FACTORIZATION THROUGH LORENTZ SPACES FOR OPERATORS ACTING IN BANACH FUNCTION SPACES

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Abstract. We show a factorization through Lorentz spaces for Banach-space-valued operators defined in Banach function spaces. Although our results are inspired in the classical factorization theorem for operators from L^s-spaces through Lorentz spaces $L^{q,1}$ due to Pisier, our arguments are different and essentially connected with Maurey's theorem for operators that factor through L^p-spaces. As a consequence, we obtain a new characterization of Lorentz $L^{q,1}$-spaces in terms of lattice geometric properties, in the line of the (isomorphic) description of L^p-spaces as the unique ones that are p-convex and p-concave.

1. Introduction

A well-known relevant fact in the theory of Banach lattices establishes that a Banach function space over a measure μ that is both p-convex and p-concave, is isomorphic to $L^p(gd\mu)$ for a certain measurable function g. This fact is related to some classical results on Banach lattices, and an easy proof can be given by applying the so called Maurey-Rosenthal factorization of operators (see [3, 4, 9]) to the identity map in the space (see [7, 9, 11, 14]). As in the case of L^p-spaces over a measure μ, in this paper we are concerned with the description of the lattice-geometric-type properties that characterize Lorentz spaces. The main results on factorization of operators through Lorentz spaces are due to Pisier, and can be found in [13]. Some generalizations for the case of sublinear operators are also known (see [1]). More results in this direction can be found in the paper [6] of Kalton and Montgomery-Smith although the underlying techniques in it are essentially different, since this paper is based on some domination results among scalar valued capacities and measures. In our case, we use for solving the problem a separation argument that is also applied in some of the papers mentioned above. Some ideas of Mastyło and Szwedek presented in [10] has also influenced the present paper.

The starting point of our analysis is given by the results by Pisier in [13], in which there is a characterization of the operators from an L^s-space that factor through a Lorentz space $L^{q,1}$ ([13, Theorem 2.1]). We use a different procedure for showing a different factorization theorem for operators acting in general order continuous Banach function spaces. Concretely, we show that a factorization through a Lorentz space $L^{q,1}$ is equivalent to a concavity-type inequality, different than the one proposed by Pisier. This kind of arguments have been used

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in [9] for the case of Orlicz spaces; some related ideas involving Lorentz spaces can also be found in [4].

In the final part of the paper, using the previously obtained factorization and asking for an additional convexity-type requirement, we give the following characterization of the Lorentz spaces $L^{q,1}$ (Theorem 4.3).

A Banach function space $X(\mu)$ is isomorphic to a Lorentz space $L^{q,1}(\lambda)$ via the identification map —where μ and λ are equivalent— if there are a Banach function space $Z(\lambda)$ over λ and constants $K, Q > 0$ such that

(1) for every $f_1, ..., f_n \in L^\infty(\mu)$,

$$\Big(\sum_{i=1}^n \|f_i\|_{X(\mu)}^q \Big)^{1/q} \leq K \Big\| \sum_{i=1}^n |f_i| \|f_i\|_{L^\infty(\mu)}^{q-1} \Big\|_{Z(\lambda)}^{1/q},$$

and

(2) there is an L^1-space L associated to λ such that for all $f \in L^\infty(\lambda)$ there are $f_1, ..., f_n \in L^\infty(\lambda)$ such that $\sum_{i=1}^n f_i = f$ and

$$\Big\| \sum_{i=1}^n \, |f_i| \, \big(\|f_i\|_{L^\infty} / \|f_i\|_L \big)^{1/q'} \, \Big\|_L \leq Q \|f\|_{X(\mu)}.$$

Actually, the space $Z(\lambda)$ in (i) can be chosen to be $X(\mu)$. Thus, both inequalities appearing in this result play the role of q-concavity —the first one— and q-convexity —the second one— in the characterization of L^q-spaces: they are the lattice-geometric counterpart for the case of Lorentz spaces.

2. Preliminaries

Through all the paper (Ω, Σ, μ) will be a *finite* measure space. We say that $(X(\mu), \|\cdot\|_{X(\mu)})$ is a Banach function space over μ (Köthe function space in the terminology of [8, p.28]) if it is a Banach lattice with the μ-a.e order that is an ideal of $L^0(\mu)$ —the space of μ-measurable functions—, satisfying that

$$L^\infty(\mu) \subseteq X(\mu) \subseteq L^1(\mu).$$

Note that since all the spaces involved are Banach lattices these inclusions are continuous. We write $X(\mu)^+$ for the positive cone of the lattice $X(\mu)$. If λ is other measure defined on the measurable space (Ω, Σ), we say that λ is equivalent to μ —and we write $\lambda \sim \mu$— if they have the same null sets.

Some fundamental geometric properties for Banach function spaces —that are defined in fact for general Banach lattices— are the q-concavity and the q-convexity. A Banach space valued operator $T : X(\mu) \to E$ is *q-concave* if there exists $C > 0$ such that

$$\Big(\sum_{i=1}^n \|T(f_i)\|_E^q \Big)^{1/q} \leq C \Big\| \Big(\sum_{i=1}^n |f_i|^q \Big)^{1/q} \Big\|_X$$

for $f_1, \ldots, f_n \in X(\mu)$. The Banach function space $X(\mu)$ is p-convex if there is a constant Q such that

$$\Big\| \Big(\sum_{i=1}^n |f_i|^p \Big)^{1/p} \Big\|_X \leq Q \Big(\sum_{i=1}^n \|f_i\|_X^p \Big)^{1/p}$$

for $x_1, \dots, x_n \in X(\mu)$. The reader can find all the information that would be needed about Banach lattices and Banach function spaces in [8, Ch. 1.d] or [12, Ch.2].

The Köthe dual —or associate space— $X(\mu)'$ of a Banach function space $X(\mu)$ is the subspace of the dual space $X(\mu)^*$ defined by the functionals that can be represented by integrals. This means that are these elements of the dual space for which there is a measurable (class of) function(s) $g \in L^0(\mu)$ such that $f \mapsto \int fg\, d\mu$ for all $f \in X(\mu)$. We will say that $X(\mu)$ is *order continuous* if for every $f_n \in X$ such that $0 \leq f_n \uparrow f \in X(\mu)$ μ-a.e., we have that $f_n \to f$ in norm. It is well known that a Banach function space $X(\mu)$ is order continuous if and only if the dual space $X(\mu)^*$ coincides with the Köthe dual $X(\mu)'$ ([8, p.28]).

Let $1 \leq p$. The p-th power $X(\mu)_{[p]}$ of a Banach function space $X(\mu)$ is defined as the set of functions

$$X(\mu)_{[p]} := \{f \in L^0(\mu) : |f|^{1/p} \in X(\mu)\}.$$

It is a quasi-Banach space with the quasi-norm $\| \cdot \|_{X(\mu)_{[p]}} := \| | \cdot |^{1/p} \|^p_{X(\mu)}$ that is in fact equivalent to a norm if $X(\mu)$ is p-convex (see [12, Ch.2]).

We will consider classical Lorentz function spaces, that are a particular example of (quasi)-Banach function spaces. Let $1 \leq q < \infty$. The norm in the Lorentz space $L^{q,1}(\mu)$ is given by the expression

$$\|f\|_{q,1} := \int_0^{\mu(\Omega)} t^{(1/q)-1} f^*(t)\, dt, \quad f \in L^{q,1}(\mu),$$

where $f^* : [0,1] \to \mathbb{R}$ is the decreasing rearrangement of f defined as

$$t \mapsto \inf\{s > 0 : \mu(|f| > s) \leq t\}.$$

For a simple function $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$, where A_i, $i = 1, ..., n$ are disjoint measurable sets and $|\alpha_1| \geq |\alpha_2| \geq ... \geq |\alpha_n|$, the Lorentz $(q,1)$-norm can be computed by

$$\|f\|_{q,1} := q \sum_{i=1}^n |\alpha_i| (t_i^{1/q} - t_{i-1}^{1/q}),$$

where $t_i := \mu(\cup_{k=1}^i A_k)$, $i = 1, ..., n$ (see for example the proof of Theorem 10.9 in [5]; see also [6, §3]).

The spaces $L^{q,1}(\mu)$ are not p-convex for any $p > 1$ (see [3, p.159]). There is a well-known representation formula using real interpolation spaces (see [2]) for the family of classical Lorentz spaces of functions on the Lebesgue measure space $([0,1], \mathcal{B}([0,1]), \mu)$. Suppose that $1 \leq p_0, p_1$ such that $p_0 \neq p_1$, and suppose that $1 \leq r$ and $0 < \sigma < 1$. We define p by the formula $1/p = (1-\sigma)/p_0 + \sigma/p_1$. Then, for every $1 \leq q_0, q_1$, we have $(L^{p_0,q_0}, L^{p_1,q_1})_{\sigma,r} = L^{p,r}$. Although we are not explicitly using this formula in the present paper, it suggests how interpolated spaces are related for the case $q_0 = q_1 = r = 1$: real interpolation of spaces with parameters $\sigma, 1$ gives a Lorentz space with second index equal to 1 too.

3. Factorizations of operators acting in Banach function spaces through Lorentz spaces

If (Ω, Σ, μ) is a finite measure space, let $X(\mu)$ and $Y(\mu)$ be two Banach function spaces which satisfy that simple functions $S(\mu)$ are dense in them. Note that this happens in particular if they are order continuous.

Lemma 3.1. *Let $1 \le s \le q$. Let μ be a finite measure. Let $X(\mu)$ be an order continuous s-convex Banach function space and $T : X(\mu) \to E$ an operator. Assume also that $Y(\mu)$ is a Banach function space included in $X(\mu)$ in which simple functions are dense. The following assertions are equivalent.*

(i) *For $f_1, ..., f_n \in Y(\mu)$,*

$$\Big(\sum_{i=1}^n \|T(f_i)\|^q \Big)^{1/q} \le K \Big\| \big(\sum_{i=1}^n |f_i| \|f_i\|_{Y(\mu)}^{q-1} \big)^{1/s} \Big\|_{X(\mu)}^{s/q}.$$

(ii) *There is a non-negative function $g_0 \in B_{(X(\mu)_{[s]})'}$ such that*

$$\|T(f)\| \le K \,\big(\int_\Omega |f|\, g_0 \, d\mu \big)^{1/q} \;\; \|f\|_{Y(\mu)}^{1/q'}, \quad f \in X(\mu) \cap Y(\mu).$$

Moreover, the function g_0 appearing in (ii) *can be assumed to satisfy $g_0 > 0$ by changing the constant K by $K + \varepsilon$ if necessary.*

Proof. (i) $\Rightarrow$ (ii) First note that, since the measure μ is finite and $1 \le s$ we have that $X(\mu) \subseteq X(\mu)_{[s]}$ (see [12, Lemma 2.21]). A standard separation argument gives the result (see for example the proof of Theorem 1 in [9]). Indeed, since $X(\mu)$ is s-convex, we can consider the s-th power $X(\mu)_{[s]}$, which is a Banach function space (see [12, Proposition 2.23]) after the convexification of the quasi-norm $\|\cdot\|_{X(\mu)_{[s]}} := \|\, |\cdot|^{1/s} \,\|_{X(\mu)}^s$, that gives an equivalent norm. In fact, it is a norm is the s-convexity constant of the space is equal to 1. For fixed functions $f_1, ..., f_n \in L^\infty(\mu)$, the inequality in (i) can then be rewritten as

$$\Big(\sum_{i=1}^n \|T(f_i)\|^q \Big)^{1/q} \le K \Big\| \sum_{i=1}^n |f_i| \|f_i\|_{Y(\mu)}^{q-1} \Big\|_{X(\mu)_{[s]}}^{1/q}. \tag{3.1}$$

Note that, since $X(\mu)$ is order continuous, $X(\mu)_{[s]}$ is so and then the dual of this space is defined by integrable functions. Consider the functions $\phi : B_{(X(\mu)_{[s]})'} \to \mathbb{R}$ given by

$$\phi(g) := \sum_{i=1}^n \|T(f_i)\|^q - K^q \int_\Omega \sum_{i=1}^n |f_i| \, \|f_i\|_{Y(\mu)}^{q-1} \, g \, d\mu.$$

All the requirements for applying Ky Fan's Lemma to the family of all such functions ϕ are satisfied, as can be easily checked. The inequality (3.1) is used for proving, using Hahn-Banach Theorem, that for every ϕ there is a function g_ϕ such that $\phi(g_\phi) \le 0$. We can assume that the final function g_0 provided by Ky Fan's Lemma is non-negative

The converse implication (ii) $\Rightarrow$ (i) is given by a direct computation.

Finally, note that all the computations still work —also the ones for the converse implication— if we change g_0 by the normalization of the function $0 < \varepsilon g_0 + (1-\varepsilon)\chi_\Omega$, for ε as small as we want. Note also that the new function is still in $(X(\mu)_{[s]})'$ due to the inclusion $X(\mu)_{[s]} \subseteq L^1(\mu)$ –and then $L^\infty(\mu) \subseteq (X(\mu)_{[s]})'$, that holds by the s-convexity of $X(\mu)$. □

Remark 3.2. Let $Y(\mu)$ be a Banach function space in which simple functions are dense and such that $Y(\mu) \subseteq X(\mu)$, with the norm of the inclusion being equal to 1. Let $0 \le g \in B_{X(\mu)'}$ and $0 \le \sigma < 1$. Consider the functional $\Psi_{1,\sigma,g} : Y(\mu) \to \mathbb{R}^+$ defined by

$$\Psi_{1,\sigma,g}(f) := \Big(\int_\Omega |f| g d\mu \Big)^\sigma \cdot \|f\|_{Y(\mu)}^{1-\sigma}, \quad f \in Y(\mu),$$

and consider its convexification, that gives the seminorm

$$\|f\|_{1,\sigma,g} := \inf\Big\{ \sum_{i=1}^{n} \Psi_{1,\sigma,g}(f_i) : \sum_{i=1}^{n} f_i = f \Big\} \tag{3.2}$$

$$= \inf\Big\{ \sum_{i=1}^{n} \Big(\int_\Omega |f_i| g d\mu \Big)^{\sigma} \cdot \|f_i\|_{Y(\mu)}^{1-\sigma} : \sum_{i=1}^{n} f_i = f \Big\},$$

$f, f_1, ..., f_n \in Y(\mu)$. Note that

$$\|f\|_{L^1(gd\mu)} \le \|f\|_{1,\sigma,g} \le \|f\|_{Y(\mu)}, \quad f \in Y(\mu),$$

and so $\|\cdot\|_{1,\sigma,g}$ is a norm whenever $g > 0$. Note also that due to the inclusion in $L^1(gd\mu)$ of the normed space $(Y(\mu), \|\cdot\|_{1,\sigma,g})$, we can say that the completion of this space is again a function space. Indeed, the convergent limit of a sequence of functions in it converges to a measurable function f also in $L^1(gd\mu)$, and so there is a subsequence that converges μ-a.e. to f, what allows to identify the limit with the function f. Let us write $\{Y(\mu), L^1(\mu)\}_{1,\sigma,g}$ for this space ($g > 0$) with the norm $\|\cdot\|_{1,\sigma,g}$. Note that simple functions are dense in it.

By the construction, the space $\{Y(\mu), L^1(\mu)\}_{1,\sigma,g}$ can be identified with the real interpolation space $(Y(\mu), L^1(gd\mu))_{\sigma,1}$. However, due to the simplicity of the description given by the formula (3.2), we prefer to use it for the computations of this section. In the next section the description as a real interpolation space will be relevant, since this will be the link with the Lorentz spaces $L^{q,1}(\mu)$.

Following the same ideas, we can consider the following general definition: if $Y(\mu)$ and $Z(\mu)$ are Banach function spaces and $Y(\mu) \subseteq Z(\mu)$, we define the space $\{Y(\mu), Z(\mu)\}_{1,\sigma}$ as the Banach function space given by the completion of the normed space $Y(\mu)$ with the norm

$$\|f\|_{\{Y(\mu),Z(\mu)\}_{1,\sigma}} := \inf\Big\{ \sum_{i=1}^{n} \|f_i\|_{Z(\mu)}^{\sigma} \cdot \|f_i\|_{Y(\mu)}^{1-\sigma} : \sum_{i=1}^{n} f_i = f \Big\}, \quad f_i, f \in Y(\mu).$$

Proposition 3.3. *Let $1 \le s \le q$. Let μ be a finite measure and $Y(\mu)$ a Banach function space over μ in which simple functions are dense. Let $X(\mu)$ be an order continuous s-convex Banach function space containing $Y(\mu)$ (norm of the inclusion equal to 1) and let $T : X(\mu) \to E$ be an operator. The following statements are equivalent.*

(i) *For $f_1, ..., f_n \in Y(\mu)$,*

$$\Big(\sum_{i=1}^{n} \|T(f_i)\|^q \Big)^{1/q} \le K \Big\| \Big(\sum_{i=1}^{n} |f_i| \|f_i\|_{Y(\mu)}^{q-1} \Big)^{1/s} \Big\|_{X(\mu)}^{s/q}.$$

(ii) *There is a function $0 < g_0 \in B_{(X(\mu)_{[s]})'}$ such that $T : \{Y(\mu), X(\mu)_{[s]}\}_{1,\sigma} \to E$ is well defined and factors through the space $\{Y(\mu), L^1(\mu)\}_{1,\sigma,g_0}$ as*

$$\begin{array}{ccc} \{Y(\mu), X(\mu)_{[s]}\}_{1,\sigma} & \xrightarrow{T} & E, \\ {\scriptstyle i}\downarrow & \nearrow_{T_0} & \\ \{Y(\mu), L^1(\mu)\}_{1,\sigma,g_0} & & \end{array}$$

where $\sigma = 1/q$ and T_0 is a continuous operator.

Proof. For (i) $\Rightarrow$ (ii), note first that by [12, Lemma 2.21] we have that $X(\mu) \subseteq X(\mu)_{[s]}$, and $X(\mu)$ is dense in $X(\mu)_{[s]}$ by the order continuity of these spaces. By Lemma 3.1 we have that the inequality in (i) gives a positive function g_0 such that the inequality

$$\|T(f)\| \leq K \Big(\int_\Omega |f| \, g_0 \, d\mu \Big)^{1/q} \ \ \|f\|_{Y(\mu)}^{1/q'}$$

holds for all $f \in Y(\mu)$ "up to an ε" (that is, changing K by $K+\varepsilon$ if necessary). Taking into account Remark 3.2, this gives

$$\|T(f)\|_E \leq K \|f\|_{1,\sigma,g}, \quad f \in Y(\mu).$$

Since $X(\mu) \subseteq X(\mu)_{[s]}$ and $g_0 \in (X(\mu)_{[s]})'$, we have that for $\sigma = 1/q$

$$\|f\|_{1,\sigma,g_0} \leq \|g_0\|^{\sigma}_{(X(\mu)_{[s]})'} \, \|f\|^{\sigma}_{X(\mu)_{[s]}} \, \|f\|^{1-\sigma}_{Y(\mu)}, \quad f \in Y(\mu).$$

Using the density of the space $Y(\mu)$ in $X(\mu)_{[s]}$, and convexifying this inequality, we get

$$\|f\|_{1,\sigma,g_0} \leq \|g_0\|^{\sigma}_{(X(\mu)_{[s]})'} \, \|f\|_{\{Y(\mu), X(\mu)_{[s]}\}_{1,\sigma}}$$

for all $f \in \{Y(\mu), X(\mu)_{[s]}\}_{1,\sigma}$. Note that this implies in particular that T is well-defined as an operator from $\{Y(\mu), X(\mu)_{[s]}\}_{1,\sigma}$ to E.

The converse is given by a straightforward calculation using the continuity of the factorization and the density of $Y(\mu)$ in all the spaces involved. □

To extreme cases are relevant in the Proposition 3.3: the case $Y(\mu) = X(\mu)$ and the case $Y(\mu) = L^\infty(\mu)$. The first one gives the following results. The second one will be analyzed in the next section.

Corollary 3.4. *Let $1 \leq s \leq q$, μ a finite measure and let $X(\mu)$ be an order continuous s-convex Banach function space. Let $T : X(\mu) \to E$ be an operator. The following statements are equivalent.*

(i) *For $f_1, ..., f_n \in X(\mu)$,*

$$\Big(\sum_{i=1}^n \|T(f_i)\|^q \Big)^{1/q} \leq K \Big\| \big(\sum_{i=1}^n |f_i| \|f_i\|^{q-1}_{X(\mu)} \big)^{1/s} \Big\|^{s/q}_{X(\mu)}.$$

(ii) *There is a function $0 < g_0 \in B_{(X(\mu)_{[s]})'}$ such that T factors as*

$$\begin{array}{ccc} X(\mu) & \xrightarrow{\quad T \quad} & E, \\ {\scriptstyle i}\downarrow & & \uparrow{\scriptstyle T_0} \\ \{X(\mu), X(\mu)_{[s]}\}_{1,\sigma} & \xrightarrow[i]{} & \{X(\mu), L^1(\mu)\}_{1,\sigma,g_0} \end{array}$$

where $\sigma = 1/q$, the "i"s are inclusion maps and T_0 is a continuous operator.

Corollary 3.5. *Let $1 \leq q$, μ a finite measure and let $X(\mu)$ be an order continuous Banach function space. Let $T : X(\mu) \to E$ be an operator. The following statements are equivalent.*

(i) *For $f_1, ..., f_n \in X(\mu)$,*

$$\Big(\sum_{i=1}^n \|T(f_i)\|^q \Big)^{1/q} \leq K \Big\| \sum_{i=1}^n |f_i| \|f_i\|^{q-1}_{X(\mu)} \Big\|^{1/q}_{X(\mu)}.$$

(ii) *There is a function* $0 < g_0 \in B_{X(\mu)'}$ *such that* T *factors as*

$$\begin{array}{ccc} X(\mu) & \xrightarrow{\quad T \quad} & E, \\ {\scriptstyle i}\downarrow & \nearrow {\scriptstyle T_0} & \\ \{X(\mu), L^1(\mu)\}_{1,1/q,g_0} & & \end{array}$$

where i *is an inclusion map and* T_0 *is a continuous operator.*

Example 3.6. Let $1 < q$ and $X(\mu) = Y(\mu) = L^q(\mu)$. Then we have that an operator $T : L^q(\mu) \to E$ that satisfy the inequalities

$$\Big(\sum_{i=1}^{n} \|T(f_i)\|^q \Big)^{1/q} \le K \Big\| \sum_{i=1}^{n} |f_i| \|f_i\|_{L^q(\mu)}^{q-1} \Big\|_{L^q(\mu)}^{1/q}$$

$$= K \Big(\int_\Omega \Big[\sum_{i=1}^{n} |f_i| \cdot \big(\int_\Omega |f_i|^q d\mu \big)^{1/q'} \Big]^q d\mu \Big)^{1/q},$$

$f_1, ..., f_n \in X(\mu)$, factors through the space $\{L^q(\mu), L^1(\mu)\}_{1,1/q,g_0}$ for a certain $g_0 \in L^{q'}(\mu)$. Using the well-known representation formulas for the real interpolation of L^p-spaces, we obtain that this space coincides with the real interpolation space $(L^q(\mu), L^1(g_0\, d\mu))_{1/q,1}$. For example, if μ is Lebesgue measure in $[0,1]$ and $g_0 = \chi_{[0,1]}$, we have that the factorization space is the Lorentz space $L^{p,1}[0,1]$, for $p = (q' + q)/(q' + 1)$.

This inequality is different —but of course equivalent— to the one given in Theorem 2.1 of the paper [13] of Pisier.

Example 3.7. Suppose now that $Y(\mu) = X(\mu)$ is an order continuous Banach function space, and consider the identity map $T = i : X(\mu) \to X(\mu)$. Suppose that the inequalities

$$\Big(\sum_{i=1}^{n} \|f_i\|^q \Big)^{1/q} \le K \Big\| \sum_{i=1}^{n} |f_i| \|f_i\|_{X(\mu)}^{q-1} \Big\|_{X(\mu)}^{1/q},$$

$f_1, ..., f_n \in X(\mu)$, hold for each finite set of functions. They are equivalent, by Corollary 3.5, to a factorization through an interpolation space. The continuity of the first arrow in this scheme gives the inequality

$$\|f\|_{X(\mu)} \le K_1 \Big(\int_\Omega |f|\, g\, d\mu \Big)^{1/q} \|f\|_{X(\mu)}^{1/q'}, \quad f \in X(\mu),$$

where $g \in B_{(X(\mu))'}$. We then have $\int_\Omega |f|\, g\, d\mu \le \|g\|_{(X(\mu))'} \|f\|_{X(\mu)} \le \|f\|_{X(\mu)}$, and so

$$\|f\|_{X(\mu)} \le K_1 \Big(\int_\Omega |f|\, g\, d\mu \Big)^{1/q} \|f\|_{X(\mu)}^{1/q'} \le K_1 \|f\|_{X(\mu)}^{1/q} \|f\|_{X(\mu)}^{1/q'} = K_1 \|f\|_{X(\mu)},$$

$f \in X(\mu)$. Therefore,

$$\|f\|_{X(\mu)} \le K_1^q \int_\Omega |f|\, g\, d\mu \le K_1^q \|f\|_{X(\mu)}, \quad f \in X(\mu).$$

This gives that $X(\mu) = L^1(g d\mu)$ isomorphically. Reading the same inequalities other way round we get that the converse is also true. Note also that a direct calculation on the original inequality shows that this requirement is equivalent to 1-concavity.

4. Geometric characterization of the Lorentz spaces $L^{q,1}$

A fundamental result in the theory of Banach spaces is the characterization of the L^p-spaces in terms of the geometric inequality that it satisfies. Having the roots in the classical theory developed by Kakutani, and due to more recent contributions by Krivine, Rosenthal, Maurey and Reisner, among others, this characterization can be written as follows: *if $X(\mu)$ is a Banach function space such that it is q-convex and q-concave, then $X(\mu)$ is the space $L^q(g\,d\mu)$, where g is a measurable function such that $|f|^q g \in L^1(\mu)$ for all $f \in X(\mu)$.*

In this section we show which are the geometric inequalities that provide the equivalent result for the case of the Lorentz function spaces $L^{q,1}(\mu)$. In order to do it we use the results previously obtained for the case $Y(\mu) = L^\infty(\mu)$ in Proposition 3.3. After Section 3, the attentive reader has already noticed the type of inequalities —extending the notions of q-convexity and q-concavity— that may occur in this context. However, we will need to introduce stronger versions of the properties provided in the previous section for giving a complete characterization of Lorentz spaces by means of lattice-geometric inequalities.

Throughout the section note that we are just proving the results for functions in $L^\infty(\mu)$. The reason is that we are always assuming that simple functions are dense in all the spaces appearing in it: order continuous Banach function spaces and L^∞-spaces.

Lemma 4.1. *Let $X(\mu)$ be an order continuous Banach function space over the finite measure μ. Consider another finite measure λ such that $\lambda \sim \mu$ and another Banach function space $Z(\lambda)$. Let $T : X(\mu) \to E$ be an operator. The following assertions are equivalent.*

(i) *For $f_1, ..., f_n \in L^\infty(\mu)$,*

$$\Big(\sum_{i=1}^n \|T(f_i)\|^q \Big)^{1/q} \le K \Big\| \sum_{i=1}^n |f_i| \|f_i\|_{L^\infty(\mu)}^{q-1} \Big\|_{Z(\lambda)}^{1/q}. \tag{4.1}$$

(ii) *There is a function $g_0 \in B_{(Z(\mu))'}$ such that for each $f \in L^\infty(\mu)$,*

$$\|T(f)\| \le K \Big(\int_\Omega |f| g_0 d\lambda \Big)^{1/q} \|f\|_{L^\infty(\mu)}^{1/q'}.$$

Proof. Note that $L^\infty(\mu) = L^\infty(\lambda)$, and so $L^\infty(\mu) \subseteq Z(\lambda)$ due to the fact that $Z(\lambda)$ is a Banach function space. The same proof that works in the case of Lemma 3.1 —for $s = 1$, $Y(\mu) = L^\infty(\mu)$ and changing $B_{(X(\mu))'}$ by $B_{(Z(\mu))'}$— is also valid in this case. □

Remark 4.2. If $X(\mu)$ is an order continuous Banach function space, consider the identity map $i : X(\mu) \to X(\mu)$ and suppose that (4.1) holds for it. Let us show that in this case the (continuous) inclusion $L^{q,1}(\lambda) \hookrightarrow X(\mu)$ holds, where $\lambda(A) = \int_A g_0 d\mu$, $A \in \Sigma$.

First notice that a direct application of Lemma 4.1 for the identity map and the convexity of the norm of $X(\mu)$ give the inclusion of the space

$$\{L^\infty(\mu), L^1(\mu)\}_{1,1/q,g_0}$$

in $X(\mu)$. This is a consequence of the fact that simple functions are dense in both spaces, and for such a function $\|f\|_{X(\mu)} \le K\|f\|_{1,1/q,g_0}$. Note that $g_0 > 0$, since otherwise we should find a measurable set A_0 such that $\mu(A_0) > 0$ and $\int_{A_0} g_0 d\mu = 0$. This would give by the domination that for a non-null function $f\chi_{A_0}$, $\int_\Omega f\chi_{A_0} g_0 d\mu = 0$ and so $\|f\chi_{A_0}\|_{X(\mu)} = 0$, a contradiction. This gives the injectivity of the identification map.

Now we just need to show that the domination appearing in Lemma 4.1 —equivalently, the domination by the norm of $\{L^\infty(\mu), L^1(\mu)\}_{1,1/q,g_0}$— implies the domination by $L^{q,1}(\lambda)$. It is just a direct calculation; we follow the one given in the proof of Theorem 10.9 in [5]. First note that, due to the fact that $g_0 > 0$ we have that $L^\infty(\mu) = L^\infty(g_0 d\mu)$. Recall that for disjoint sets $A_i \in \Sigma$, $i = 1, ..., n$, and scalars $|\alpha_1| \geq |\alpha_2| \geq ... \geq |\alpha_n|$, the norm of the simple function $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ in the the Lorentz $(q,1)$-norm is given by

$$\|f\|_{q,1} := q \sum_{i=1}^n |\alpha_i| (t_i^{1/q} - t_{i-1}^{1/q}),$$

where $t_i := \lambda(\cup_{k=1}^i A_k)$, $i = 1, ..., n$. Note also that we can write a representation of f as

$$f = \sum_{i=1}^{n-1} \Big((|\alpha_i| - |\alpha_{i-1}|) \big(\sum_{j \leq i} (\alpha_j / |\alpha_j|) \chi_{A_j} \big) \Big) + |\alpha_n| \big(\sum_{j=1}^n (\alpha_j / |\alpha_j|) \chi_{A_j} \big).$$

Together with the triangle inequality for the norm $\|\cdot\|_{X(\mu)}$ and using the domination given by Lemma 4.1, it can be easily seen that there is a constant $K > 0$ such that

$$\|f\|_{X(\mu)} \leq K \Big(\big(\sum_{i \leq n-1} (|\alpha_i| - |\alpha_{i+1}|) t_i^{1/q} \big) + |\alpha_n| t_n^{1/q} \Big)$$

$$= K \Big(\sum_{i=1}^n |\alpha_i| (t_i^{1/q} - t_{i-1}^{1/q}) \Big) = \frac{K}{q} \|f\|_{L^{q,1}(\lambda)}.$$

Since this holds for simple functions, we get the result.

Note that we have shown in particular that if $g_0 > 0$, then

$$\|f\|_{\{L^\infty(\mu), L^1(\mu)\}_{1,1/q,g_0}} \leq \|f\|_{L^{q,1}(\lambda)}$$

for all functions in $L^{q,1}(\lambda)$. The converse inequality also holds, since an straightforward calculation shows that there is a constant $c > 0$ such that for each function $f \in L^\infty(g_0 d\mu)$,

$$\|f\|_{L^{q,1}(\lambda)} \leq c \|f\|_{L^1(\lambda)}^{1/q} \|f\|_{L^\infty(\lambda)}^{1/q'}$$

(see [5, p.205]).

Let us consider now the converse geometric property, in order to close the diagram. It is related to the fact that $X(\mu)$ must be included in the Lorentz space associated to λ —a measure equivalent to μ—. Concretely, it must exist a function space $Z(\lambda)$ and a constant $Q > 0$ such that for every $f \in L^\infty(\lambda)$ there is a decomposition $f = \sum_{i=1}^n f_i$ in $L^\infty(\lambda)$ which satisfies

$$\sum_{i=1}^n \|f_i\|_{Z(\lambda)}^{1/q} \|f_i\|_{L^\infty(\lambda)}^{1/q'} \leq Q \, \|f\|_{X(\mu)}.$$

In the case that $Z(\lambda)$ is an L^1-space, we obtain the inequality (4.3) below, that can be understood as a dual version of the geometric inequality (4.1).

Theorem 4.3. *Let $X(\mu)$ be an order continuous Banach function space over the finite measure μ. The following assertions are equivalent.*

(i) *There is an order continuous Banach function space* $Z(\lambda)$ *over a finite measure* $\lambda \sim \mu$ *such that for every* $f_1, ..., f_n \in L^\infty(\mu)$,

$$\Big(\sum_{i=1}^n \|f_i\|_{X(\mu)}^q \Big)^{1/q} \leq K \Big\| \sum_{i=1}^n |f_i| \|f_i\|_{L^\infty(\mu)}^{q-1} \Big\|_{Z(\lambda)}^{1/q}, \tag{4.2}$$

and

$$\inf \Big\{ \sum_{i=1}^n \|f_i\|_{Z(\lambda)}^{1/q} \|f_i\|_{L^\infty(\mu)}^{1/q'} : \sum_{i=1}^n f_i = f \Big\} \leq Q \|f\|_{X(\mu)}, \quad f_i, f \in L^\infty(\mu)$$

for two fixed positive constants $K, Q > 0$.

(ii) *There is an* L^1*-space* L *over a finite measure* $\lambda \sim \mu$ *such that for every* $f_1, ..., f_n \in L^\infty(\mu)$,

$$\Big(\sum_{i=1}^n \|f_i\|_{X(\mu)}^q \Big)^{1/q} \leq K \Big\| \sum_{i=1}^n |f_i| \|f_i\|_{L^\infty(\mu)}^{q-1} \Big\|_{L}^{1/q},$$

and for all $f \in L^\infty(\lambda)$ *there are* $f_1, ..., f_n \in L^\infty(\lambda)$ *such that* $\sum_{i=1}^n f_i = f$ *and*

$$\Big\| \sum_{i=1}^n \; |f_i| \; \big(\|f_i\|_{L^\infty} / \|f_i\|_L \big)^{1/q'} \; \Big\|_L \leq Q \|f\|_{X(\mu)}, \tag{4.3}$$

for two fixed positive constants $K, Q > 0$.

(iii) *There is a function* $g_0 > 0$ *such that* $X(\mu)$ *and* $L^{q,1}(g_0 d\lambda)$ *are isomorphic.*

Proof. For (i) $\Rightarrow$ (ii), use Lemma 4.1 for obtaining a function $0 < g \in B_{(Z(\lambda))'}$ such that for all $f \in L^\infty(\mu)$,

$$\|f\|_{X(\mu)} \leq K \Big(\int_\Omega |f| g d\lambda \Big)^{1/q} \|f\|_{L^\infty(\mu)}^{1/q'}.$$

This clearly gives the first inequality in (i). For the second one, fix $Q < Q'$. Take a function $f \in L^\infty(\mu)$ and find a decomposition of f in this space, $f = \sum_{i=1}^n f_i$, such that

$$\sum_{i=1}^n \|f_i\|_{Z(\lambda)}^{1/q} \|f_i\|_{L^\infty(\mu)}^{1/q'} \leq Q' \|f\|_{X(\mu)}.$$

Then

$$\int_\Omega \Big(\sum_{i=1}^n \; |f_i| \; \|f_i\|_{L^\infty}^{1/q'} \; \big(\int_\Omega |f_i| g d\lambda \big)^{-1/q'} \; \Big) \, g \, d\lambda = \sum_{i=1}^n \; \big(\int_\Omega |f_i| g d\lambda \big)^{1-1/q'} \; \|f_i\|_{L^\infty}^{1/q'}$$

$$\leq \sum_{i=1}^n \; \|f_i\|_{Z(\lambda)}^{1/q} \; \|f_i\|_{L^\infty}^{1/q'} \leq Q' \|f\|_{X(\mu)}.$$

(ii) $\Rightarrow$ (iii). Note that $g d\lambda \sim \mu$, and so $g > 0$ and there is a μ-integrable function $h > 0$ such that $g h d\lambda = \mu$. Using again Lemma 4.1 we obtain that the inequalities in (ii) implies that the norm of $\{L^\infty(\mu), L^1(g d\lambda)\}_{1,1/q,h} = \{L^\infty(\lambda), L^1(\lambda)\}_{1,1/q,gh}$ is equivalent to the norm of $X(\mu)$. By Remark 4.2, we have that the identity gives the isomorphism of $L^{q,1}(g_0 d\lambda)$ and $X(\mu)$, where $g_0 = gh$.

Finally, for (iii) $\Rightarrow$ (i) we only have to take $\lambda = g_0 d\lambda$ and $Z(\lambda) = L^1(g_0 d\lambda)$. Then for $f_1, ..., f_n \in L^\infty(\mu) = L^\infty(\lambda)$,

$$\Big(\sum_{i=1}^n \|f_i\|_{X(\mu)}^q \Big)^{1/q}$$

$$\leq K\Big(\sum_{i=1}^{n}\big(\|f_i\|_{Z(\lambda)}^{1/q}\,\|f_i\|_{L^\infty(\lambda)}^{1/q'}\big)^q\Big)^{1/q} = K\Big(\sum_{i=1}^{n}\big(\|f_i\|_{L^1(g_0 d\lambda)}^{1/q}\,\|f_i\|_{L^\infty(\lambda)}^{1/q'}\big)^q\Big)^{1/q}$$

$$= K\Big(\sum_{i=1}^{n}\|f_i\|_{L^1(g_0 d\lambda)}\|f_i\|_{L^\infty(\mu)}^{q-1}\Big)^{1/q} = K\Big\|\sum_{i=1}^{n}|f_i|\|f_i\|_{L^\infty(\mu)}^{q-1}\Big\|_{Z(\lambda)}^{1/q}.$$

The second requirement is a direct consequence of the representation of $L^{q,1}(\lambda)$ given in Remark 4.2.

□

Note that $Z(\lambda)$ in (i) of Theorem 4.3 can be chosen in particular to be $X(\mu)$, obtaining in this way a non-standard concavity-type property for $X(\mu)$ involving just this space and $L^\infty(\mu)$. To finish this paper, let us remark also that the inequalities appearing in (i) and (ii) of Theorem 4.3 are of (lattice) geometric nature: the first one is some sort of "$(q,1)$-concavity", while the second one is a dual notion, associated to a convexity-type property for an expression involving the norm in L.

References

[1] Achour, D., Mezrag, L.: Factorisation des opèrateurs sous-linéaires par $L^{p,\infty}(\Omega,\nu)$ et $L^{q,1}(\Omega,\nu)$. Ann. Sci. Math. Québec. 29, 109–121 (2002)

[2] Berg, J., Löfström, J.: Interpolation spaces. An introduction. Springer, Heidelberg (1976)

[3] Defant, A.: Variants of the Maurey-Rosenthal theorem for quasi-Köthe function spaces. Positivity 5, 153–175 (2001)

[4] Defant, A., Sánchez Pérez, E. A.: Domination of operators on function spaces. Math. Proc. Cambridge Philos. Soc. 146, 57–66 (2009)

[5] Diestel, J., Jarchow, H., Tonge, A.: Absolutely summing operators. Cam. Univ. Pres, Cambridge (1995)

[6] Kalton, N.J., Montgomery-Smith, S.J.: Set-functions and factorization. Arch. Math. 61, 183–200 (1993)

[7] Krivine, J. L.: Théorèmes de factorisation dans les espaces réticulés. Sém. Anal. Fonct. Maurey-Schwartz, 1 (1973)

[8] Lindenstrauss, J., Tzafriri, L.:Classical Banach Spaces II. Springer, Berlin (1979)

[9] Mastyło, M., Sánchez Pérez, E.A.: Factorization of operators through Orlicz spaces. Bull. Malays. Math. Sci. Soc. 40, 1653–1675 (2017)

[10] Mastyło, M., Szwedek, R. Interpolative construction and factorization of operators. J. Math. Anal. Appl. 401, 198–208 (2013)

[11] Maurey, B.: Theorémes de factorisation pour les opèrateurs linéaires à valeurs dans les spaces L^p. Asterisque 11, (1974)

[12] Okada, S., Ricker, W.J., Sánchez Pérez, E.A.: Optimal Domain and Integral Extension of Operators acting in Function Spaces: Birkhäuser, Basel (2008)

[13] Pisier, G.: Factorization of operators through $L_{p\infty}$ or L_{p1} and noncommutative generalizations, Math. Ann. 276, 105–136 (1986)

[14] Rosenthal H. P.: On subspaces of L p. Ann. Math. 97, 344–373 (1973)

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