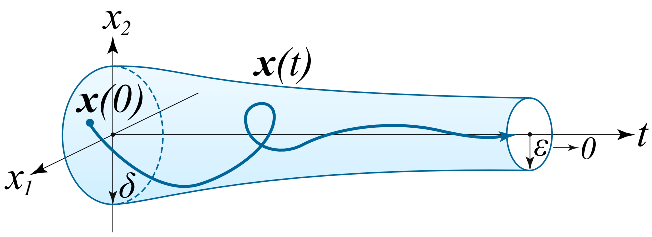
For autonomous systems, all the stability concepts in Definition 3.2.1 are uniform in time. This means that the choice of δ does not depend on t. Nevertheless, this does not hold true for nonautonomous systems. So, new definitions are required to distinguish uniform and non-uniform type of stability [17].

3.2.2. Lyapunov Stability Assessment

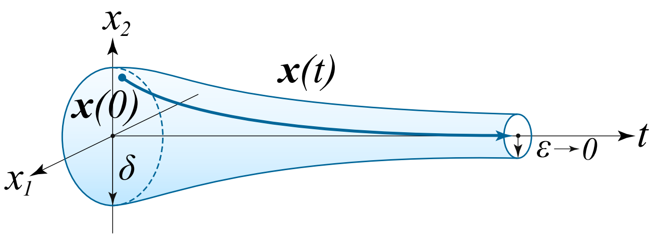
In addition to the definitions of stability, Lyapunov proposed two methods to demostrate stability of a given solution. The first method was developed with base on a "standard" perturbative analysis due to the need to characterise property the perturbation dynamics. The result of this analysis conducted to a called *"characteristic number"* later termed as *"Lyapunov characteristic exponent"*, or simply *"Lyapunov exponent"*. The second method, the most known, deals with introducing a pseudo-energy function that vanishes in the equilibrium point and is otherwise positive, and decreases (or does not increase) along a generic trajectory. Both methods are described in the following sections.



(a) Lyapunov stability.



(b) Asymptotic stability.



(c) Exponential stability.

Figure 3.2: Graphical description of different types of stability of an equilibrium point according to Definition 3.2.1 (Source: [86], own work).

3.2.2.1. Lyapunov First Method and LEs

The identified as *first method* in the Lyapunov's thesis, it consisted on the study of the stability of the solutions writing the equation as a perturbation of a linear system. As result of that analysis, Lyapunov defined the characteristic number λ_L. Assuming that $\delta(t) \approx e^{\lambda t}$ for a given perturbation $\delta(t)$, Lyapunov defined λ_L as the value such that $\delta(t)e^{\lambda_L t}$ neither diverges nor converges exponentially. This characteristic number is basically the opposite of the today number known as Lyapunov exponent. Nevertheless, all the stability theory behind that motivated the definition of λ_L, holds unchanged in the later developments that took place in connection with this method, specifically the definitions of Lyapunov exponent and regularity.

Our analysis of the concept of Lyapunov exponents starts by recalling the general system (3.7). A linearization of this set along the solutions, known as the variational system of equations or just *variational equations*, can be formulated as

$$\dot{\boldsymbol{v}}_t = \boldsymbol{J}(\boldsymbol{x}_t, t)\boldsymbol{v}_t, \qquad \boldsymbol{v}_{t_0} = \boldsymbol{v}_0, \qquad t \in \mathbb{I}, \tag{3.8}$$

with $\boldsymbol{J}_j(\boldsymbol{x}_t, t) := \left(\frac{\partial \boldsymbol{f}_j}{\partial \boldsymbol{x}}\right)\Big|_{\boldsymbol{x}(t;t_0)}$ is the Jacobian of the vector field $\boldsymbol{f}(\boldsymbol{x}_t, t)$. Here, $\boldsymbol{v}_t \in \mathbb{R}^d$, and $\boldsymbol{J}(\boldsymbol{x}_t, t) \in (\mathbb{I}, \mathbb{R}^{d\times d})$ is a matrix depending continuously on $t \in \mathbb{I}$. Boundedness of $\boldsymbol{J}(\boldsymbol{x}_t, t)$ is assumed, i.e.

$$\sup\{\|\boldsymbol{J}(\boldsymbol{x}_t, t)\| : t \in \mathbb{I}\} < \infty.$$

If we integrate the equation (3.8), we obtain

$$\boldsymbol{v}_t = \boldsymbol{H}(\boldsymbol{x}_0, t)\boldsymbol{v}_0, \tag{3.9}$$

where

$$\boldsymbol{H}(\boldsymbol{x}_0, t) = \exp\left(\int_0^t \boldsymbol{J}(\boldsymbol{x}_s, s)ds\right)$$

depends on the trajectory $\boldsymbol{x}(t; t_0)$ at all intermediate times. In practice, the matrix $\boldsymbol{H}(\boldsymbol{x}_0, t)$ is obtained by solving the ODE system (3.8).

Intending to determine the stability of the solution of (3.8), we introduce the characteristic numbers $\lambda : \mathbb{R}^d \to \mathbb{R} \cup \{0\}$ such that $\lambda_1 < \ldots < \lambda_s$ (i.e., λ attains only finitely distinct values) on $\mathbb{R}\backslash\{0\}$ where $p \leq d$. Each number λ_i (with $1 \leq i \leq d$) occurs with some multiplicity d_i so that $\sum_{i=1}^{p} d_i = d$. There numbers are identified as *Lyapunov exponents* (LEs) of (3.8) and are given by the formula

$$\lambda(\boldsymbol{v}) = \lim_{t\to\infty} \frac{1}{t} \ln\|\boldsymbol{v}_t\| \tag{3.10}$$

for each $\boldsymbol{v} \in \mathbb{R}^d$, where is the unique solution of (3.8) that satisfies the initial condition $\boldsymbol{v}_0 = \boldsymbol{v}$. Assuming properties like (Lyapunov-Perron) regularity and smoothness, the existence and well-posedness of the LEs are guaranteed. The usefulness of the LEs as measurer of dynamical systems' stability lies in the fact that when all λ's of a linear dynamics are negative, all solutions are asymptotically stable.

As will be seen in Section 3.3, the theory of LEs is developed substantially further in the framework of the ergodic theory given by V. Oseledets and his *"multiplicative ergodic theorem"* (MET), see [69]. Even the result of ergodic theory defines the real symmetric matrix

$$\boldsymbol{M}(\boldsymbol{x}_0, t) = \boldsymbol{H}(\boldsymbol{x}_0, t)^T \boldsymbol{H}(\boldsymbol{x}_0, t),$$

whose limit

$$\boldsymbol{\Lambda}(\boldsymbol{x}_0, t) = \lim_{t \to \infty} \frac{1}{2t} \log(\boldsymbol{M}(\boldsymbol{x}_0, t)),$$

defines a matrix $\boldsymbol{\Lambda}$, where the LEs λ_i (with $1 \leq i \leq d$) are defined by the eigenvalues of $\boldsymbol{\Lambda}$. In recent years, modem exposition of LEs context of the MET has allowed the development of numerical studies and applications initially for the assessment of chaotic systems, nowadays extended to the test of dynamical systems in general.

3.2.2.2. Lyapunov's Second Method

Also known as *Lyapunov's direct method.* Although it was the second method proposed by Lyapunov, from the beginning, it became more important than the first method. The Lyapunov's direct method assesses the stability of a solution by constructing a called *Lyapunov fuction*, which vanishes in the equilibrium point and is otherwise positive, and decreases (or does not increase) along a generic trajectory. Lyapunov's direct method gives sufficient conditions for Lyapunov, asymptotic, and exponential stability of a nonlinear dynamical system. The spirit of the method comes from the following Theorem.

Theorem 3.2.2 (Lyapunov's Theorem [41])**.** *Consider the nonlinear dynamical system (3.7) and assume that there exists a continuously differentiable function $\mathcal{V} \in \mathcal{C}^1(\mathbb{D}_{\boldsymbol{x}}, \mathbb{R})$ such that*

$$\mathcal{V}(0) = 0, \tag{3.11}$$

$$\mathcal{V}(\boldsymbol{x}) > 0, \qquad \boldsymbol{x} \neq 0, \tag{3.12}$$

$$\dot{\mathcal{V}}(\boldsymbol{x}) f(\boldsymbol{x}, t) \leq 0, \tag{3.13}$$

with $\dot{\mathcal{V}} := \frac{d\mathcal{V}}{dt}$. Then the zero solution $\boldsymbol{x}(t) \equiv 0$ to (3.7) is Lyapunov stable. If, in addition,

$$\dot{\mathcal{V}}(\boldsymbol{x}) f(\boldsymbol{x}) < 0, \qquad \boldsymbol{x} \neq 0, \tag{3.14}$$

then the zero solution $\boldsymbol{x}_t \equiv 0$ *to (3.7) is asymptotically stable. Finally, if there exist scalars* α, β, $\varepsilon > 0$, *and* $p \geq 1$, *such that* $\mathcal{V} \in \mathcal{C}^1(\mathbb{D}_{\boldsymbol{x}}, \mathbb{R})$ *satisfies*

$$\alpha \|\boldsymbol{x}_t\|^{\,p} \leq \mathcal{V}(\boldsymbol{x}) \leq \beta \|\boldsymbol{x}_t\|^{\,p}, \tag{3.15}$$

$$\dot{\mathcal{V}}(\boldsymbol{x}) f(\boldsymbol{x}) \leq \varepsilon \mathcal{V}(\boldsymbol{x}), \tag{3.16}$$

then the zero solution $\boldsymbol{x}_t \equiv 0$ *to (3.7) is exponentially stable.*

A key idea in stability theory is that the qualitative behavior of the smooth dynamical system given by the n-dimentional ODE (3.7) at an equilibrium point $\boldsymbol{x}^*$, can be analyzed by using the linearization of the system at that equilibrium point. Then, the linearized system of (3.7) has the form

$$\dot{\boldsymbol{v}}_t = \boldsymbol{A}\boldsymbol{v}_t, \qquad \boldsymbol{v}_{t_0} = \boldsymbol{v}_0, \qquad t \in \mathbb{I}, \tag{3.17}$$

with $\boldsymbol{A}_j(\boldsymbol{x}_t) := \left(\frac{\partial \boldsymbol{f}_j}{\partial \boldsymbol{x}}\right)\Big|_{\boldsymbol{x}^*}$. Where $\boldsymbol{v}_t \in \mathbb{R}^n$ are de unknown variables on the linearized system, and the $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ in the Jacobian matrix of the system at the equilibrium point $\boldsymbol{x}^*$ (this is also denoted as *state matrix* overall in the control area). Here, the *eigenvalues* $\boldsymbol{\lambda} \in \mathbb{C}^n$ of $\boldsymbol{A}$ characterize the behavior of the nearby points. Specifically, if $\mathfrak{Re}(\boldsymbol{\lambda}) < 0$ then the point is a stable attracting fixed point, and the nearby points converge to it asymptotically. This condition is equivalent to

$$\boldsymbol{A}^T\boldsymbol{M} + \boldsymbol{M}\boldsymbol{A} \preceq 0 \quad \text{for some} \quad \boldsymbol{M} = \boldsymbol{M}^T \succeq 0,$$

for the relevant Lyapunov function $\mathcal{V}(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}$. If there exists an eigenvalue λ of A with $\mathfrak{Re}(\lambda) > 0$ then the system is unstable at that point.

3.3. LEs of Ergodic RDSs

Next to the survey given above regarding to stability theory in the Lyapunov's framework, we will continue with the theory that supports the existence and well-possedness of LEs in RDSs generated by SDEs. In this regard, we will start by citing an important result of the theory of RDSs known as *"multiplicative ergodic theorem"*, developed by V. Odelets in [69] with previous work on muliplication of random matrices by Furstenberg and Kesten in [34]. The MET provides us with suitable spectral objects, such as invariant subspaces (Lyapunov subspaces) and exponential growth rates (LEs), that permit the lift to nonlinear RDS and hence a local theory for nonlinear RDS. In this manner, we have with the MET the right substitute for deterministic linear algebra.

The MET concept permits a suitable definition of LEs for linear cocycles over an ergodic MDS. First, let us recall the SDE system and the RDSs defined for the Theorem 3.1.3. As duscussed before, both SDE as well as RDE systems can define ergodic RDSs. The version of the MET described below compiles concepts from [7], [19]. In line with the Oseledets' theory, it proposes the decomposition of $\mathbb{R}^d$ into random subspaces (called Lyapunov vectors), with equal exponential growth rate (Lyapunov exponents) with probability one.

Theorem 3.3.1 (Multiplicative Ergodic Theorem)**.** *Let $\boldsymbol{\Phi}$ be a linear cocycle over the RDS $\Theta = (\theta, \boldsymbol{\varphi})$. Assuming the integrability condition*

$$\log^+\|\boldsymbol{\Phi}(t, \omega, \boldsymbol{x})\| \in \mathcal{L}^1,$$

where $\log^+\|\boldsymbol{\Phi}\| = \max\{\log(\boldsymbol{\Phi}), 0\}$ denotes the positive part of $\log$, is satisfied. Here $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ is the Banach space of all (equivalence classes of) functions $f : \Omega \to \mathbb{R}$, which are measurable with respect to the σ-algebra $\mathcal{F}$ and the Borel σ-algebra on $\mathbb{R}$, and $\mathbb{P}$-integrable. Ergodicity of the probability measure $\mathbb{P}$ is also assumed. Additionally, let μ be an ergodic invariant measure with respect to the cocycle $\boldsymbol{\varphi}$ [7]. Then, there exists an invariant set $\hat{\Omega} \subset \Omega$ of full μ-measure, such that for each $\omega \in \hat{\Omega}$ the following statements hold:

1. *There exists a measurable decomposition*

$$\mathbb{R}^d = L_1(\omega) \oplus \cdots \oplus L_p(\omega),$$

   of $\mathbb{R}^d$ into random linear subspaces $L_i(\omega)$, which are invariant under Θ. Here $p \leq d$, where $d_i \in \mathbb{N}$ denotes the dimension of the subspace $L_i(\omega)$ (with $1 \leq i \leq p$), and $\sum_{i=1}^{p} d_i = d$. This splitting is characterized by the following properties:
2. *Lyapunov exponents $\lambda(\omega, \boldsymbol{x})$ quantify the exponential growth rate of the subspaces $L_i(\omega)$.*
3. *There are real numbers $\lambda_1 > \ldots > \lambda_p$, such that for each $\boldsymbol{x} \in \mathbb{R}^d \backslash \{0\}$ the LEs $\lambda(\omega, \boldsymbol{x}) \in \{\lambda_1 > \ldots > \lambda_p\}$ exists as a limit with*

$$\lambda(\omega, \boldsymbol{x}) = \lim_{t\to\infty} \frac{1}{t} \log\|\boldsymbol{\Phi}(t, \omega, \boldsymbol{x})\| = \lambda_i \quad \Longleftrightarrow \quad \boldsymbol{x} \in L_i(\omega) \setminus \{0\}.$$

4. *The maps $L_i :\longrightarrow \mathbb{G}_{d_i}$, with the Borel σ-algebra on the Grassmannian $\mathbb{G}_{d_i}$, are measurable (with $1 \leq i \leq p$).*

5. *The matrix* $\mathbf{\Lambda}(\omega)$ *defined by the limit*

$$\mathbf{\Lambda}(\omega) := \lim_{t\to\infty} \left(\boldsymbol{\Phi}(t,\omega)^T \boldsymbol{\Phi}(t,\omega)\right)^{\frac{1}{2t}}$$

exists and is a positive definite random matrix. The different eigenvalues of $\mathbf{\Lambda}(\omega)$ *are constants and can be written as* $e^{\lambda_1} > \ldots > e^{\lambda_p}$. *The corresponding random eigenspaces are given by* $L_1(\omega), \ldots, L_p(\omega)$. *Furthermore, the LEs are obtained as limits from the singular values* δ_j *of* $\boldsymbol{\Phi}(t,\omega)$. *The set of indices* $1, \ldots, d$ *can be decomposed into subsets* Σ_i, *with* $j = 1, \ldots, p$, *such that for all* $j \in \Sigma_i$,

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log \delta_j(\boldsymbol{\Phi}(t,\omega))$$

A proof of this theorem is given in [7]. It is need to remark that if the underlying MDS is not ergodic, the MET still holds, but in a weaker form.

According to [6, pag. 118], the LEs λ_i are independent of $(\omega, \boldsymbol{x})$ and thus they are universal constants of the cocycle generated by (3.3) under the ergodic invariant probability measure μ. Finally, if the following identity holds for $\boldsymbol{\Phi}$

$$\sum_{i=1}^{d} \lambda_i = \lim_{t\to\infty} \frac{1}{t} \log |\det \boldsymbol{\Phi}(t,\omega,\boldsymbol{x})|, \tag{3.18}$$

the system is said to be *Lyapunov regular* (Lyapunov regularity condition) [18], [71]. In practice, it is hard (if not impossible) to verify Lyapunov regularity for a particular system [7]. One of the key statements of the MET is that linear RDS (whether these are constant, periodic, quasi-periodic, or almost-periodic) are a.s. Lyapunov regular.

The concept of LEs plays an important role in the asymptotic stability assessment of dynamical systems subjected to stochastic disturbances. Under appropriate regularity assumptions, the negativity of all LEs of the system of variational equations implies the exponential asymptotic stability of both the linear SDE and the original nonlinear SDE system. Furthermore, the asymptotic behavior analysis of the stochastic models may be done based on every realization of the system rather than statistically in the mean or mean square sense.

CHAPTER 4

Numerical Methods for computing LEs

In this Chapter, we derive the numerical techniques to compute the finite-time approximation of the LEs. Inspired by [18], our work proposes an adaptation of the ideas from the purely deterministic case [26], [28], [29] to noise-driven dynamical systems. The methods take advantage on the existence of a Lyapunov transformation of the linear RDS to an upper-triangular structure and the feasibility to retrieve a numerical approximation of the LEs from that form. The transformation is performed through an orthogonal change of variables. The approach is made under the assumption of Lyapunov regularity of the system. In order to explain the methods, let us consider the SDE again as an initial value problem of the form

$$d\boldsymbol{x}_t = \sum_{j=0}^{m} \boldsymbol{f}_j(\boldsymbol{x}_t) dw_t^j, \quad t \in \mathbb{I}, \quad \boldsymbol{x}_{t_0} = \boldsymbol{x}_{t_0}, \tag{4.1}$$

where $\boldsymbol{f}_j$ are sufficiently smooth functions. The corresponding variational equation of (4.1) along with the solutions $\boldsymbol{x}_t(\boldsymbol{x}_0)$, turned into a matrix initial value problem, is given by

$$d\boldsymbol{V}_t = \sum_{j=0}^{m} \boldsymbol{J}_j(\boldsymbol{x}_t) \boldsymbol{V}_t dw_t^j, \qquad \boldsymbol{V}_0 = \boldsymbol{I}_d, \tag{4.2}$$

with the identity matrix $\boldsymbol{I}_d \in \mathbb{R}^{d\times d}$ as initial value, where $\boldsymbol{J}_j(\boldsymbol{x}_t) := \frac{\partial \boldsymbol{f}_j}{\partial \boldsymbol{x}}$ are the Jacobians of the vector functions $\boldsymbol{f}_j(\boldsymbol{x}_t)$, and $\boldsymbol{V} \in \mathcal{C}^1(\mathbb{I} \times \mathbb{R}^{d\times d})$ is the fundamental solution matrix, whose columns are linearly independent solutions of the variational equation. A key theoretical tool for determining the LEs is the computation of the continuous QR factorization of $\boldsymbol{V}_t$,

$$\boldsymbol{V}_t = \boldsymbol{Q}_t \boldsymbol{R}_t,$$

where $\boldsymbol{Q}_t$ is orthogonal, i.e., $\boldsymbol{Q}_t^T \boldsymbol{Q}_t = \boldsymbol{I}_d$, and $\boldsymbol{R}_t$ is upper triangular with positive diagonal elements R_t^{ii} for $i = 1, \ldots, d$. Applying the MET theory presented in Subsection 3.3, and taking into account the norm-preserving property of the orthogonal matrix function $\boldsymbol{Q}_t$, we have

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log \|\boldsymbol{V}_t \boldsymbol{p}_i\| = \lim_{t\to\infty} \frac{1}{t} \log \|\boldsymbol{R}_t \boldsymbol{p}_i\|, \tag{4.3}$$

where $\{\boldsymbol{p}_i\}$ is an orthonormal basis associated with the splitting of $\mathbb{R}^d$. Lyapunov regular systems preserve their regularity under kinematic similarity transformations. Then, considering the regularity condition (3.18), the Liouville equation (3.5), and performing some algebraic manipulations (see details in [18, pag. 150]), the LEs are given by

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log |R_t^{ii}| \quad \text{a.s.,} \quad \text{for} \quad i = 1, \ldots, d. \tag{4.4}$$

The QR-based methods require to perform the QR decomposition of $\boldsymbol{V}_t$ for a long enough time, so that the R_t^{ii} have started to converge. Depending on whether the decomposition is performed after or before integrating numerically the variational equation, the method is called *discrete or continuous QR method.*

4.1. Discrete QR Method

The discrete QR method is a popular method for computing LEs in ODEs and DAEs. In this approach, the fundamental solution matrix $\boldsymbol{V}_t$ and its triangular factor $\boldsymbol{R}_t$ are indirectly computed by a reorthogonalized integration of the variational equation (4.2) through an appropriate QR decomposition. Thus, given grid points $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$, we can write $\boldsymbol{V}_{t_\ell}$ in terms of the state-transition matrices as

$$\boldsymbol{V}_{t_\ell} = \boldsymbol{Z}_{(t_\ell, t_{\ell-1})} \boldsymbol{Z}_{(t_{\ell-1}, t_{\ell-2})} \cdots \boldsymbol{Z}_{(t_2, t_1)} \boldsymbol{Z}_{(t_1, t_0)} \boldsymbol{V}_{t_0}. \tag{4.5}$$

At $t_0 = 0$, we perform a standard matrix QR decomposition

$$\boldsymbol{V}_{t_0} = \boldsymbol{Q}_{t_0}\boldsymbol{R}_{t_0},$$

and for $\ell = 1, 2, \ldots, N$, we determine $\boldsymbol{Z}_{(t_\ell, t_{\ell-1})}$ as the numerical solution (via numerical integration) of the matrix initial value problem

$$\begin{aligned} d\boldsymbol{Z}_{(t_\ell, t_{\ell-1})} &= \sum_{j=0}^{m} \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Z}_{(t_\ell, t_{\ell-1})} dw_t^j, \\ \boldsymbol{Z}_{(t_\ell, t_{\ell-1})} &= \boldsymbol{Q}_{t_{\ell-1}}, \quad t_{\ell-1} \le t \le t_\ell, \end{aligned} \tag{4.6}$$

and then compute the QR decomposition

$$\boldsymbol{Z}_{(t_\ell, t_{\ell-1})} = \boldsymbol{Q}_{t_\ell}\boldsymbol{R}_{(t_\ell, t_{\ell-1})},$$

where $\boldsymbol{R}_{(t_\ell, t_{\ell-1})}$ has positive diagonal elements. From (4.5), the value of the fundamental matrix $\boldsymbol{V}_{t_\ell}$ is determined via

$$\boldsymbol{V}_{t_\ell} = \boldsymbol{Q}_{t_\ell}\boldsymbol{R}_{(t_\ell, t_{\ell-1})}\boldsymbol{R}_{(t_{\ell-1}, t_{\ell-2})} \cdots \boldsymbol{R}_{(t_2, t_1)}\boldsymbol{R}_{(t_1, t_0)}\boldsymbol{R}_{t_0},$$

which is again a QR factorization with positive diagonal elements. Since this is unique, for the QR decomposition $\boldsymbol{V}_{t_\ell} = \boldsymbol{Q}_{t_\ell}\boldsymbol{R}_{t_\ell}$, we have

$$\boldsymbol{R}_{t_\ell} = \boldsymbol{R}_{(t_\ell, t_{\ell-1})}\boldsymbol{R}_{(t_{\ell-1}, t_{\ell-2})} \cdots \boldsymbol{R}_{(t_2, t_1)}\boldsymbol{R}_{(t_1, t_0)}\boldsymbol{R}_{t_0} = \prod_{\kappa=0}^{\ell} \boldsymbol{R}_\kappa.$$

Here we denote as $\boldsymbol{R}_\kappa$ the triangular transition matrices $\boldsymbol{R}_{(t_\ell, t_{\ell-1})}$ with $\kappa = 0, 1, \ldots, \ell$. From (4.4), the LEs are thus computed as

$$\lambda_i = \lim_{\ell\to\infty} \frac{1}{t_\ell} \log \left| \prod_{\kappa=0}^{\ell} R_\kappa^{ii} \right| = \lim_{\ell\to\infty} \frac{1}{t_\ell} \sum_{\kappa=0}^{\ell} \log |R_\kappa^{ii}|, \quad i = 1, \ldots, d. \tag{4.7}$$

4.2. Continuous $\boldsymbol{QR}$ Method

The implementation of the continuous QR technique requires to determine a system of SDEs for the $\boldsymbol{Q}$ factor and the scalar equations for the logarithms of the diagonal elements of the $\boldsymbol{R}$ factor elementwise. Then, once the orthogonal matrix $\boldsymbol{Q}$ is computed by numerical integration, the logarithms of the diagonal elements of $\boldsymbol{R}$ can also be obtained.

By differentiating in the Itô sense the decomposition $\boldsymbol{V}_t = \boldsymbol{Q}_t \boldsymbol{R}_t$ and using the orthogonality $\boldsymbol{Q}_t^T \boldsymbol{Q}_t = \boldsymbol{I}_d$, we get

$$d\boldsymbol{V}_t = (d\boldsymbol{Q}_t)\boldsymbol{R}_t + \boldsymbol{Q}_t(d\boldsymbol{R}_t), \tag{4.8}$$

$$\boldsymbol{0} = (d\boldsymbol{Q}_t^T)\boldsymbol{Q}_t + \boldsymbol{Q}_t^T(d\boldsymbol{Q}_t). \tag{4.9}$$

Inserting (4.8) into the variational equation (4.2), and multiplying by $\boldsymbol{Q}_t^T$ from the left and by $\boldsymbol{R}_t^{-1}$ from the right, we obtain

$$\boldsymbol{Q}_t^T(d\boldsymbol{Q}_t) + (d\boldsymbol{R}_t)\boldsymbol{R}_t^{-1} = \sum_{j=0}^{m} \boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t dw_t^j. \tag{4.10}$$

Since $(d\boldsymbol{R}_t)\boldsymbol{R}_t^{-1}$ is upper triangular, the skew-symmetric matrix $d\boldsymbol{S}_t := \boldsymbol{Q}_t^T(d\boldsymbol{Q}_t)$ satisfies

$$d\boldsymbol{S}_t^{il} = \begin{cases} \sum_{j=0}^{m} \left(\boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t\right)^{jl} dw_t^j, & i > l, \\ 0, & i = l, \\ -\sum_{j=0}^{m} \left(\boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t\right)^{jl} dw_t^j, & i < l. \end{cases} \tag{4.11}$$

This results in an SDE for $\boldsymbol{Q}_t$ given by

$$d\boldsymbol{Q}_t = \boldsymbol{Q}_t d\boldsymbol{S}_t = \sum_{j=0}^{m} \boldsymbol{Q}_t \boldsymbol{T}_t^j(\boldsymbol{x}_t, \boldsymbol{Q}_t) dw_t^j, \tag{4.12}$$

where the matrices $\boldsymbol{T}_t^j(\boldsymbol{x}_t, \boldsymbol{Q}_t)$ (for $j = 0, \ldots, m$) are defined via

$$\left(\boldsymbol{T}_t^j(\boldsymbol{x}_t, \boldsymbol{Q}_t)\right)^{il} = \begin{cases} \left(\boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t\right)^{jl}, & i > l, \\ 0, & i = l, \\ -\left(\boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t\right)^{jl}, & i < l. \end{cases} \tag{4.13}$$

A corresponding SDE for $\boldsymbol{R}_t$ can be obtained from (4.10) and (4.11) via

$$d\boldsymbol{R}_t = \sum_{j=0}^{m} (\boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t - \boldsymbol{T}_t^i(\boldsymbol{x}_t, \boldsymbol{Q}_t))\boldsymbol{R}_t dw_t^j, \tag{4.14}$$

and the equation for the ith diagonal element R_t^{ii} is given by

$$dR_t^{ii} = \sum_{j=0}^{m} (\boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t)^{ii} R_t^{ii} dw_t^j, \quad \text{for } i = 1, \ldots, d. \tag{4.15}$$

Since the computed LEs can be obtained from (4.4), we make use of the Itô Lemma to introduce the following SDE for the function $\psi_t^i = \log R_t^{ii}$ from (4.15),

$$\begin{aligned} d\psi_t^i = d(\log R_t^{ii}) = &\sum_{j=0}^{m} (\boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t)^{ii} dw_t^j \\ &- \frac{1}{2}\left[\sum_{j=0}^{m} (\boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t)^{ii} dw_t^j\right]^2 . \end{aligned} \tag{4.16}$$

If we assume there are no correlations between the diffusion terms in the SDE system; then we do not have terms $dw_t^k\, dw_t^\ell$ (for $1 < k < m$, and $1 < \ell < m$, with $k \neq \ell$) in the SDE (4.16). Also, using that $dt\, dt \equiv 0$, $dt\, dw_t^k \equiv 0$, and $dw_t^j\, dw_t^j \equiv dt$ for $1 < k < m$, the SDE (4.16) is reduced to

$$d\psi_t^i = \sum_{j=0}^{m} (\boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t)^{ii} dw_t^j - \frac{1}{2}\sum_{j=1}^{m} \left[(\boldsymbol{Q}_t^T \boldsymbol{J}_j(\boldsymbol{x}_t)\boldsymbol{Q}_t)^{ii}\right]^2 dt. \tag{4.17}$$

By integrating this SDE, it is possible to obtain the LEs λ_i from

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t}\psi_t^i, \quad i = 1, \ldots, d. \tag{4.18}$$

As it will be illustrated in the numerical examples, the alternative expressions (4.17) and (4.18), while easy to implement numerically, improve the numerical results in comparison to (4.15), and they lead to better robustness for large time intervals.

In summary, the difference between the discrete and the continuous QR method is that for the first one, the orthonormalization is performed numerically at every discrete time step, while the continuous QR method maintains the orthogonality via solving differential equations that encode the orthogonality continuously.

4.3. Computational Considerations

In this section, we discuss additional aspects of the computational implementation of discrete and continuous QR methods to calculate LEs. The application of the discrete QR technique mainly requires the numerical integration of the SDEs (4.1) and (4.6). This task is performed by using standard weak Euler-Maruyama and Milstein schemes, integrators for whom the ergodicity preservation property has been proved (see [40], [87]).

On the other hand, the numerical integration of the SDEs (4.1), (4.12), and (4.17) in the computational implementation of the continuous QR technique must be performed in such a way that it preserves the orthogonality of the factor $\boldsymbol{Q}$ in each integration step. This can be achieved via projected orthogonal schemes, which consist of a two-step process in which first an approximation is computed via any standard scheme, and then the result is projected into the set of orthogonal matrices [27]. Again we use the Euler-Maruyama and Milstein method as in the discrete case.

We have implemented the two QR-methods in MATLAB. However, to obtain a unique QR factorization in each step, we have modified the QR decomposition provided by MATLAB to ensure this uniqueness, by forming a diagonal matrix $\boldsymbol{\mathcal{I}}$ with $\mathcal{I}_{i,i} = \text{sign}(R_{i,i})$, for $i = 1, \ldots, d$; and then setting $\boldsymbol{Q} := \boldsymbol{Q}\boldsymbol{\mathcal{I}}$ and $\boldsymbol{R} := \boldsymbol{\mathcal{I}}\boldsymbol{R}$.

4.4. Numerical Examples

This Section illustrates the described the described QR-based procedures via two strangeness-free SDAE systems in order to compare the computational efficiency, accuracy, and robustness of both the discrete and continuous QR methods using the numerical integration schemes Euler-Maruyama and Milstein. The four numerical methods will be denoted as *D-EM, D-Milstein, C-EM, C-Milstein*, respectively. The computations are carried out with MATLAB Version 9.7.0(R2019b) on a computer with CPU Intel Core i7 composed by 6 cores of 2.20GHz, and 16 GB of RAM.

4.4.1. Example 1

Let us consider the simple SDAE equation system

$$\boldsymbol{E}d\boldsymbol{x}_t = \boldsymbol{f}_0(\boldsymbol{x}_t)dt + \boldsymbol{f}_1(\boldsymbol{x}_t)dw_t, \tag{4.19}$$

where

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{x}_t = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \boldsymbol{f}_0 = \begin{bmatrix} -x_2 \\ -\alpha x_1 + \arctan(x_1) + x_2 \end{bmatrix},$$

and

$$\boldsymbol{f}_1 = \begin{bmatrix} (x_1^2+1)^{\frac{1}{2}} \\ 0 \end{bmatrix},$$

with $\alpha \in \mathbb{R}^+$. The nonlinear functions in both the drift and diffusion parts are continuous on $\mathbb{R}^+$, with continuous and bounded derivatives, and w_t is a one-dimensional Wiener process. The underlying SDE of (4.19) is

$$d\hat{x}_t = [-\alpha \hat{x}_t + \arctan(\hat{x}_t)]dt + (\hat{x}_t^2+1)^{\frac{1}{2}} dw_t, \tag{4.20}$$

whose LE exists and can be explicitly represented as the following integral with respect to the solution of a stationary Fokker-Planck equation (see further details in [7])

$$\lambda = -\alpha + \frac{1}{2}\int_{\mathbb{R}} \frac{\hat{x}^2-2}{\hat{x}^2+1} p(x)dx, \tag{4.21}$$

where $p(x)$ is the stationary density of the unique invariant probability law of $\hat{x}_t$. By solving numerically (4.21) for $\alpha = 2$, we obtain the *exact* value of the LE associated to (4.20 and its original SDAE (4.19, which is $\lambda = -1.3385$. The accuracy of the QR-based methods will be assessed by comparing their computed results with this value as reference.

A large number of simulations have been carried out for stepsizes $h =$ 1e−2, 9e−3, …, 1e−3 with $T =$ 1000, 2000, …, 12000; to obtain computed approximations of the LE truncated at the *final time* $t_f := T$, denoted by λ_T. To complete our stochastic numerical analysis of the LE, we have calculated the values of expectation $\mathbb{E}[\lambda_T]$, standard deviation $\sigma[\lambda_T]$, and variance $\mathbb{V}[\lambda_T]$; estimated from 100 independent realizations. Some results are presented in Tables (4.1) to (4.4), taking $T = 6000, 12000, 20000$ and $h =$ 1e−1, 1e−2, 1e−3, 1e−4.

Observe that the time scale in Figure 4.1 has been conveniently adjusted to the range [0, 250], in order to show the exponential drop of the LE for the different realizations in the four methods, along with the time evolution. While in Figure 4.2 the time scale has been adjusted to the range [0, 10000], to better display the convergence of the mean and variance of the LE.

Based on the analytic expression of the LE, given by equation (4.21), the LE λ can be considered as a deterministic quantity. According to the numerical results obtained from the four QR-based methods, the sequences of random variables λ_{t_ℓ} reveal a trend towards null variance and convergence to the mean as ℓ tends to infinity. Such evolution

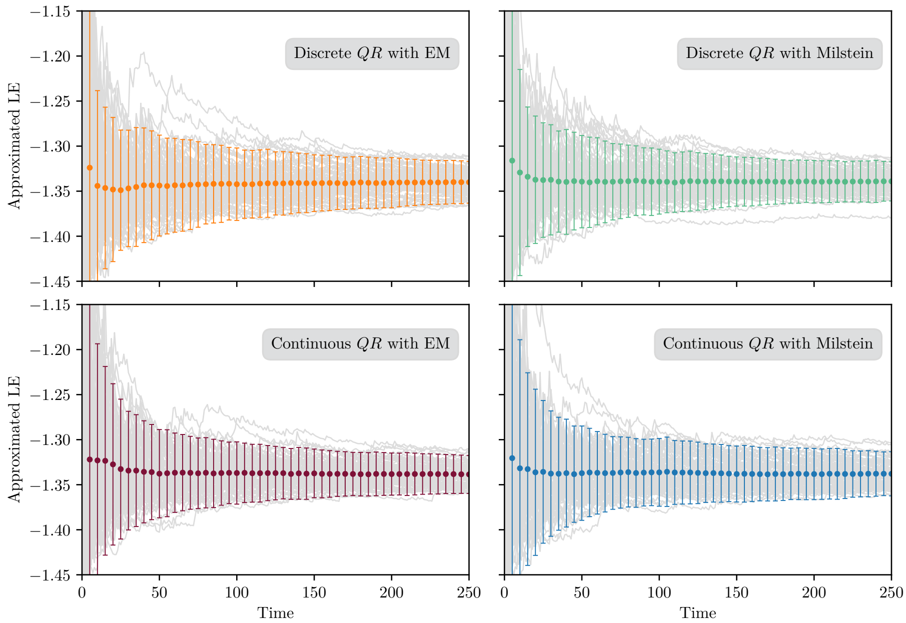


Figure 4.1: Discrete and continuous QR-based approximations of the LE corresponding to SDAE (4.19) via Euler-Maruyama and Milstein integrators, with a stepsize $h = 1\mathrm{e}{-3}$ *and* $T = 250$*. The solid circles show the mean and the whiskers the* 95% *confidence intervals of the trajectories.*

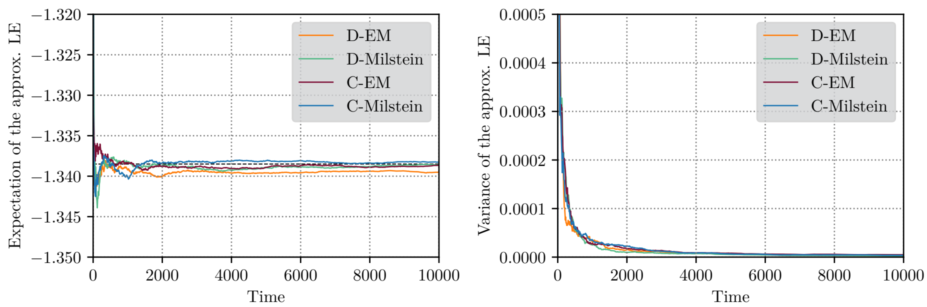


Figure 4.2: Discrete and continuous QR-based approximations of the LE corresponding to SDAE (4.19) via Euler-Maruyama and Milstein integrators, with a stepsize $h = 1\mathrm{e}{-3}$ *and* $T = 10000$*. The black dashed line in the left-hand side subplot shows the analytic value of* λ*.*

can be seen in Figure 4.1, and more obviously in Figure 4.2. For all the methods, an exponential decay is illustrated in $\mathbb{E}[\lambda_{t_\ell}]$ and $\mathbb{V}[\lambda_{t_\ell}]$ as ℓ tends to infinite. This behavior indicates a mean square (m.s.) conver-

T	h	$\mathbb{E}[\lambda_T]$	$\sigma[\lambda_T]$	$\mathbb{V}[\lambda_T]$	Rel. error [%]	CPU-time [sec]
6000	1e−1	−1.51906	0.00476	2.266e−5	13.48994	0.9827
6000	1e−2	−1.35231	0.00296	8.734e−6	1.03179	12.6422
6000	1e−3	−1.33874	0.00234	5.483e−6	0.01807	125.6875
6000	1e−4	−1.33903	0.00225	5.042e−6	0.03958	4367.2676
12000	1e−1	−1.51870	0.00334	1.116e−5	13.46268	2.0336
12000	1e−2	−1.35217	0.00184	3.392e−6	1.02160	24.9517
12000	1e−3	−1.33920	0.00161	2.579e−6	0.05206	252.4366
12000	1e−4	−1.33902	0.00134	1.795e−6	0.03848	8815.9035
20000	1e−1	−1.51780	0.00262	6.858e−6	13.39551	3.3806
20000	1e−2	−1.35236	0.00139	1.944e−6	1.03533	41.2975
20000	1e−3	−1.33936	0.00133	1.781e−6	0.06437	416.3228
20000	1e−4	−1.33902	0.00119	1.415e−6	0.03922	13870.1475

Table 4.1: Numerical results of the calculated LE for SDAE system (4.19) computed via Discrete QR-EM method.

T	h	$\mathbb{E}[\lambda_T]$	$\sigma[\lambda_T]$	$\mathbb{V}[\lambda_T]$	Rel. error [%]	CPU-time [sec]
6000	1e−1	−1.47000	0.00356	1.267e−5	9.82471	1.3947
6000	1e−2	−1.34911	0.00205	4.217e−6	0.79239	14.2033
6000	1e−3	−1.33883	0.00185	3.415e−6	0.02480	139.4657
6000	1e−4	−1.33901	0.00224	5.039e−6	0.03812	4786.6657
12000	1e−1	−1.46914	0.00249	6.202e−6	9.75996	2.8596
12000	1e−2	−1.34925	0.00186	3.466e−6	0.80302	27.8426
12000	1e−3	−1.33889	0.00176	3.093e−6	0.02924	280.9451
12000	1e−4	−1.33900	0.00134	1.794e−6	0.03700	9629.9437
20000	1e−1	−1.46973	0.00159	2.517e−6	9.80448	4.6544
20000	1e−2	−1.34924	0.00141	1.982e−6	0.80274	46.9282
20000	1e−3	−1.33915	0.00130	1.699e−6	0.04854	465.2272
20000	1e−4	−1.33900	0.00119	1.416e−6	0.03770	15121.3377

Table 4.2: Numerical results of the calculated LE for SDAE system (4.19) computed via Discrete QR-Milstein method.

gence of those sequences to a degenerate random variable, based on the implication that if

$$\lambda_{t_\ell} \text{ such that:} \begin{cases} \mathbb{E}[\lambda_{t_\ell}] = \mu_\lambda, \ \forall \ell, \\ \mathbb{V}[\lambda_{t_\ell}] \xrightarrow[\ell\to\infty]{} 0, \end{cases} \quad \Rightarrow \ \lambda_{t_\ell} \xrightarrow[\ell\to\infty]{\text{m.s.}} \mu_\lambda.$$

This means that the limit of λ_{t_ℓ} can be interpreted as a deterministic value with probability 1. This enables us to state that the stochastic approximations λ_{t_ℓ} converge in m.s. sense to a number (a degenerate

T	h	$\mathbb{E}[\lambda_T]$	$\sigma[\lambda_T]$	$\mathbb{V}[\lambda_T]$	Rel. error [%]	CPU-time [sec]
6000	1e−1	−1.35864	0.00326	1.061e−5	1.50459	1.3841
6000	1e−2	−1.34005	0.00278	7.743e−6	0.11616	13.6211
6000	1e−3	−1.33822	0.00277	7.651e−6	0.02128	135.0360
6000	1e−4	−1.33892	0.00224	5.025e−6	0.03152	4642.6202
12000	1e−1	−1.35932	0.00226	5.091e−6	1.55512	2.7334
12000	1e−2	−1.33999	0.00186	3.459e−6	0.11134	26.9335
12000	1e−3	−1.33813	0.00159	2.535e−6	0.02786	272.8842
12000	1e−4	−1.33891	0.00134	1.794e−6	0.03052	9217.7371
20000	1e−1	−1.35888	0.00196	3.835e−6	1.52252	4.4753
20000	1e−2	−1.34010	0.00096	9.306e−7	0.11965	45.0326
20000	1e−3	−1.33807	0.00148	2.187e−6	0.03224	465.5119
20000	1e−4	−1.33892	0.00119	1.417e−6	0.03121	14583.8282

Table 4.3: Numerical results of the calculated LE for SDAE system (4.19) computed via Continuous QR-EM method.

T	h	$\mathbb{E}[\lambda_T]$	$\sigma[\lambda_T]$	$\mathbb{V}[\lambda_T]$	Rel. error [%]	CPU-time [sec]
6000	1e−1	−1.33950	0.00287	8.228e−6	0.07468	1.5009
6000	1e−2	−1.33767	0.00253	6.386e−6	0.06235	14.8586
6000	1e−3	−1.33810	0.00232	5.402e−6	0.02998	147.5276
6000	1e−4	−1.33890	0.00225	5.052e−6	0.02988	5082.6248
12000	1e−1	−1.33931	0.00259	6.692e−6	0.06075	3.0427
12000	1e−2	−1.33864	0.00121	1.460e−6	0.01053	29.3666
12000	1e−3	−1.33769	0.00153	2.329e−6	0.06087	299.2588
12000	1e−4	−1.33889	0.00134	1.802e−6	0.02878	10035.9100
20000	1e−1	−1.33990	0.00182	3.296e−6	0.10453	4.9331
20000	1e−2	−1.33828	0.00152	2.310e−6	0.01654	49.5662
20000	1e−3	−1.33853	0.00140	1.960e−6	0.00258	505.4304
20000	1e−4	−1.33889	0.00119	1.416e−6	0.02948	15917.1861

Table 4.4: Numerical results of the calculated LE for SDAE system (4.19) computed via Continuous QR-Milstein method.

random variable), which is expected to represent the LE λ.

In Figure 4.3 we compare the relative error of the accuracy of the four numerical methods for different stepsize h and time interval $[0, T]$. From this graphical representation, we observe that continuous methods obtain better results than discrete ones, as expected. We also observe that the Milstein method has, in general, better accuracy than the Euler-Maruyama scheme, since its convergence order is higher, but requires more computational time. This latter fact is evidenced in Figure 4.4,

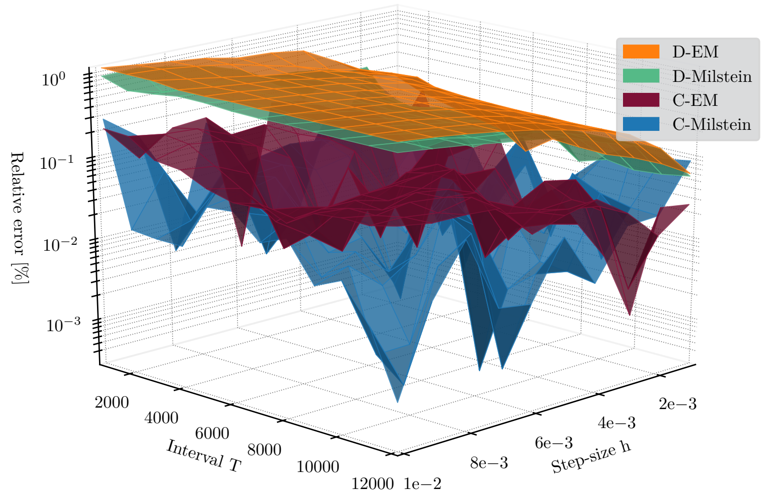


Figure 4.3: Comparison of relative errors for discrete and continuous QR-based approximations of the LE corresponding to SDAE (4.19) via Euler-Maruyama and Milstein integrators, with a range of stepsizes between $h = 1\mathrm{e}{-2}, \ldots, 1\mathrm{e}{-3}$*; and with* $T = 1000, \ldots, 12000$.

where a comparison of CPU time (in seconds) is shown for different values of h and T. Here, we observe that all the methods are affected to the same extent by incrementing the simulation interval T, via a logarithmic increment, and by narrowing the stepsizes h, via an exponential increment. A more pronounced difference between the methods should be observed in higher-dimensional systems.

4.4.2. Example 2

In this example, we make use of the *Chua's circuit* perturbed by noise, showed in Figure 4.5. The Chua's circuit is a simple electronic system that exhibits chaotic behavior due to its nonlinear negative resistance called *Chua's diode*.

For the purpose of our example, we consider the circuit is affected by an external noisy interference [78]. The noise, assumed to be coupled to the circuit in its left-hand side loop, is modeled as an additive voltage source, see Figure 4.5. On the other hand, the nonlinear relation between the voltage v_D with the current i_D in the Chua's diode is modeled

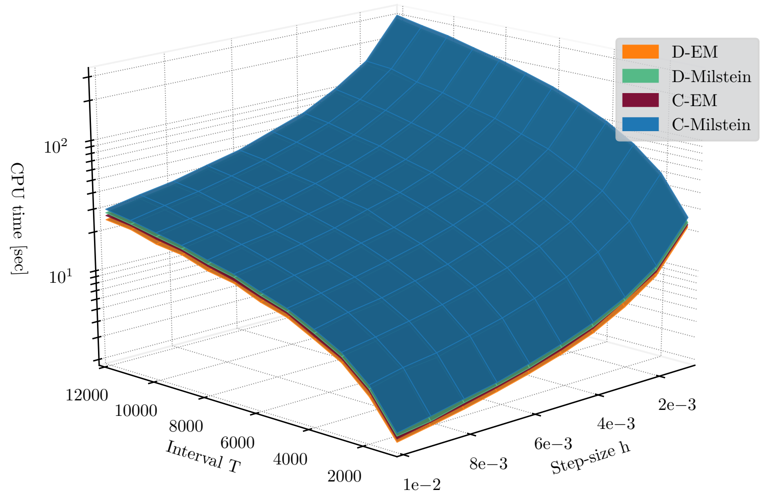


Figure 4.4: Comparison of the computing-time for discrete and continuous QR-based approximations of the LE corresponding to SDAE (4.19) via Euler-Maruyama and Milstein integrators, with a range of stepsizes between $h = 1\mathrm{e}{-2}, \dots, 1\mathrm{e}{-3}$*; and with* $T = 1000, \dots, 12000$.

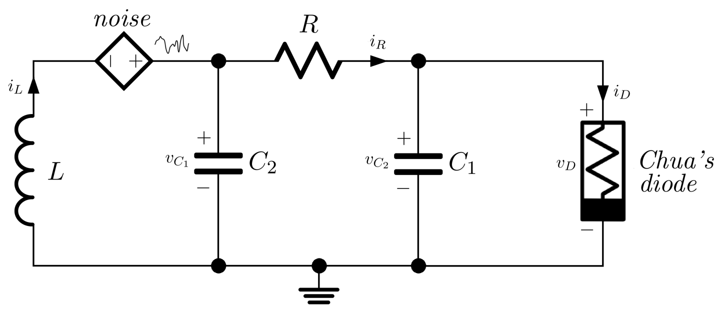


Figure 4.5: Chua's circuit diagram.

through the continuous cubic polynomial function $i_D := K_a v_D^3 - K_b v_D$, where K_a and K_b are positive constants, see [46]. By means of Kirchhoff's circuit laws, the Chua's circuit can be naturally written as an

Itô-type strangeness-free SDAE system as follows

$$\begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot d \begin{bmatrix} v_{C_1} \\ v_{C_2} \\ i_L \\ i_R \\ v_D \\ i_D \end{bmatrix} = \begin{bmatrix} -i_R - i_D \\ i_R + i_L \\ -v_{C_2} \\ v_{C_1} - v_{C_2} - Ri_R \\ v_{C_1} - v_D \\ K_a v_D^3 - K_b v_D - i_D \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ \varepsilon \\ 0 \\ 0 \\ 0 \end{bmatrix} dw_t, \tag{4.22}$$

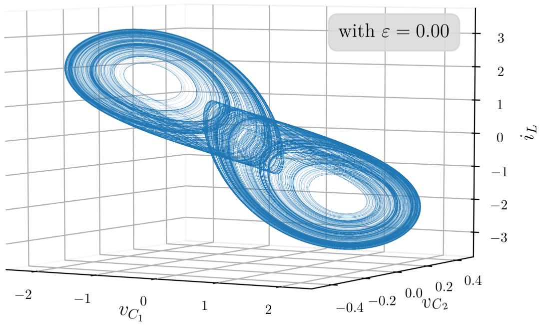
where $[v_{C_1}\ v_{C_2}\ i_L]^T$ are state variables and $[i_R\ v_D\ i_D]^T$ are algebraic variables, the subscript "t" has been omitted in this formulation for simplicity. The noise intensity constant is represented by ε. The SDAE (4.22) can be reduced to its underlying SDE form

$$\begin{aligned} RC_1 dv_{C_1} &= [-RK_a v_{C_1}^3 + (RK_b - 1)v_{C_1} + v_{C_2}]dt, \\ RC_2 dv_{C_2} &= [v_{C_1} - v_{C_2} + Ri_L]dt, \\ Ldi_L &= -v_{C_2}dt + \varepsilon dw_t. \end{aligned} \tag{4.23}$$

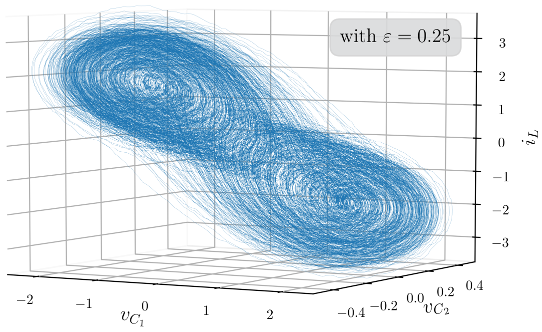
For the numerical simulations, we have chosen as constants $R = 1.00$; $C_1 = 0.0915$; $C_2 = 1.00$; $L = 0.0714$; $K_a = 0.0625$; $K_b = 1.00$. Additionally, we have assigned the value $\varepsilon = 0.25$ to the noise intensity constant. Under these values, the Chua's system still tends to evolve around its characteristic double scroll attractor, but in a different way compared to the deterministic case (i.e., with $\varepsilon = 0.00$). The system presents a stochastic bifurcation, specifically a *phenomenological bifurcation* (or P-bifurcation), because of the qualitative changes in the stationary probability distributions of the system's states [7], [8]. Despite the qualitative changes in the Chua's system due to the stochastic perturbation, its chaotic behavior remains as can be seen in Figure 4.6.

Since the positiveness of the largest Lyapunov exponent (LLE) in a nonlinear dynamical is usually an indication (although not sufficient) that the system is chaotic, we make use of the QR-based methods to compute the LEs in order to test for the presence of chaos in the dynamics of the noise-driven oscillator (4.22). Furthermore, it is known the sum of all LEs allows for identifying dissipative dynamical systems. If $\mathcal{S} := \sum_{i=1}^{d} \lambda_i$ in negative, the system is *dissipative* [71].

Unfortunately, it is unfeasible to obtain analytically LEs for the present example, doing a precision test of the methods impossible to perform. Therefore, the assessment of the QR-based techniques this time is rather focused on the observation of consistent, time-convergent, and



(a) without stochastic perturbation.



(b) with stochastic perturbation.

Figure 4.6: Chua's system phase-portraits in chaotic regime.

homogeneous computed LEs for all the numerical methods. In addition, the preservation of the dissipative characteristic of the system for all the simulations through the computed value $\mathcal{S}$ is verified as well.

Figure 4.7 shows the time evolution computed LEs by using the four QR-based methods. The simulation was carried out with the stepsize $h = 1\text{e}{-4}$ and the interval $T = 6000$. It can be seen the convergence of each LE for a single realization. We additionally present the numerical values of $\mathcal{S}$. Here, the negativeness of $\mathcal{S}$ evidences that, in our example, the stochastic Chua's circuit is a dissipative system.

As it can be seen, the computed LEs of the Chua's system (4.22), and hence the sum of LEs $\mathcal{S}$ as well, exhibit a satisfactory convergence along the time even with a single realization. This is a suitable indicator

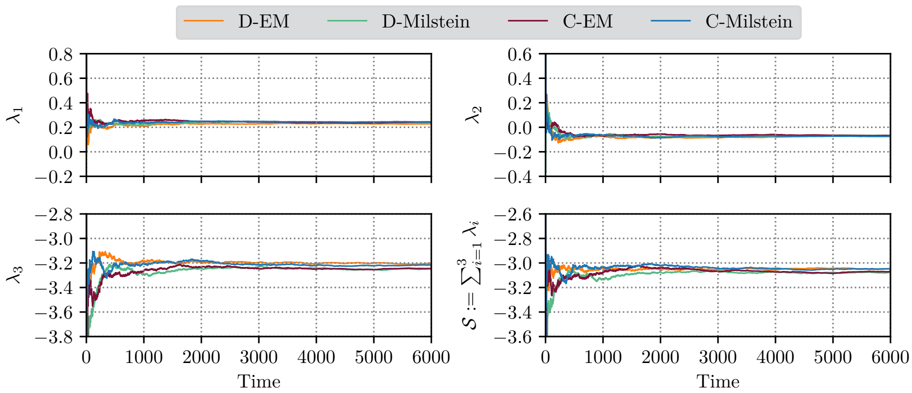


Figure 4.7: Time evolution of the computed LEs in stochastic Chua's system (4.22) using the four QR-based methods for a stepsize $h = 1\text{e}{-4}$ and an interval $T = 6000$.

of the numerical robustness and a good performance of the proposed QR-based methods when they are implemented in complex situations such as the positiveness of the computed LLE. Even though in Figure 4.7 we present results only for a stepsize $h = 1\text{e}{-4}$, similar satisfactory results can be obtained for wider stepsizes. In Table 4.5, we collect the numerical results corresponding to $h = 1\text{e}{-3}$, $5\text{e}{-4}$, $1\text{e}{-4}$.

Method	h	λ_1	λ_2	λ_3	$\mathcal{S}$
D-EM	1e−3	0.23994	−0.06890	−3.22041	−3.04936
	5e−4	0.24207	−0.07063	−3.23055	−3.05911
	1e−4	0.24824	−0.07093	−3.20410	−3.05678
D-Mil	1e−3	0.24166	−0.07304	−3.23649	−3.06788
	5e−4	0.24486	−0.06837	−3.24114	−3.06465
	1e−4	0.24738	−0.06726	−3.24892	−3.05880
C-EM	1e−3	0.22970	−0.06103	−3.22547	−3.05680
	5e−4	0.22962	−0.06231	−3.23562	−3.06831
	1e−4	0.23400	−0.06837	−3.24523	−3.06960
C-Mil	1e−3	0.23514	−0.06963	−3.23238	−3.05686
	5e−4	0.23169	−0.06858	−3.21973	−3.05662
	1e−4	0.23287	−0.07468	−3.21613	−3.05794

Table 4.5: Numerical results of the calculated LEs for the Chua's system (4.22) computed via the four QR-based methods for $T = 6000$.

CHAPTER 5

Application to Power Systems

5.1. General Structure of Power Systems

To understand the overall behavior of power systems, we need to know the power grids' underlying topology. The mostly alternating current (AC) power systems have been generally composed of three-phase generation, transmission and distributions networks, and loads, see Figure 5.1. The networks supply massive three-phase industrial loads at different distribution and transmission voltages as well as single-phase residential and commercial loads. In United States and many countries in Latin-America, the term sub-transmission denotes networks with voltage ranges between transmission and distribution. Distribution voltages are typically 10-60 kV, sub-transmission voltages are typically 66-138 kV, and transmission voltages are typically above 138 kV. Generated voltages are up to 35 kV for generators used in large electrical power stations. Power station auxiliary supply systems and industrial power systems supply a substantial amount of induction motor loads. On the other hand, residential and commercial loads include a significant amount of single-phase induction motor loads, see [88].

For over a century, power systems have employed synchronous ma-

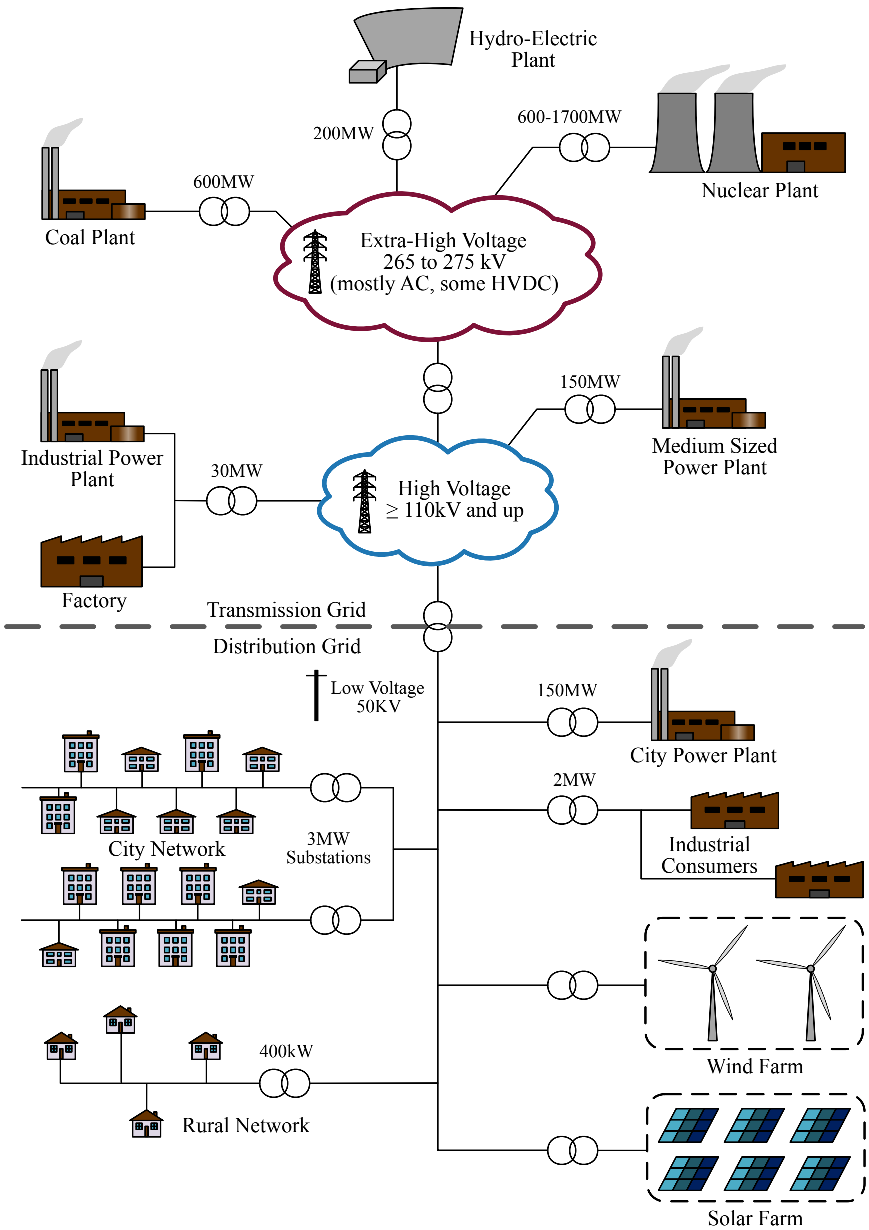


Figure 5.1: Classical Structure of a Power System (Source: [51], own work).

chines for electricity generation. However, in the last decades the technological development, the strong presence of renewable energy generation (e.g., wind power, which has begun to expand at a large pace), and the deregulation of the electricity sector; have led to the appearance of new concepts such as distributed generation (DG), demand-side management (DSM), virtual power plants (VPP), electric vehicles (EV), smart metering, etc. which changed the classical structure. These new elements in the power network structure, along with other concepts, constitute nowadays widely concept of *Smartgrid*. A suitable definition taken from [89], states that Smartgrid is a type of electrical grid which attempts to predict and intelligently respond to the behavior and actions of all electric power users connected to it (suppliers, consumers, and those that do both) in order to efficiently deliver reliable, economical, and sustainable electricity services.

5.2. Power System Stability

The concept of stability in power systems is, in essence, the same as that for any dynamical system in general. For this reason, on base of concepts as the ones surveyed in Section 3.2, one can find in the literature some previous efforts in order to define, classify, and assess the stability in power systems. The most of them led by IEEE (Institute of Electrical and Electronics Engineers) and CIGRE (Conseil International des Grands Réseaux Electriques). As result, the today's reference document is the technical report made by the *IEEE/CIGRE Joint Task Force on Stability Terms and Definitions*, entitled "*Definition and Classification of Power System Stability*" [49], set up jointly by the CIGRE Study Committee 38 and the IEEE Power System Dynamic Performance Committee. This report addresses the issues of stability definition and classification in power systems from a fundamental viewpoint and closely examines the practical ramifications. The report aims to define *power system stability* more precisely, provide a systematic basis for its classification, and discuss linkages to related issues such as power system *reliability* and *security*, see [49], [50], [60]. The definitions are addressed below.

5.2.1. Definition of Stability

The stability of a power system refers to the continuance of intact operation following a disturbance and depends on the operating condition and the nature of the physical disturbance that affects to it. Formal definition given by IEEE/CIGRE in [49] which says: *Power system stability* *is the ability of an electric power system, for a given initial operating condition, to regain a state of operating equilibrium after being subjected to a physical disturbance, with most system variables bounded so that practically the entire system remains intact.*

The previous definition refers mainly to an interconnected power system as a whole and is essentially a single problem. However, the various types of stability problems of a power system cannot be adequately understood and effectively dealt with by treating them as a single problem. Furthermore, because of the high dimensionality and complexity of the power system, which involves a large number of variables, simplifying assumptions are made in order to allow the analysis of specific types of problems with satisfactorily accuracy[31]. Stability analysis is greatly facilitated by classification of stability into appropriate categories. Thereby, the study of the stability in power systems is split into three significant divisions:

- rotor angle stability,
- voltage stability,
- frequency stability.

Based on this triplet, there is an entire classification given by IEEE/-CIGRE technical report. This categorization is based on the following considerations:

- The physical nature of the resulting mode of instability as indicated by the main system variable in which instability can be observed.
- The size of the disturbance considered, which influences the method of calculation and prediction of stability.
- The devices, processes, and the time span that must be taken into consideration in order to assess stability.

This main taxonomy is a meaningful and widely accepted placement due to high dimensionality and complexity of the stability problems. Figure 5.2 gives an overall graphic that explains of power system stability classification based on the dynamics of the phenomenon and identifying its categories and subcategories.

A further description of the subcategories related to the stability concepts for the categories voltage, frequency, and rotor angle; may be found in [32], [49], [50], [60], [66], [79] and references therein. Another

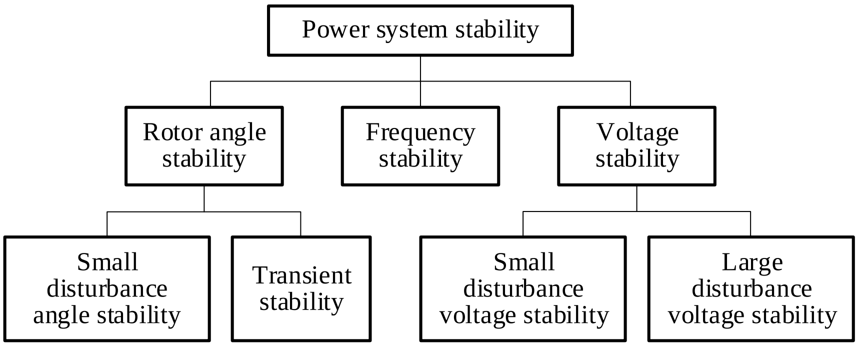


Figure 5.2: IEEE/CIGRE Power systems stability classification [49].

possible classification of the power system stability can be done in terms of the duration of the phenomenon (time-domain) and the components that influence the phenomenon. The idea is illustrated in Table (5.1).

Time Domain	**Generator Influenced Phenomenon**		**Load Influenced Phenomenon**
Short-term	Rotor angle stability		Small-disturbance voltage stability
	Small-disturbance stability	Transient stability	
Long-term	Frequency stability		Large-disturbance voltage stability

Table 5.1: PS stability classification based on the time-domain [31].

5.2.2. Definition of Reliability

The reliability of a power system refers to the probability of its satisfactory operation over the long-run. It denotes the ability to supply adequate electric service on a nearly continuous basis, with few interruptions over an extended time period. A formal definition of power system reliability given by NERC (North American Electric Reliability Council) and subscribed by [49] is the follow: *Power system reliability is the degree to which the performance of the elements in a electric power system results in electricity being delivered to customers (final consumers or retailers) within accepted standards and in the amount desired.*

Reliability can be addressed by considering two basic functional aspects of the power systems such as *Adequacy*, that is the ability of the power system to supply the aggregate electric power and energy requirements of the customer at all times, taking into account scheduled and unscheduled outages of system components; and *Security*, explained below.

Unlike the stability and security concepts, which are time-varying attributes which can be assessed by studying the power system performance under a particular set of conditions. Reliability, is an indicator of the time-average performance of the power system. It can only be studied by testing the system behavior over an appreciable time interval. Reliability evaluation techniques are well developed, the most of them based on a probabilistic framework. The interested reader can review [4] for further information this issue.

5.2.3. Definition of Security

Defined in [49] as *the ability of the power system to withstand sudden disturbances such as electric short circuits or unanticipated loss of system components.* Security of a power system refers to the degree of risk in its ability to survive imminent disturbances (contingencies) without the customer service interruption. It relates to the robustness of the system to imminent disturbances and, hence, depends on the system operating condition as well as the contingent probability of disturbances.

System security may be further distinguished from stability in terms of the resulting consequences. For example, two systems may both be stable with equal stability margins, but one may be relatively more secure because the consequences of instability are less severe.

The security analysis concerns about the estimation of power system robustness to imminent disturbances (small or large). These analysis considers two important aspects: the static security analysis, it is a steady-state assessment which verifies the right operating conditions of the system after a disturbance ocurrs; and the dynamic security analysis, which implies the examination of the three power system stability categories described in Section 5.2.1.

As can be seen in these brief sections, there are important linkages between reliability, security, and stability in power systems. These have been widely discussed in several papers and books. We refer to [32], [49], [50], [60], [66], [79] for further information.

5.2.4. Tools for Dynamic System Analysis

The whole dynamic system analysis (DSA) of power system (where dynamic stability analysis is included) can be seen in general way as a set of methodologies and mathematical modeling techniques to frame, understand, and discuss complex issues and problems which occurs in time horizons of the order of small fractions of a second to minutes. The purpose of the DSA is the assessment of transient and dynamical behavior of the network and its components when it is affected by disturbances. The DSA involves carrying out studies in the time- and frequency-domain, it includes studies such as transient stability, critical clearing time, dynamic voltage step/control, fault ride through, etc. Following these scheme, several software packages such as: PowerFactory, NEPLAN, DINIS, ERACS, ETAP, IPSA, Power World, PSS/E, SKM Power Tools [43], as well as non-commercial choices such as: PSAT, Dome, MATPOWER, MatDyn, PYPOWER-Dynamics, InterPSS, GridCal, UWPFLOW, PyPSA, among others; can be found nowadays. Their key features for the dynamic stability assessment usually are:

- transient stability,
- small–signal stability,
- voltage stability,
- power system stability controls
- power system security assessment,
- fault ride through.

5.3. Applying LEs as DSA Method

In the DSA of power systems, a significant majority of study-cases are oriented to evaluate the angle and voltage stability of the system when it is subjected to small or large disturbances. In this regard, studies considering small disturbances are commonly known as small-signal stability assessment (SSSA). Here, linear stability analysis via the computation of the eigenvalues has been one of the traditional modal approaches to predict the degree of stability of a power system [50], [79]. However, eigenvalue analysis is limited to linear time-invariant systems or systems close to a stationary solution. When time-varying systems are tested, as is the case of many systems subjected to stochastic disturbances, then eigenvalue approach is no longer applicable. On the other hand, the DSA of power systems affected by large disturbances, called transient stability assessment (TSA), is mainly performed with verification strategies based on time-domain integration [49], [60]. Figure 5.3 illustrates, coloured in grey, the subcategories from the IEEE/CIGRE power systems stability classification where the computed LEs can be implemented as stability assessment method.

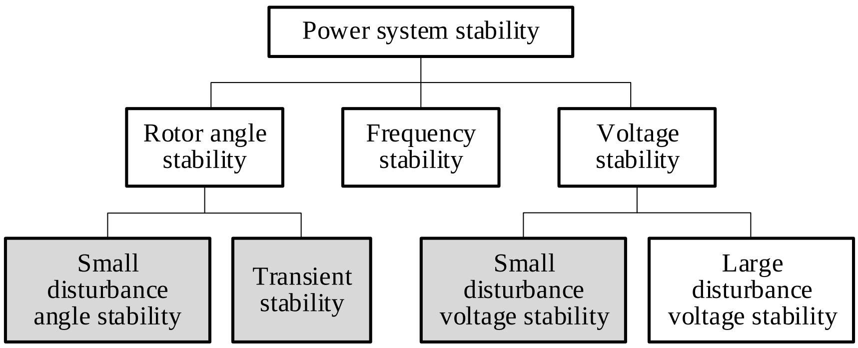


Figure 5.3: Subcategories of the IEEE/CIGRE power systems stability classification where LEs can be implemented as a stability assessment technique (in grey).

Since the concept of LEs is based on the trajectories of the dynamical systems, the method is an interesting measure of dynamic stability for power systems under stochastic disturbances in general. So, testing asymptotic stability of power systems via LEs has become an attractive approach for the two areas mentioned before, namely, the SSSA of rotor angles and voltages, by using the linearized set of SDAEs which model the system [90], [92]; and strategies for the rotor angles via TSA using the nonlinear SDAE system and its variational equation [44], [93] to determine stability regions of the power system under analysis. For both cases, asymptotic (exponential) stability is checked via approximations of the LLE of the system. In particular, a negative LLE indicates that the dynamics of the system is asymptotically stable. In the next subsections, test cases of power systems modeled through strangeness-free SDAE are presented. They illustrate the suitability of the computed LEs to approach (exponential) stability for SSSA.

5.4. Modeling Power Systems through SDAEs

Under the assumption of deterministic dynamic behavior, power systems are typically modeled via a system of quasi-linear DAEs with partitioned variables, see [50], [79], of the form

$$\boldsymbol{E}_{11} d\boldsymbol{x}_t^{D_1} = \boldsymbol{f}_0^{D_1}(\boldsymbol{x}_t^{D_1}, \boldsymbol{x}_t^{A})dt, \tag{5.1a}$$

$$0 = \boldsymbol{f}_0^{A}(\boldsymbol{x}_t^{D_1}, \boldsymbol{x}_t^{A}), \tag{5.1b}$$

where $\boldsymbol{E}_{11} \in \mathbb{R}^{d_1 \times d_1}$ is a diagonal block matrix, $\boldsymbol{f}_0^{D_1} \in \mathcal{C}^1(\mathbb{R}^{d_1+a}, \mathbb{R}^{d_1})$, $\boldsymbol{f}_0^{A} \in \mathcal{C}^1(\mathbb{R}^{d_1+a}, \mathbb{R}^{a})$, $\boldsymbol{x}_t^{D_1} \in \mathbb{R}^{d_1}$ are the dynamic state variables, and $\boldsymbol{x}_t^{A} \in \mathbb{R}^{a}$ are the algebraic state variables and we set $n_1 = d_1 + a$. The DAE system (5.1) is strangeness-free (or differentiation index-1).

The dynamic behavior of synchronous machines, system controllers, power converters, transmission lines, or power loads are adequately represented through such a DAE formulation. But, in current real-world systems, the dynamic behavior of power systems is affected by disturbances of a stochastic nature such as renewable stochastic power generation, rotor vibrations in synchronous machines, stochastic variations of loads, electromagnetic transients, or perturbations originated by the measurement errors of control devices, see [65]. Such disturbances can be modeled through Itô SDEs of the form

$$d\boldsymbol{x}_t^{D_2} = \boldsymbol{f}_0^{D_2}(\boldsymbol{x}_t^{D_1}, \boldsymbol{x}_t^{D_2}, \boldsymbol{x}_t^{A})dt + \boldsymbol{f}_1^{D_2}(\boldsymbol{x}_t^{D_1}, \boldsymbol{x}_t^{D_2}, \boldsymbol{x}_t^{A})dw_t. \tag{5.2}$$

Here, $\boldsymbol{f}_0^{D_2} \in \mathcal{C}^1(\mathbb{R}^{d_2+a}, \mathbb{R}^{d_2})$ is the drift, $\boldsymbol{f}_1^{D_2} \in \mathcal{C}^2(\mathbb{R}^{d_2+a}, \mathbb{R}^{a})$ is the diffusion, $\boldsymbol{x}_t^{D_2} \in \mathbb{R}^{d_2}$ are the stochastic variables, and w_t is the Wiener process. By combining (5.2) and (5.1), and assuming that $\boldsymbol{x}_t^{D_2}$ perturbs (5.1a) and (5.1b), we obtain a strangeness-free SDAE system of the form

$$\boldsymbol{E}_1 d\boldsymbol{x}_t^{D_1} = \boldsymbol{f}_0^{D_1}(\boldsymbol{x}_t^{D_1}, \boldsymbol{x}_t^{D_2}, \boldsymbol{x}_t^{A})dt, \tag{5.3a}$$

$$d\boldsymbol{x}_t^{D_2} = \boldsymbol{f}_0^{D_2}(\boldsymbol{x}_t^{D_1}, \boldsymbol{x}_t^{D_2}, \boldsymbol{x}_t^{A})dt + \boldsymbol{f}_1^{D_2}(\boldsymbol{x}_t^{D_1}, \boldsymbol{x}_t^{D_2}, \boldsymbol{x}_t^{A})dw_t, \tag{5.3b}$$

$$0 = \boldsymbol{f}_0^{A}(\boldsymbol{x}_t^{D_1}, \boldsymbol{x}_t^{D_2}, \boldsymbol{x}_t^{A}), \tag{5.3c}$$

or in simplified notation as

$$\boldsymbol{E} d\boldsymbol{x}_t = \boldsymbol{f}_0(\boldsymbol{x}_t)dt + \boldsymbol{f}_1(\boldsymbol{x}_t)dw_t, \quad \boldsymbol{x}_{t_0} = \boldsymbol{x}_0, \tag{5.4}$$

with

$$\boldsymbol{E} := \begin{bmatrix} \boldsymbol{E}_{11} & 0 & 0 \\ 0 & \boldsymbol{I}_{d_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \boldsymbol{x}_t := \begin{bmatrix} \boldsymbol{x}_t^{D_1} \\ \boldsymbol{x}_t^{D_2} \\ \boldsymbol{x}_t^{A} \end{bmatrix},$$

drift $\boldsymbol{f}_0 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, and diffusion $\boldsymbol{f}_1 \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$, where $n = d_1 + d_2 + a$.

The study-cases presented below are formulated as the form (5.3). An alternative approach for including the stochastic disturbances is to implement the Wiener process directly in the underlying ODE of the system, turning them into SDEs (see [17], [36] for examples).

5.4.1. Modeling Stochastic Perturbations

In this subsection, we discuss the modeling of stochastic variations via SDEs. We employ the well known mean-reverting process termed *Ornstein-Uhlenbeck* (OU) process [38], [65]. The SDE, which defines the OU process, has the form

$$d\eta_t = \alpha(\mu - \eta_t)dt + \beta dw_t, \qquad \eta_{t_0} = \eta_0, \qquad t \in \mathbb{I}, \tag{5.5}$$

where $\alpha, \mu, \beta \in \mathbb{R}^+$. The OU process is a stationary autocorrelated Gaussian diffusion process distributed as $\mathcal{N}(\mu, \beta^2/2\alpha)$. Another mean-reverting choice, similar to the OU process, would be the *Cox-Ingersoll-Ross* (CIR) process, whose realizations are always nonnegative; in fact, it is a sum of squared OU process [5].

It is usually recommended to ensure the boundedness of the stochastic variations for the numerical implementations. In this regard, suitable resources are odd trigonometric functions such as a sin or arctan to guarantee boundedness. For example, if from (5.5) we generate a process with a normal distribution $\mathcal{N}(\mu, \sigma^2)$, for $\mu = 0$ and $\sigma^2 = 0.16$, this value of variance enables us to generate a mean-reverting stochastic trajectory, whose confidence interval of 95% ($\pm 2\sigma$) is inside the threshold of ± 1. Then, through the functions

$$\xi(\eta_t) = \sin \eta_t, \qquad \text{or} \qquad \chi(\eta_t) = \frac{2}{\pi} \arctan \eta_t, \tag{5.6}$$

we obtain a bounded stochastic variation inside the interval $[-1, 1]$, and the OU SDEs, that generate the stochastic variations are represented by (5.3b).

To couple the parameters of the system in (5.3a) and (5.3c) with a bounded stochastic disturbance, we use

$$p(\eta_t) = p_0 + \rho\xi(\eta_t),$$

where p_0 is a constant parameter, η_t is the stochastic process that describes the variations of the parameter, and $\rho \in \mathbb{R}^+$ is a factor that controls the magnitude of the perturbation.

5.5. Study-Cases

In this Section, we present results of our implementation of the *QR*-based methods for the calculation of LEs at the hand of several test cases of power systems represented by strangeness-free SDAEs models of so-called single-machine-infinite-bus (SMIB) systems. This simplified model is frequently used in the area of power systems in order to understand the local dynamic behavior of a specific machine connected to a complex power network. The SMIB consists of a synchronous generator connected through a transmission line to a bus with a fixed bus voltage magnitude and angle, called infinite bus, which represents the grid. A diagram of the system is shown in Figure 5.4.

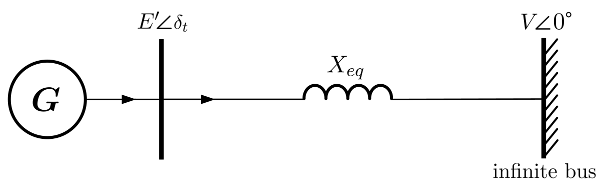


Figure 5.4: Single-machine infinite-bus (SMIB) scheme.

In each test case, we consider a different type of disturbance. For Case 1, the disturbance is a stochastic load connected to the system. In Case 2, the disturbance is due to noise caused by a measurement error in a transducer of the machine control system. In both cases, the maximum disturbance that the system can admit without losing stability is analyzed, as well as the effect (positive or negative) of the disturbance for the system in the stable region. The whole SMIB system, i.e., the synchronous machine, system constraints, and stochastic disturbances; are modeled by a strangeness-free SDAE system. The dimension of this system is mainly defined by the type of model used in the synchronous machine; we use a *classical model* and a *flux-decay model*, see [50], [60], [75], [79] for detailed descriptions.

5.5.1. Case 1: SMIB with stochastic load

In this test case, we make use of the LEs to assess the impact of stochastic disturbances associated with an active power load, over the rotor angle stability of a synchronous generator. Both the machine and load are connected to the same bus, and this bus in turn, is linked to the grid through a transmission line. This kind of SMIB model is typically used to analyze the effects of renewable energy sources, or aggregated random power consumption, see Figure 5.5. For this version of SMIB

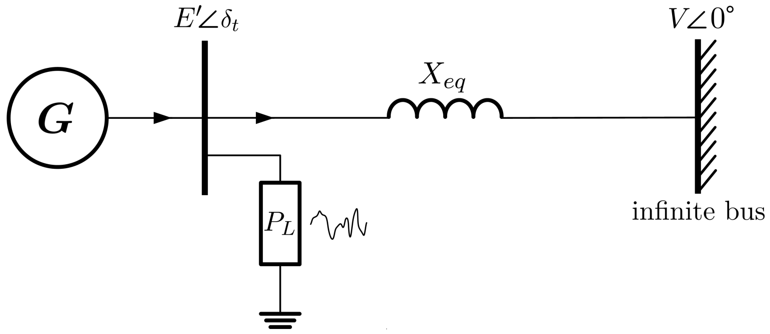


Figure 5.5: Scheme of a SMIB system with a stochastic load used in Test Case 1.

system called *classical model*, the dynamic behavior of the synchronous machine is represented by the swing equations where the rotor angle δ_t and the rotor speed ω_t are the state variables, see [60], [79]. The algebraic constraint in the system is given by the active power balance, expressed in terms of P_m the mechanical power, P_e the electrical power, and P_L the constant power consumed by the load. A stochastic process η_t is modeled by an OU SDE. We consider that $\rho\eta_t$ is the stochastic component of power consumption that perturbs additively the active power balance of the system, where ρ is the size of the disturbance. This leads to the system

$$d\delta_t = [\omega_t - \omega_s]dt, \tag{5.7a}$$

$$2Hd\omega_t = [P_m - P_e - K_D(\omega_t - \omega_s)]dt, \tag{5.7b}$$

$$\eta_t = -\alpha\eta_t dt + \beta dw_t, \tag{5.7c}$$

$$0 = \frac{E'V}{X_{eq}}\cos\delta_t + (P_L + \rho\eta_t) - P_e. \tag{5.7d}$$

By computing the LEs of this SDAE system and checking the LLE, we can determine the maximal perturbation size ρ (via successive increments of ρ) admitted by the SMIB system before losing rotor angle stability. The numerical tests are performed for the values $P_m = 0.8$; $P_L = 0.3$; $X_{eq} = 0.8$; $H = 3.5$; $K_D = 0.4$; $\omega_s = 2\pi 50$; $V = 1.0$; $E' = 1.05$; $\alpha = 1.0$; $\beta = 0.4$. Most of the values are expressed in the per-unit system (pu) [75]. The QR methods are executed with step size $h = 1\text{e}{-3}$ and a simulation time $T = 20000$. Figure 5.6 displays

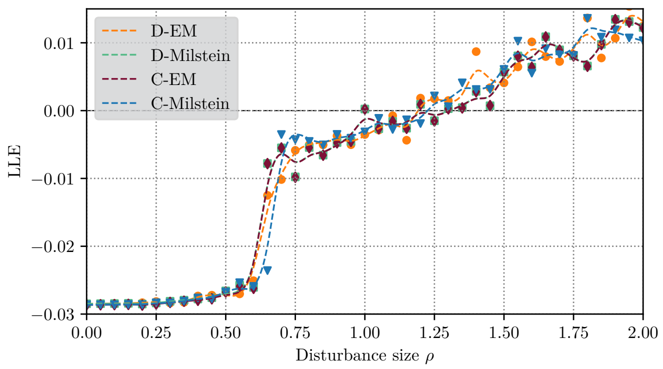


Figure 5.6: LLE considering different disturbance sizes ρ for the SMIB Test Case 1, tested with four QR-based methods.

the computed LLE utilizing the four QR-based methods for incremental disturbance sizes $\rho = 0.00, 0.05, \ldots, 2.00$. As expected, at $\rho = 0.0$ when the system is not affected by a stochastic disturbance, i.e., the system is deterministic, the computed LEs match closely with the real parts of the eigenvalues obtained from the Jacobian matrix of the linearization of (5.7). When increasing ρ, all methods reveal the same monotonically increasing behavior of the calculated value of the LLE towards the unstable region. First, there is a slow increase for $0.00 < \rho < 0.60$, and then an abrupt increase of the LLE in the interval $0.60 < \rho < 0.75$. In the interval $0.75 < \rho < 1.20$, even though the LLE has not yet reached the instability region, for this particular case, the characteristics such as a low damping coefficient and the presence of the stochastic disturbance, provokes a behavior in the system called *pole slipping*. This is, in a certain sense, a different kind of instability because the system loses synchronism as it reaches another equilibrium point near another attractor, see [79, sec. 5.8] for further details. The different aspects of this study-case are better illustrated with the phase portraits in Figure 5.7. The charts display the trajectories of the dynamical system (5.7) projected onto the $\delta_t - \omega_t/\omega_s$ plane for the disturbance sizes $\rho = 0.0$, 0.3, 0.7, 1.0, 1.5. Detailed numerical data for this case are presented in Table 5.2.

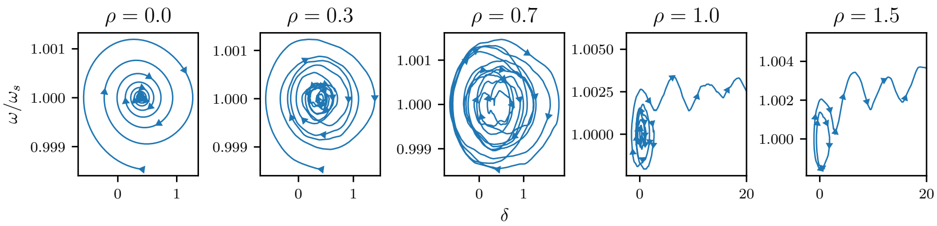


Figure 5.7: Phase portraits of the system (5.7) considering different disturbance sizes ρ for the SMIB Test Case 1.

This test case shows the immense potential of using LEs as an indicator of instability for nonlinear power systems. These could also be used in multi-machine study cases where the computational complexity has to be reduced, e.g., by model reduction.

5.5.2. Case 2: SMIB with regulator perturbed by noise

In this subsection we consider an SMIB system with a synchronous machine described by a third-order *flux-decay model*. Here, in addition to the rotor angle δ_t and the rotor speed ω_t associated to the swing equations, the system includes the effect of the field flux ψ_{fd} described by the field circuit dynamic equations and constraints. In this model the machine is equipped with an automatic voltage regulator (AVR) to keep the generator output voltage magnitude in a desirable range, and a power system stabilizer (PSS) to damp out low-frequency oscillations, see Figure 5.8. The AVR and PSS add to the system three more

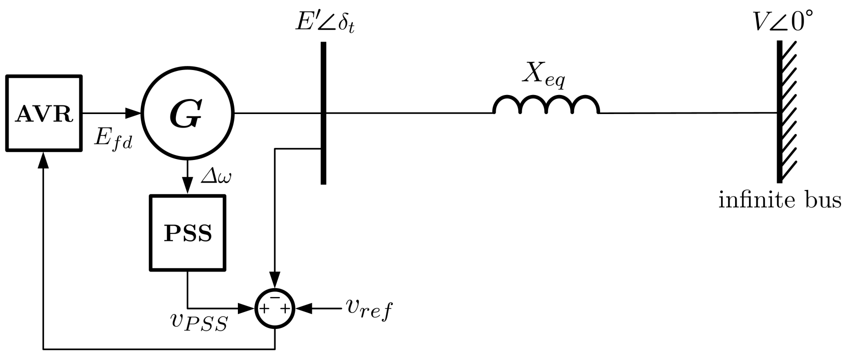


Figure 5.8: SMIB system scheme equipped with AVR and PSS, corresponding to Test Case 2.

ρ	D-EM	D-Mil	C-EM	C-Mil
0.00	−0.02849	−0.02849	−0.02864	−0.02864
0.05	−0.02848	−0.02847	−0.02863	−0.02863
0.10	−0.02843	−0.02845	−0.02860	−0.02863
0.15	−0.02845	−0.02841	−0.02857	−0.02859
0.20	−0.02832	−0.02852	−0.02867	−0.02863
0.25	−0.02815	−0.02836	−0.02852	−0.02833
0.30	−0.02837	−0.02811	−0.02828	−0.02837
0.35	−0.02827	−0.02795	−0.02811	−0.02811
0.40	−0.02734	−0.02778	−0.02797	−0.02762
0.45	−0.02722	−0.02758	−0.02775	−0.02773
0.50	−0.02676	−0.02658	−0.02674	−0.02675
0.55	−0.02702	−0.02575	−0.02590	−0.02537
0.60	−0.02508	−0.02606	−0.02620	−0.02591
0.65	−0.01250	−0.00779	−0.00781	−0.02359
0.70	−0.01016	−0.00549	−0.00550	−0.00353
0.75	−0.00412	−0.01408	−0.01417	−0.00428
0.80	−0.00503	−0.00544	−0.00546	−0.00454
0.85	−0.00505	−0.00656	−0.00658	−0.00505
0.90	−0.00372	−0.00471	−0.00475	−0.00350
0.95	−0.00503	−0.00452	−0.00454	−0.00406
1.00	−0.00353	0.00024	0.00023	−0.00314
1.05	−0.00287	−0.00267	−0.00268	−0.00114
1.10	−0.00075	−0.00159	−0.00158	−0.00278
1.15	−0.00436	−0.00259	−0.00261	−0.00136
1.20	0.00184	0.00093	0.00091	−0.00191
1.25	0.00175	−0.00149	−0.00154	0.00217
1.30	0.00149	0.00029	0.00026	0.00059
1.35	0.00038	0.00044	0.00040	0.00409
1.40	0.00870	0.00284	0.00279	0.00311
1.45	0.00338	0.00072	0.00075	0.00314
1.50	0.00409	0.00570	0.00564	0.00607
1.55	0.00644	0.00806	0.00802	0.01024
1.60	0.01014	0.00642	0.00638	0.00553
1.65	0.00797	0.01089	0.01086	0.00915
1.70	0.00724	0.00896	0.00892	0.00826
1.75	0.00828	0.00808	0.00805	0.00815
1.80	0.01366	0.00658	0.00654	0.01361
1.85	0.00776	0.00977	0.00974	0.01083
1.90	0.01068	0.01346	0.01341	0.01192
1.95	0.01537	0.01313	0.01305	0.01072
2.00	0.01248	0.01225	0.01219	0.01031

Table 5.2: Numerical results of the approximated LLE of SMIB system (5.7) corresponding to the study-case 1, computed via the four QR-based techniques.

state variables v_1, v_2, and v_s; together with their corresponding DAEs, which describe the dynamic behavior and constraints of the controllers into the SMIB system. The resulting model is a nonlinear system of strangeness-free DAEs. We use the LEs to analyze the system stability at a specific operation point in the state-space when it is subjected to small-disturbances. Using the small-signal stability assessment (SSSA), the set of DAEs that describes the dynamics of the power system is linearized around the desired operating point. The final result is a linear DAE system. A comprehensive explanation of this model, its linearization, and reduction to an underlying ODE system can be found in [50, ch. 12]. We consider a disturbance of stochastic nature entering in the exciter block of the AVR as an error of the reference signal [50], [92], by adding the stochastic variable η to v_1 in equation (5.8c). Resolving the algebraic constraints leads to the linearized system of SDEs

$$\begin{aligned}
d\Delta\delta &= \omega_s \Delta\omega dt, && (5.8a)\\
2H d\Delta\omega &= \left[-K_1\Delta\delta - K_D\Delta\omega - K_2\Delta\psi_{fd} + \Delta T_m\right] dt, && (5.8b)\\
T_3 d\Delta\psi_{fd} &= \left[-K_3K_4\Delta\delta - (1 + K_3K_6K_A)\Delta\psi_{fd}\right.\\
&\quad \left. -K_3K_A(1+\rho\eta)\Delta v_1 + K_3K_A\Delta v_s\right] dt, && (5.8c)\\
T_R d\Delta v_1 &= \left[-K_5\Delta\delta + K_6\Delta\psi_{fd} - \Delta v_1\right] dt, && (5.8d)\\
d\Delta v_2 &= \left[-K_1K_{ST}\Delta\delta - K_DK_{ST}\Delta\omega - K_2K_{ST}\Delta\psi_{fd}\right.\\
&\quad \left. -\frac{1}{T_W}\Delta v_2 + \frac{K_{ST}}{2H}\Delta T_m\right] dt, && (5.8e)\\
T_2 d\Delta v_s &= \left[-K_1K_{ST}T_1\Delta\delta - K_DK_{ST}T_1\Delta\omega - K_2K_{ST}T_1\Delta\psi_{fd}\right.\\
&\quad \left. +\left(\frac{T_1}{T_W}+1\right)\Delta v_2 - \frac{1}{T_2}\Delta v_s + \frac{K_{ST}T_1}{2H}\Delta T_m\right] dt, && (5.8f)\\
d\eta &= -\alpha\eta dt + \beta dw, && (5.8g)
\end{aligned}$$

where $\Delta\delta$, $\Delta\omega$, $\Delta\psi_{fd}$, Δv_1, Δv_2, Δv_s, and η are the state variables of the linear underlying SDE system (in the same way than Example 2 in Section 4.4.2, the subscript t has been omitted in the formulation for simplicity). Once again, the stochastic perturbation is generated via an OU SDE, and the size of the perturbation is controlled by the parameter ρ. The numerical analysis is done for the values $\omega_s = 2\pi 60$; $H = 3.0$; $K_1 = 1.591$; $K_2 = 1.50$; $K_D = 0.0$; $K_3 = 0.333$; $K_4 = 1.8$; $K_5 = -0.12$; $K_6 = 0.3$; $K_A = 200.0$; $T_R = 0.02$; $K_{ST} = 9.5$; $T_1 = 0.154$; $T_2 = 0.033$; $T_3 = 1.91$; $T_W = 1.4$; $\alpha = 1.0$; $\beta = 0.4$; $\Delta T_m = 0.0$.

Based on the analysis of Section (4.4.1), we only consider the continuous Euler-Maruyama QR method. The results of computing the LLE of the SMIB system for incremental values of the perturbation size ρ, are presented graphically in Figure 5.9. The values of the LLE when

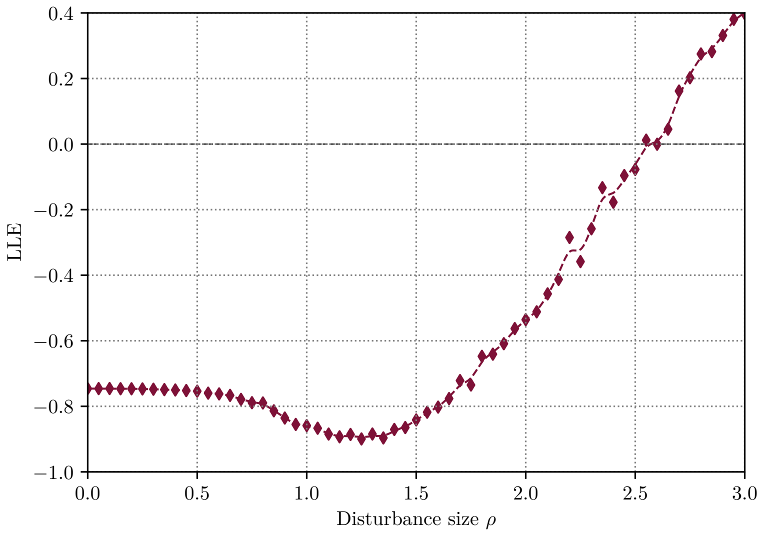


Figure 5.9: Computed LLE for the dimension 7 SMIB system of Test Case 2, considering different disturbance sizes and using the continuous Euler-Maruyama QR method.

increasing perturbation size ρ clearly mark four defined intervals. In the leftmost interval with $0.00 < \rho < 0.40$, the calculated LLE is practically constant and equal to the real part of the right-most eigenvalue from the deterministic system. In this region, there is no impact of the disturbance on the system stability. In the interval $0.40 < \rho < 1.30$ a curious situation occurs, as the size of the disturbance increases, the distance from the LLE to the positive region increases, in other words, the noise improves the stability of the system. In the interval $1.30 < \rho < 2.60$, the situation changes completely, and the LLE converges to zero. Finally, from $\rho \approx 2.60$ onwards, the system is unstable. Table 5.3 shows the numerical values of this test case.

Finally, we have evaluated the computing-times for this 7-dimensional test case. The results are shown in Figure 5.10. Although the computational cost for all method is similar for the different methods as a factor of the step sizes h and time interval $[0, T]$, the computational costs strongly increase.

Even though the present work has been oriented for testing the asymptotic stability of transmission power systems under uncertainty through the computation of LEs, as showed in the last two study-cases, the numerical QR-based techniques for computing the LEs in stochastic dynamical systems can be also a suitable assessment tool in a vast range of fields such as physics, chemistry, biology, sociology, economics, etc. Moreover, beyond its use as a tool for the asymptotic stability assessment, LEs are useful for other quantitative studies such as the characterization of synchronization or chaos.

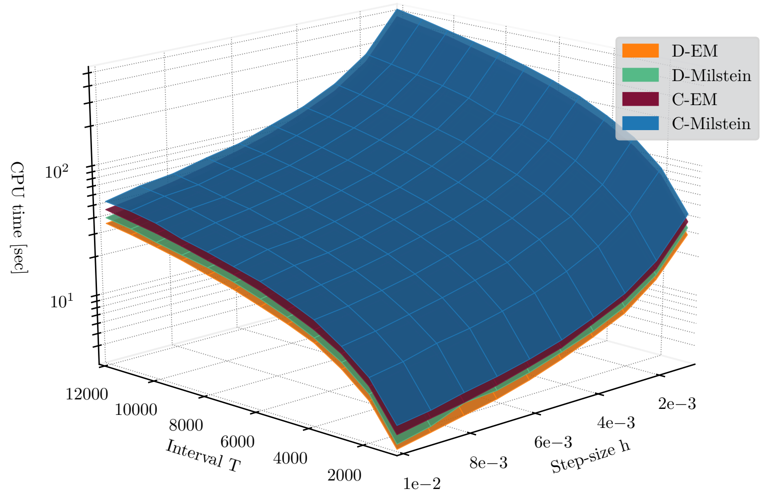


Figure 5.10: Computing-time comparison of LE calculation for the dimension 7 SMIB Test Case 2. Comparison performed for the four QR methods in a range of step sizes between $h = [1\text{e}{-2}, 1\text{e}{-3}]$ *and with* $T = [1000, 12000]$.

ρ	LLE	ρ	LLE	ρ	LLE	ρ	LLE
0.00	−0.74586	0.75	−0.78894	1.50	−0.84152	2.25	−0.35850
0.05	−0.74593	0.80	−0.78972	1.55	−0.81798	2.30	−0.25838
0.10	−0.74572	0.85	−0.81366	1.60	−0.80238	2.35	−0.13307
0.15	−0.74632	0.90	−0.83571	1.65	−0.77636	2.40	−0.17751
0.20	−0.74647	0.95	−0.85522	1.70	−0.72155	2.45	−0.09607
0.25	−0.74694	1.00	−0.85872	1.75	−0.73500	2.50	−0.07736
0.30	−0.74786	1.05	−0.86683	1.80	−0.64744	2.55	0.01194
0.35	−0.74901	1.10	−0.88458	1.85	−0.64093	2.60	−0.00056
0.40	−0.75015	1.15	−0.89293	1.90	−0.60895	2.65	0.04541
0.45	−0.75187	1.20	−0.88585	1.95	−0.56286	2.70	0.16176
0.50	−0.75370	1.25	−0.89951	2.00	−0.53564	2.75	0.20227
0.55	−0.75980	1.30	−0.88447	2.05	−0.51171	2.80	0.27489
0.60	−0.76208	1.35	−0.89645	2.10	−0.45690	2.85	0.28201
0.65	−0.76663	1.40	−0.87056	2.15	−0.41292	2.90	0.33127
0.70	−0.77893	1.45	−0.86469	2.20	−0.28496	2.95	0.38053

Table 5.3: Numerical results of the approximated LLE of SMIB system (5.8) corresponding to the study-case 2, computed via C-EM method.

CHAPTER 6

Final Discussion

6.1. Conclusions

In the present doctoral thesis, the modeling of multi-physical systems perturbed by noisy disturbances through SDAEs and their stability analysis via LEs have been studied.

We have revisited the theory of strangeness-free SDAE systems, as well as the concepts of LEs associated with the RDSs generated via such SDAEs. We have adapted and implemented stochastic versions of continuous and discrete QR-based methods to calculate approximations of the LEs, and assessed them by using Euler-Maruyama and Milstein schemes over the corresponding underlying SDEs.

The results obtained from our numerical experiments illustrate the approximations of the corresponding LE converge to degenerate random variables, i.e., the LE can be interpreted as a deterministic value. In the limit, the approximations' variance tends to zero. This finding is a quite relevant justification to avoid a Monte-Carlo-driven assessment by computing many LEs for different realizations of the stochastic variables, and rather determine the Lyapunov spectra through just a single simulation.

Both QR-based methods provide reliable results, but in general, continuous methods provide better accuracy than the discrete counterpart at the expenses of higher computational cost and higher memory requirement. Even though other discretization schemes could be carried out as well, the ergodicity property of both Milstein and EM schemes has already been proved.

Despite the accuracy limitations of the projected orthogonal methods, which are in some way mitigated by narrowing the integration stepsize, these are still a better choice over unitary integrators. Although the use of automatic unitary integrators instead could improve the precision of the computed LEs, in exchange for a substantial increase in computational cost. This represents a significant drawback in applications to high-dimension system; see [26]. However, the implementation of automatic unitary integrators and their comparison with projected orthogonal methods look like an interesting research topic for future studies, as long as we restrict the applications to small-dimension systems.

We have illustrated the QR-based methods for SMIB power system problems and shown the usefulness of the LEs as a stability indicator for the rotor angle and voltage stability analysis of power systems affected by bounded stochastic disturbances. As shown in the study-case 2, the LLE works like an index in the maximum input parameter noise the system admits before losing stability. On the other hand, in the power systems study-cases, it is possible to check the numerical methods' robustness when some positive LLEs are computed; these cases' results are coherent. Note that any numerical method for computing LEs has limited robustness in tricky situations such as positive (large) LEs. Or even in large negative LEs as is the case of discrete QR methods [26].

Regarding the contributions of this research work, it is valuable to point out that these are mainly focused on improving computational aspects to calculate reliable approximations of LEs associated with SDAE systems. Our most important proposal relies on the continuous QR-based techniques obtained after applying Itô calculus (see expressions (4.16) to (4.18)) to compute the LEs for SDAE systems with non-correlated random noises. This ansatz improves the numerical results compared to (4.15) and leads to better robustness for large time intervals. All these concepts are for the first time used for the computation of LEs and applied to the specific challenging examples arising in power system applications.

Also, to reinforce the contributions of this research work, for the sake of completeness and since the extant literature about this topic is relatively scarce, in Chapters 2 and 3, we have included a complete survey about the SDAE systems, their underlying SDEs, the RDS generated by such systems, and the existence of well-defined LEs. Moreover, Chapter 4 present a didactic explanation of the continuous and discrete QR-based methods for computing LEs. We think that the final result of our contribution may be useful for a wide audience interested in studying further advances about the computation of LEs associated to SDAEs. In a nutshell, the contribution of the thesis is indeed significantly on the numerical side and focuses on the development of computational methods to compute LEs for mid- and large-scale complex examples modeled as SDAE systems.

6.2. Future Developments

As future work, we suggest using discretization schemes for SDAEs to directly apply the numerical integration to the SDAE system. This implementation will represent an evident improvement compared to the use of SDE oriented discretization schemes, whose application is limited to strangeness-free SDAEs.

An appealing idea could be testing methods for computing the LEs based on *Singular Value Decompositions* (SVD). Since the relationship between the LEs and the evolution of the phase volume leads to a scalar optimization problem for the LLE, which allows to control the growth of the LE, the implementation of a norm preserving decomposition such as SVD and a careful comparison with QR-based methods is a choice that deserves to be assessed.

It would be of interest a combination of norm preserving decomposition techniques with model reduction, overall for the assessment of large-scale systems. Due to the high computational cost of the every-step decomposition of the fundamental solution matrix, both continuous or discrete techniques could become inapplicable in large-scale systems. Here, reducing the order of the model by using stability preserving reduction techniques is of interest.

Although the aim of this research work has been to develop reliable computational methods to compute the LEs associated to SDAEs with no correlated noises modeling power systems, as future work it would be interesting to extend our analysis to SDAEs with correlated noises including their applications to other engineering problems.

Concerning the applications to power systems and dynamical network systems, the stability assessment of large-scale study-cases is a remarkable work to be performed.

6.3. List of Publications

6.3.1. Journal Publications

- A. González, P. Fernández de Córdoba, J.C. Cortés, V. Mehrmann. *"Stability Assessment of Stochastic Differential-Algebraic Systems via Lyapunov Exponents with an Application to Power Systems"*. Mathematics 2020, 8, 1393. (doi: 10.3390/math8091393). JCR category rank (2020): 28/324 (Q1) in "Mathematics".

6.3.2. Conference Publications

- A. González, P. Fernández de Córdoba, V. Mehrmann. *"Stochastic Stability Assessment of Power Systems Modeled as SDAEs Through Lyapunov Exponents"*. Isaac Newton Institute of Mathematical Sciences (University of Cambridge). Workshop in Energy Systems: Looking Forward to 2050. Cambridge (England), May 2019. (Poster).
- A. González, P. Fernández de Córdoba, V. Mehrmann. *"Modelado y Análisis de Estabilidad Dinámica de Sistemas de Redes de Enería a través de DAEs"*. (Modeling and Dynamic Stability Analysis of Network Energy Systems Through DAEs). 3ra Conferencia de Matemáticos Ecuatorianos en París (CONMATE-P). Paris (France), April 2017. (Invited speaker).
- A. González, P. Fernández de Córdoba, V. Mehrmann. *"Dynamic Stability Analysis of Electric Power Systems Under Uncertainties"*. Workshop on Mathematical solutions for Industry: Success stories and perspectives. Madrid (Spain), October 2016. (Poster).
- A. González, P. Fernández de Córdoba, V. Mehrmann. *"New Techniques of Dynamic Modelling and Stability Analysis for Stochastic Network Systems Applied to Energy Systems"*. Workshop on Optimization Challenges in the Evolution of Energy Networks to Smart Grids. Coimbra (Portugal), October 2016. (Oral presentation).

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