On the minus partial order in control systems

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Abstract

In this paper, the minus matrix partial order is considered to introduce the concept of minus partial ordered control systems. The transmission of the reachability property under this binary relation is investigated. Furthermore, the analysis of compartmental systems leads us to consider block triangular matrices. Hence, the existence and computation of partially ordered matrices having a similar block structure are studied. These results are applied to compartmental systems to get, via feedback, related systems with the same block structure and ordered under the minus partial order.

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1. Introduction and background

Matrix partial orders have been object of research in the literature having an increasing attention lastly due to their potential for real applications in areas such as electrical networks or statistical problems. In particular, the minus partial order plays an important role in solving problems that involve shorted operators or modified matrices by adding/deleting a row or a column [8, 14]. Some results on theoretical, applied, and numerical aspects of generalized inverses and partial orders can be found in [6, 7, 9, 15, 16, 17, 19, 20, 21].

Algebraic relations and properties of partially ordered matrices motivate research on the usefulness of partial orders in the field of linear dynamic systems. A first approach to this research appeared in [11], where the sharp partial order was applied to study linear autonomous systems.

Systems in which we are interested in the present work are compartmental systems, as for example the models related to population dynamic behaviour or the evolution of an infectious disease [1, 10]. Usually, these models are repre-

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sented by discrete/continuous time compartmental control systems whose state matrix has a prescribed structure, for instance, a block triangular matrix [4].

Moreover, we are interested in applying the minus partial order to these models to improve their performance and to analyze properties that guarantee the efficiency of certain techniques in the medium and long term. Previously, we need to introduce the novel concept of minus partial ordered systems and to make the first general considerations that derive from them.

Furthermore, we present the study of the effect of a feedback on the state coefficient matrix such that the obtained system is related to the initial system under the minus partial order. Since a model represents a real process, it is fundamental to keep the structure of the state coefficient matrix. We look for successor systems to the initial system by improving the characteristics of it. Specifically, we focus our attention on the new state coefficient matrix of the system and on the transmission of the reachability property.

In order to ensure that the structure is maintained for the state coefficient matrix of a compartmental system, we have to analyze the minus partial order for block triangular matrices under a perturbation.

We recall that for a given $n \times n$ real matrix M, the matrix M^- is a $\{1\}$-generalized inverse of M if $MM^-M = M$, and $M^\dagger$ is the Moore-Penrose inverse of M if it satisfies $MM^\dagger M = M$, $M^\dagger M M^\dagger = M^\dagger$, and $MM^\dagger$ and $M^\dagger M$ are symmetric matrices (see [3]).

Let $\mathbb{R}^{n\times n}$ be the set of $n \times n$ real matrices. For two given matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$, it is well known that M_2 is a successor of M_1 under the minus partial order if there exists a $\{1\}$-generalized inverse M_1^- of M_1 such that $M_1 M_1^- = M_2 M_1^-$ and $M_1^- M_1 = M_1^- M_2$. This binary relation will be denoted by $M_1 \overset{-}{\leq} M_2$ (see [14]). The following result gives a characterization of the minus partial order.

Proposition 1. *[2] Let $M_1, M_2 \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(M_1) = r$. Then, the following assertions are equivalent:*

(a) $M_1 \overset{-}{\leq} M_2$.

(b) There exist nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$PM_1Q = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} \quad \text{and} \quad PM_2Q = \begin{pmatrix} I_r & O \\ O & Y \end{pmatrix}, \tag{1}$$

for some matrix $Y \in \mathbb{R}^{(n-r)\times(n-r)}$.

(c) $\operatorname{rank}(M_2) = \operatorname{rank}(M_1) + \operatorname{rank}(M_2 - M_1)$ *(rank substractivity condition).*

A discrete-time linear control system is given by

$$x(k+1) = Mx(k) + Bu(k),\ k \in \mathbb{Z}, \tag{2}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $k \in \mathbb{Z}$. The matrix $M \in \mathbb{R}^{n\times n}$ is the state coefficient matrix and $B \in \mathbb{R}^{n\times m}$ is the input coefficient matrix. This system is denoted by (M, B).

We recall that the system (M, B) is reachable if for any state x there exists a control sequence transferring the trajectory of the system from the origin state 0 into x. The reachability property is characterized by the condition that the reachability matrix defined by

$$\mathrm{R}(M, B) = (B \ MB \ M^2B \cdots M^{n-1}B),$$

has full rank. Moreover, (M, B) is controllable if for any state x there exists a control sequence transferring the trajectory of the system from the state x into the origin state 0. In the discrete-time case, these properties are equivalent if the state coefficient matrix M is nonsingular. In [5], the authors studied the reachability property for higher order linear systems.

We highlight that the main contribution of this paper is to state a link between linear control systems and matrix partial orders. The authors think that this idea can be fruitful and will provide an interesting new research line where areas are enriching each other.

This paper is organized as follows. In Section 2, we introduce the minus partial order for control linear systems and analyze the reachability property. In Section 3, we consider the minus partial order for block triangular matrices and we establish conditions for the existence and computation of successor matrices, under this order, of a given matrix. Finally, compartmental systems are considered in Section 4. We compute the explicit expression of a feedback in order to obtain successor closed-loop systems preserving the block triangular structure. Moreover, the reachability property is analyzed in an interesting case of compartmental systems.

2. Minus partial ordered control systems

In this section we consider control systems and we are going to introduce the minus partial order relation between two given systems. Inspired by the definition given in [11] for the sharp partial order, we can give the following definition.

Definition 1. *We say that two autonomous linear control systems $x(k+1) = M_1x(k)$ and $x(k+1) = M_2x(k)$, $k \in \mathbb{Z}$, are ordered under the minus partial order if $M_1 \stackrel{-}{\leq} M_2$.*

Notice that two ordered autonomous systems under the sharp partial order are equivalent under similarities [11], while this situation is not true, in general, for two ordered autonomous systems under the minus partial order. In this last case, we can construct chained systems satisfying the condition $\mathrm{rank}(M_i) = \mathrm{rank}(M_{i-1}) + \mathrm{rank}(M_i - M_{i-1})$, $i \geq 1$. The final system of the sequence will be a system whose state coefficient matrix is nonsingular, that is, a reversible system.

If we consider a linear control system with inputs as in (2), the natural extension of this concept allows us to obtain a successor system where the reachability from 0 and controllability to 0 properties are equivalent.

Definition 2. *We say that two linear control systems* (M_1, B) *and* (M_2, B) *defined as in (2) are minus partially ordered if* $M_1 \stackrel{-}{\leq} M_2$*. This relation is denoted by* $(M_1, B) \stackrel{-}{\leq} (M_2, B)$*.*

Given two minus partially ordered systems $(M_1, B) \stackrel{-}{\leq} (M_2, B)$, by Proposition 1, there exist nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that M_1 and M_2 satisfy the relation (1). Hence, with respect to the system (M, B) we can write

$$PQQ^{-1}x(k+1) = PM_1QQ^{-1}x(k) + PBu(k), \quad k \in \mathbb{Z},$$

which can be rewritten as

$$y(k+1) = \bar{M}_1 y(k) + Q^{-1}Bu(k),$$

where $y(k) = Q^{-1}x(k) \in \mathbb{R}^{n\times 1}$. The state coefficient matrix can be partitioned as

$$\bar{M}_1 = (PQ)^{-1} \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} = \begin{pmatrix} A_{11} & O \\ A_{21} & O \end{pmatrix}, \tag{3}$$

where $(PQ)^{-1}$ has been denoted by $(PQ)^{-1} = (A_{ij})$ with $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $i = 1, 2$ and $n_1 = r$ and $n_2 = n-r$. In the same way, the transformation $y(k) = Q^{-1}x(k)$ applied to the system (M_2, B) leads us to the following system

$$y(k+1) = \bar{M}_2 y(k) + Q^{-1}Bu(k), \tag{4}$$

with

$$\begin{aligned} \bar{M}_2 &= (PQ)^{-1} \begin{pmatrix} I_r & O \\ O & Y \end{pmatrix} = \bar{M}_1 + \Delta, \\ \Delta &= (PQ)^{-1} \begin{pmatrix} O & O \\ O & Y \end{pmatrix} = \begin{pmatrix} O & A_{12}Y \\ O & A_{22}Y \end{pmatrix}. \end{aligned} \tag{5}$$

Thus, $\bar{M}_i = (PQ)^{-1}PM_iQ = Q^{-1}M_iQ,\ i = 1, 2$.

Summarizing, we have shown the following result.

Lemma 2. *For two given ordered systems, that is* $(M_1, B) \stackrel{-}{\leq} (M_2, B)$*, the following statements hold.*

(a) The system (M_i, B) *is similar to the system* $(\bar{M}_i, Q^{-1}B)$ *with* $\bar{M}_i$ *defined in (3) and (5),* $i = 1, 2$*.*

(b) $\sigma(M_1) = \sigma(\bar{M}_1) = \sigma(A_{11}) \cup \{0\}$ *and* $\sigma(M_2) = \sigma(\bar{M}_2)$*, where* $\sigma(\cdot)$ *denotes the spectrum of the matrix.*

(c) $(\bar{M}_1, Q^{-1}B) \stackrel{-}{\leq} (\bar{M}_2, Q^{-1}B)$*.*

Notice that matrices $\bar{M}_1$ and $\bar{M}_2$ are related by (5). Then, the system $(\bar{M}_2, Q^{-1}B)$ can be interpreted as the result of a disturbance in the state coefficient matrix of the system $(\bar{M}_1, Q^{-1}B)$. This provides an interpretation of what determines the order relation in the field of linear control systems.

Moreover, in control theory this kind of perturbation can be considered the result of the action of a feedback. So, we can search for successor systems of a given system (M, B) via a state-feedback $u(k) = Fx(k) + v(k)$, that is, to look for systems $(M + BF, B)$ which are ordered with (M, B) under the minus partial order. In this case, $\mathrm{rank}(M + BF) = \mathrm{rank}(M) + \mathrm{rank}(BF)$ according to the characterization given in Proposition 1.

2.1. Reachability

For two given systems ordered under the minus partial order, $(M_1, B) \overset{-}{\leq} (M_2, B)$, we are interested in studying their structural properties to see if there exists some relationship between them. Specifically, we are going to focus on the reachability property. Since, by Lemma 2, system (M_i, B) is similar to system $(\bar{M}_i, Q^{-1}B)$, $i = 1, 2$, we can derive the reachability property from the last one.

We analyze several cases attending to the structure of the control coefficient matrix B :

Case 1. Assume that $B^Q = \left\{ Q \begin{pmatrix} S \\ O \end{pmatrix}, \text{ with } S \in \mathbb{R}^{r \times m} \right\}$ and let $B \in B^Q$. The reachability matrix of the systems $(\bar{M}_1, Q^{-1}B)$ and $(\bar{M}_2, Q^{-1}B)$ are given by

$$\mathrm{R}(\bar{M}_1, Q^{-1}B) = \quad H \begin{pmatrix} S & O & O & \dots & O \\ O & S & A_{11}S & \dots & A_{11}^{n-2}S \end{pmatrix} \tag{6}$$

and

$$\mathrm{R}(\bar{M}_2, Q^{-1}B) = \quad \mathrm{R}(\bar{M}_1, Q^{-1}B) + H \begin{pmatrix} O & O & \star & \dots & \star \\ O & O & \star & \dots & \star \end{pmatrix}, \tag{7}$$

where $H = \begin{pmatrix} I_r & A_{11} \\ O & A_{21} \end{pmatrix} \in \mathbb{R}^{n \times 2r}$ and where $\star$'s represent suitable block matrices obtained to construct the reachability matrix which are not needed in the remaining computations.

According to this expression we have the following result.

Proposition 3. *Let (M_1, B) and (M_2, B) be two minus partially ordered control systems, $(M_1, B) \overset{-}{\leq} (M_2, B)$, with $B \in B^Q$. If $\mathrm{rank}(S) = r$ we have*

(a) (M_1, B) is reachable if and only if $\mathrm{rank}(A_{21}) = n - r$.

(b) If (M_1, B) is reachable then (M_2, B) is also reachable.

Proof.

(a) Assume that (M_1, B) is a reachable system. By Lemma 2, $(\bar{M}_1, Q^{-1}B)$ is reachable as well. Then, $\operatorname{rank}(\mathrm{R}(\bar{M}_1, Q^{-1}B)) = n$. By using factorization (6), we have

$$n = \operatorname{rank}\left(H \begin{pmatrix} S & O & O & \dots & O \\ O & S & A_{11}S & \dots & A_{11}^{n-2}S \end{pmatrix} \right) \leq \operatorname{rank}(H) \leq n,$$

which implies $\operatorname{rank}(H) = n$. By definition of H, $\operatorname{rank}(A_{21}) = n - r$.

Conversely, if $\operatorname{rank}(A_{21}) = n - r$ and $\operatorname{rank}(B) = r$, we have (see [18])

$$\operatorname{rank}(\mathrm{R}(\bar{M}_1, Q^{-1}B)) \geq \operatorname{rank}(H) + \operatorname{rank} \begin{pmatrix} S & O & O & \dots & O \\ O & S & A_{11}S & \dots & A_{11}^{n-2}S \end{pmatrix} - 2r = n.$$

(b) By using (6) we have

$$\operatorname{rank}(\mathrm{R}(\bar{M}_1, Q^{-1}B)) = \operatorname{rank}\left(H \begin{pmatrix} S & O \\ O & S \end{pmatrix} \right).$$

Then, from (7) we have $\operatorname{rank}(\mathrm{R}(\bar{M}_2, Q^{-1}B)) = n$. □

Case 2. Assume that $B^P = \left\{ B \;/\; PB = \begin{pmatrix} S \\ O \end{pmatrix}, \text{ with } S \in \mathbb{R}^{r \times m} \right\}$ and let $B \in B^P$. The reachability matrix of the systems $(\bar{M}_1, Q^{-1}B)$ and $(\bar{M}_2, Q^{-1}B)$ are given by

$$\mathrm{R}(\bar{M}_1, Q^{-1}B) = (PQ)^{-1} \begin{pmatrix} S & A_{11}S & A_{11}^2 S & \dots & A_{11}^{n-2}S \\ O & O & O & \dots & O \end{pmatrix}$$

and

$$\mathrm{R}(\bar{M}_2, Q^{-1}B) = \mathrm{R}(\bar{M}_1, Q^{-1}B) + (PQ)^{-1} \begin{pmatrix} O & O & \star & \dots & \star \\ O & YA_{21}S & \star & \dots & \star \end{pmatrix}.$$

Then, the following result is straightforward.

Proposition 4. *Let (M_1, B) and (M_2, B) two minus partially ordered control systems, $(M_1, B) \overset{-}{\leq} (M_2, B)$, with $B \in B^P$. Then*

(a) (M_1, B) is not reachable.

(b) If $\operatorname{rank}(YA_{21}S) = n - r$ and $\operatorname{rank}(S) = r$ then (M_2, B) is reachable.

Case 3. Assume that $B_Q = \left\{ Q \begin{pmatrix} O \\ S \end{pmatrix}, \text{ with } S \in \mathbb{R}^{(n-r) \times m} \right\}$ and let $B \in B_Q$. Then, the reachability matrices are

$$\mathrm{R}(\bar{M}_1, Q^{-1}B) = \begin{pmatrix} O & O & O & \dots & O \\ S & O & O & \dots & O \end{pmatrix}$$

and

$$\mathrm{R}(\bar{M}_2, Q^{-1}B) = \mathrm{R}(\bar{M}_1, Q^{-1}B) + \begin{pmatrix} O & A_{12}YS & \star & \dots & \star \\ O & A_{22}YS & \star & \dots & \star \end{pmatrix}.$$

We can prove the following result.

Proposition 5. *Let (M_1, B) and (M_2, B) be two minus partially ordered control systems, $(M_1, B) \overset{-}{\leq} (M_2, B)$, with $B \in B_Q$.*

(a) (M_1, B) is always not reachable.

(b) If $\operatorname{rank}(A_{12}YS) = r$ and $\operatorname{rank}(S) = n - r$ then (M_2, B) is reachable.

Case 4. Assume that $B_P = \left\{ B \;/\; PB = \begin{pmatrix} O \\ S \end{pmatrix},\ \text{with } S \in \mathbb{R}^{(n-r)\times m} \right\}$ and let $B \in B_P$. The reachability matrices are

$$\mathrm{R}(\bar{M}_1, Q^{-1}B) = (PQ)^{-1} \begin{pmatrix} O & A_{12}S & A_{11}A_{12}S & \dots & A_{11}^{n-2}A_{12}S \\ S & O & O & \dots & O \end{pmatrix}$$

and

$$\mathrm{R}(\bar{M}_2, Q^{-1}B) = \mathrm{R}(\bar{M}_1, Q^{-1}B) + (PQ)^{-1} \begin{pmatrix} O & O & \star & \dots & \star \\ O & YA_{22}S & \star & \dots & \star \end{pmatrix}.$$

Now, we state the following proposition.

Proposition 6. *Let (M_1, B) and (M_2, B) be two minus partially ordered control systems, $(M_1, B) \overset{-}{\leq} (M_2, B)$, with $B \in B_P$. If $\operatorname{rank}(A_{12}S) = n - r$ and $\operatorname{rank}(S) = n - r$ then (M_1, B) and (M_2, B) are reachable.*

The study of these special classes allow us to assert that, in general, the reachability property is not preserved under the minus partial order for systems.

Interested on compartmental systems, whose state coefficient matrix has a block triangular structure, in the following section we analyze block triangular matrices under the minus partial order. We will obtain a characterization for successor matrices under the minus partial order of a given matrix M preserving the same structure as M.

3. Ordering block triangular matrices

In this section we are going to consider a fixed block triangular matrix $M \in \mathbb{R}^{n\times n}$ given by

$$M = \begin{pmatrix} M_{11} & M_{12} \\ O & M_{22} \end{pmatrix}, \tag{8}$$

where $M_{11} \in \mathbb{R}^{n_1\times n_1}$ and $M_{22} \in \mathbb{R}^{n_2\times n_2}$, with $n = n_1 + n_2$, and $\operatorname{rank}(M) = r > 0$. This matrix allows us to define the following set:

$$\mathbb{M} = \left\{ M_X = \begin{pmatrix} M_{11} & M_{12} \\ O & M_{22} + X \end{pmatrix} :\ X \in \mathbb{R}^{n_2\times n_2} \right\}.$$

Remark 1. *Notice that, for a given square real matrix, we can always use the Schur decomposition to get an orthogonally similar upper-triangular matrix M. Since, moreover, the minus partial order is preserved under nonsingular matrices [14], the above structure is not restrictive at all.*

From now on, we say that M_X is associated with $X \in \mathbb{R}^{n_2 \times n_2}$ whenever $M_X \in \mathbb{M}$.

It is well known that always there exist nonsingular matrices $P, Q \in \mathbb{R}^{n \times n}$ such that

$$PMQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}. \tag{9}$$

By partitioning these matrices P and Q as

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \tag{10}$$

with $P_{11} \in \mathbb{R}^{r \times n_1}$, $P_{22} \in \mathbb{R}^{(n-r) \times n_2}$, $Q_{11} \in \mathbb{R}^{n_1 \times r}$, and $Q_{22} \in \mathbb{R}^{n_2 \times (n-r)}$, we can state the following characterization on successor matrices of M in the set $\mathbb{M}$ under the minus partial order.

Proposition 7. *Let $M \in \mathbb{R}^{n \times n}$ be a structured matrix as in (8) with* $\operatorname{rank}(M) = r > 0$. *Then, there exists a matrix $X \in \mathbb{R}^{n_2 \times n_2}$ such that $M_X \in \mathbb{M}$ satisfies $M \overset{-}{\leq} M_X$ if and only if there exist nonsingular matrices $P, Q \in \mathbb{R}^{n \times n}$ decomposed as in (10), a matrix $Y \in \mathbb{R}^{(n-r) \times (n-r)}$ such that the relation (9) holds, and the following system*

$$P_{12} X Q_{21} = O, \quad P_{12} X Q_{22} = O, \quad P_{22} X Q_{21} = O, \quad P_{22} X Q_{22} = Y \tag{11}$$

has a solution X.

Proof. Let X be a matrix such that $M_X \in \mathbb{M}$ and $M \overset{-}{\leq} M_X$. By Proposition 1, there are nonsingular matrices $P, Q \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{(n-r) \times (n-r)}$ such that

$$PMQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} \quad \text{and} \quad PM_XQ = \begin{pmatrix} I_r & O \\ O & Y \end{pmatrix}.$$

Moreover, by definition of $\mathbb{M}$, $M_X = M + \begin{pmatrix} O & O \\ O & X \end{pmatrix}$. By combining both results and by using the decompositions of P and Q given in (10), we have

$$\begin{pmatrix} P_{12} X Q_{21} & P_{12} X Q_{22} \\ P_{22} X Q_{21} & P_{22} X Q_{22} \end{pmatrix} = P \begin{pmatrix} O & O \\ O & X \end{pmatrix} Q = \begin{pmatrix} O & O \\ O & Y \end{pmatrix}.$$

Then, we get that the system (11) has a solution.

Conversely, we consider P, Q, and Y such that M satisfies (9) and the system (11) has a solution X. Now, it is straightforward to show that the matrix $M_X = M + \begin{pmatrix} O & O \\ O & X \end{pmatrix} \in \mathbb{M}$ satisfies $M \overset{-}{\leq} M_X$.

However, the existence of a solution of the matrix system (11) for matrices P and Q satisfying (9) is not always guaranteed. This fact is illustrated in the following example.

Example 8. *Consider the matrix*

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & m_{22} & m_{23} \end{pmatrix},$$

with $m_{11} \neq 0$*,* $m_{22} \neq 0$*, and* $m_{12}m_{23} \neq m_{13}m_{22}$*. Then, for the matrices*

$$P = \begin{pmatrix} \frac{1}{m_{11}} & 0 & -\frac{m_{12}}{m_{11}m_{22}} \\ 0 & 0 & \frac{1}{m_{22}} \\ 0 & 1 & -1 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ \frac{m_{11}m_{23}}{m_{12}m_{23}-m_{13}m_{22}} & 1 & \frac{m_{11}m_{23}}{m_{13}m_{22}-m_{12}m_{23}} \\ \frac{m_{11}m_{22}}{m_{13}m_{22}-m_{12}m_{23}} & 0 & \frac{m_{11}m_{22}}{m_{12}m_{23}-m_{13}m_{22}} \end{pmatrix},$$

the equation (9) holds for $r = 2$*. The system (11) has only the trivial solution* $X = O$ *if* $Y = 0$*. And for* $Y \neq 0$*, the system (11) has no solution. Therefore, in this last case, there are not successors under the minus partial order of the matrix* M *in the set* $\mathbb{M} - \{M\}$*.*

In general, the consistency of the system (11) and its form depend on the properties of the block matrices of P and Q. In order to study them we will use the Kronecker product of two matrices which is defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{mp\times nq},$$

for $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. According to this definition the following properties can be easily stated.

Lemma 9. *Let* $A \in \mathbb{R}^{m\times n}$*,* $B \in \mathbb{R}^{p\times s}$*, and* $C \in \mathbb{R}^{q\times s}$*. Then*

(i) There exists a permutation T *such that* $A \otimes \begin{pmatrix} B \\ C \end{pmatrix} = T \begin{pmatrix} A\otimes B \\ A \otimes C \end{pmatrix}$.

(ii) $\begin{pmatrix} B \\ C \end{pmatrix} \otimes A = \begin{pmatrix} B\otimes A \\ C \otimes A \end{pmatrix}$.

Moreover, we will denote by $\mathrm{vec}(A)$ the vectorization of the matrix A, which consists of stacking the columns of A into a unique column vector.

This vectorization and the Kronecker product allow us to express the matrix equation $AXB = C$ by means of the linear system of equations $(B^T \otimes A)\,\mathrm{vec}(X) = \mathrm{vec}(C)$ (see [13]).

In addition, if S^- is a $\{1\}$-generalized inverse of S, we denote by $P_S = I - SS^-$ the oblique projector onto the null space $\mathcal{N}(SS^-)$ along the column space $\mathcal{R}(S)$.

Theorem 10. *Let $M \in \mathbb{R}^{n\times n}$ be a structured matrix as in (8) with $\mathrm{rank}(M) = r > 0$. Consider the block decomposition (10) of a pair of matrices P and Q that satisfy (9). Then, the following statements are equivalent.*

(a) There exists a matrix $X \in \mathbb{R}^{n_2\times n_2}$ such that $M_X \in \mathbb{M}$ associated with X satisfies $M \overset{-}{\leq} M_X$.

(b) There exists a matrix $U \in \mathbb{R}^{n_2\times n_2}$ such that

$$\begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} U P_{Q_{21}} Q_{22} = \begin{pmatrix} O \\ Y \end{pmatrix}, \text{ for some } Q_{21}^-.$$

(c) $P_G \mathrm{vec}\begin{pmatrix} O \\ Y \end{pmatrix} = O$ with $G = Q_{22}^T P_{Q_{21}}^T \otimes \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix}$ for some G^- and Q_{21}^-.

Proof. (a) $\Rightarrow$ (b) By Proposition 7, the system (11) has a solution. This system can be rewritten as

$$\begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} X Q_{21} = O, \quad \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} X Q_{22} = \begin{pmatrix} O \\ Y \end{pmatrix}. \tag{12}$$

Firstly, the general solution of the homogeneous equation in (12) is given by $X = U - \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix}^- \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} U Q_{21} Q_{21}^-$, for an arbitrary $U \in \mathbb{R}^{n_2\times n_2}$. Since (12) has a solution, by substituting the above expression of X into the second equation of (12), we can ensure the existence of a matrix U such that

$$\begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} U P_{Q_{21}} Q_{22} = \begin{pmatrix} O \\ Y \end{pmatrix}.$$

(b) $\Rightarrow$ (a) It is easy to check that $X = U - \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix}^- \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} U Q_{21} Q_{21}^-$ satisfies the system (12), provided that (b) holds. Now, by Proposition 7, the result follows.

(b) $\Leftrightarrow$ (c) By using the Kronecker product, the relation given in (b) can be expressed as

$$\left(Q_{22}^T P_{Q_{21}}^T \otimes \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} \right) \mathrm{vec}(U) = \mathrm{vec}\begin{pmatrix} O \\ Y \end{pmatrix}.$$

That is, $\mathrm{vec}\begin{pmatrix} O \\ Y \end{pmatrix} \in \mathcal{R}(G)$, where G is defined as in the statement (c). Since $\mathcal{R}(G) = \mathcal{R}(GG^-) = \mathcal{N}(P_G)$ (see [3]), we get that $P_G \mathrm{vec}\begin{pmatrix} O \\ Y \end{pmatrix} = O$. □

We close this section with a result that provides an explicit solution X of the system (11).

Proposition 11. *Let $M \in \mathbb{R}^{n\times n}$ be a structured matrix as in (8) with* $\mathrm{rank}(M) = r > 0$. *Consider the block decomposition (10) of a pair of matrices P and Q that satisfy (9). If the system (11) is consistent for some matrix Y, then a solution X is given by*

$$\mathrm{vec}(X) = ((\tilde{Q}^T)^\dagger \otimes \tilde{P}^\dagger) \begin{pmatrix} I & O \\ O & S \end{pmatrix} \begin{pmatrix} O \\ \mathrm{vec}(Y) \end{pmatrix},$$

with $\tilde{P} = \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix}$, $\tilde{Q} = \begin{pmatrix} Q_{21} & Q_{22} \end{pmatrix}$, and S being a suitable permutation matrix.

Proof. By applying vectorization to equations of the system (11), we obtain the system given by $(Q_{21}^T \otimes P_{12})\mathrm{vec}(X) = O$, $(Q_{22}^T \otimes P_{12})\mathrm{vec}(X) = O$, $(Q_{21}^T \otimes P_{22})\mathrm{vec}(X) = O$, $(Q_{22}^T \otimes P_{22})\mathrm{vec}(X) = \mathrm{vec}(Y)$. By using Lemma 9, we get

$$\left(\tilde{Q}^T \otimes \tilde{P}\right) \mathrm{vec}(X) = \begin{pmatrix} I & O \\ O & S \end{pmatrix} \begin{pmatrix} O \\ \mathrm{vec}(Y) \end{pmatrix},$$

with $\tilde{P}$ and $\tilde{Q}$ as in the statement, and $S \in \mathbb{R}^{n(n-r)\times n(n-r)}$ being a suitable permutation matrix. Since the matrix $R := \tilde{Q}^T \otimes \tilde{P}$ has full column rank and $R^\dagger = (\tilde{Q}^T)^\dagger \otimes \tilde{P}^\dagger$ (see [13]), a solution of the last system is given by

$$\mathrm{vec}(X) = R^\dagger \begin{pmatrix} I & O \\ O & S \end{pmatrix} \begin{pmatrix} O \\ \mathrm{vec}(Y) \end{pmatrix}.$$

□

Remark 2. Since the matrices $Q_{21}Q_{21}^T + Q_{22}Q_{22}^T$ and $P_{12}^T P_{12} + P_{22}^T P_{22}$ in

$$(\tilde{Q}^T)^\dagger = \begin{pmatrix} Q_{21}^T \\ Q_{22}^T \end{pmatrix}^\dagger = (Q_{21}Q_{21}^T + Q_{22}Q_{22}^T)^{-1} \begin{pmatrix} Q_{21} & Q_{22} \end{pmatrix}$$

and

$$\tilde{P}^\dagger = \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix}^\dagger = (P_{12}^T P_{12} + P_{22}^T P_{22})^{-1} \begin{pmatrix} P_{12}^T & P_{22}^T \end{pmatrix}$$

are positive definite, their inverses are given by a simple numerical computation.

4. Ordered compartmental systems

Compartmental systems are frequently used in real process of areas such as biology, demography, engineering; see [12] and references therein. A compartmental system consists of a finite number of connected subsystems and hence, the coefficient matrices have a specific block structure. They appear, for instance, when individuals of a specie are organized in classes depending on the stage of life [4]. In this case the state matrix shows a block triangular structure as in [1].

Consider a compartmental system (M, B) whose state coefficient matrix M has the structure given in (8). By applying a state-feedback $u(k) = Fx(k)+v(k)$, we obtain the closed-loop system

$$x(k+1) = (M + BF)x(k) + Bv(k).$$

We look for admissible feedbacks F that preserve the structure of the state matrix M and satisfy

$$BF = \begin{pmatrix} O & O \\ O & B_2F_2 \end{pmatrix}, \tag{13}$$

where

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \qquad \text{and} \qquad F = \begin{pmatrix} F_1 & F_2 \end{pmatrix}, \tag{14}$$

with $B_1 \in \mathbb{R}^{n_1\times m}$, $B_2 \in \mathbb{R}^{n_2\times m}$, $F_1 \in \mathbb{R}^{m\times n_1}$, and $F_2 \in \mathbb{R}^{m\times n_2}$. Moreover, we are interested in obtaining successor closed-loop systems of the initial system under the minus partial order. In this way, we look for systems that preserve the structure of the state matrix M and that the rank of its state matrix satisfies $\text{rank}(M+BF) = \text{rank}(M)+\text{rank}(B_2F_2)$. This problem is solved in the following result which follows from Theorem 10 and Proposition 11.

Proposition 12. *Let (M, B) be a compartmental system with M structured as in (8) and $B = \begin{pmatrix} O \\ B_2 \end{pmatrix}$ with $\text{rank}(B_2) = m$. There exists a matrix $F_2 \in \mathbb{R}^{n_2\times n_2}$ such that $M + BF$ satisfies $M \overset{-}{\leq} M + BF$ for $F = (O \;\; F_2)$ if and only if $P_G\text{vec}\begin{pmatrix} O \\ Y \end{pmatrix} = O$ with $G = Q_{22}^T P_{Q_{21}}^T \otimes \begin{pmatrix} P_{12}B_2 \\ P_{22}B_2 \end{pmatrix}$ for some G^- and Q_{21}^-.*

Moreover,

$$\text{vec}(F_2) = ((\tilde{Q}^T)^\dagger \otimes \tilde{P}^\dagger)\begin{pmatrix} I & O \\ O & S \end{pmatrix}\begin{pmatrix} O \\ \text{vec}(Y) \end{pmatrix},$$

with $\tilde{P} = \begin{pmatrix} P_{12}B_2 \\ P_{22}B_2 \end{pmatrix}$, $\tilde{Q} = \begin{pmatrix} Q_{21} & Q_{22} \end{pmatrix}$, and S being a suitable permutation matrix.

Next, we consider a particular case in which the interconnections in the first compartment of (8) have associated a nonsingular matrix, that is $0 < \mathrm{rank}(M_{11}) = n_1 < r$. Then, if $r_2 = \mathrm{rank}(M_{22})$, we have that

$$r = \mathrm{rank}(M) = \mathrm{rank}(M_{11}) + \mathrm{rank}(M_{22}) = n_1 + r_2. \tag{15}$$

Consider two nonsingular matrices P_2 and Q_2 that satisfy

$$P_2 M_{22} Q_2 = \begin{pmatrix} I_{r_2} & O \\ O & O \end{pmatrix},$$

then we can check that

$$PMQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}, \tag{16}$$

with

$$P = \begin{pmatrix} I_{n_1} & O \\ O & P_2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} M_{11}^{-1} & -M_{11}^{-1} M_{12} Q_2 \\ O & Q_2 \end{pmatrix}. \tag{17}$$

Hence, we can establish the following result.

Proposition 13. *Let (M, B) be a compartmental system with M structured as in (8) satisfying the condition (15) and $B = \begin{pmatrix} O \\ B_2 \end{pmatrix}$ such that $\mathrm{rank}(B_2) = m$. Then, the state-feedback $u(k) = (O\ F_2)x(k)$ with*

$$F_2 = (B_2^T B_2)^{-1} B_2^T P_2^{-1} \hat{Y} Q_2^{-1}, \tag{18}$$

and $\hat{Y} = \begin{pmatrix} O_{r_2} & O \\ O & Y \end{pmatrix}$ for some $Y \in \mathbb{R}^{(n-r)\times(n-r)}$, provides a block triangular matrix $M + BF$ such that $(M, B) \overset{-}{\leq} (M + BF, B)$.

Proof. In order to get a feedback such that $M + BF$ is a successor to matrix M, we impose that the equation given in (1) is verified, with the matrices P and Q given in (17). Taking into account the structure of the matrix $M + BF$, we get

$$P(M + BF)Q = \begin{pmatrix} I_{n_1} & O \\ O & P_2(M_{22} + B_2 F_2) Q_2 \end{pmatrix} = \begin{pmatrix} I_r & O \\ O & Y \end{pmatrix},$$

for some matrix $Y \in \mathbb{R}^{(n-r)\times(n-r)}$. Then, we have

$$P_2 B_2 F_2 Q_2 = \begin{pmatrix} O_{r_2} & O \\ O & Y \end{pmatrix}.$$

Since P_2 and Q_2 are nonsingular matrices and B_2 has full column rank, its Moore-Penrose inverse is given by $B_2^\dagger = (B_2^T B_2)^{-1} B_2^T$, we obtain that the feedback $u(k) = (O\ F_2)x(k)$ with F_2 as in (18), for some $Y \in \mathbb{R}^{(n-r)\times(n-r)}$, provides $(M, B) \overset{-}{\leq} (M + BF, B)$. □

Now, we consider the reachability property and the set of reachable states (reachability space), which is the space generated by the columns of the reachability matrix. Next proposition establishes that the set of reachable states remains invariant under this kind of feedback.

Proposition 14. *Consider the ordered systems (M, B) and $(M + BF, B)$ with M as in (8) satisfying the condition given in (15). Let $B = \begin{pmatrix} O \\ B_2 \end{pmatrix}$, with $B_2 \in \mathbb{R}^{n_2 \times m}$ being a full column rank matrix, and $F = (O\ F_2)$ given as in (18). Then, the reachability space of (M, B) is the same as the reachability space of $(M + BF, B)$.*

Proof. We analyze systems $(\bar{M}, Q^{-1}B)$ and $(\bar{M}_F, Q^{-1}B)$, introduced in Section 2, similar to ordered systems (M, B) and $(M + BF, B)$. From (17) we obtain that these similar systems are given by

$$\bar{M} = \begin{pmatrix} M_{11} & M_{12}M_{22}Q_2 \\ O & Q_2^{-1}M_{22}Q_2 \end{pmatrix}, \quad \bar{M}_F = \bar{M} + \Delta = \bar{M} + \begin{pmatrix} O & M_{12}B_2F_2Q_2 \\ O & Q_2^{-1}B_2F_2Q_2 \end{pmatrix},$$

and

$$Q^{-1}B = \begin{pmatrix} M_{12}B_2 \\ Q_2^{-1}B_2 \end{pmatrix}.$$

Then, the reachability matrix of the system $(\bar{M}_F, Q^{-1}B)$ is

$$\mathrm{R}(\bar{M}_F, Q^{-1}B) = \left(Q^{-1}B\ (\bar{M} + \Delta)Q^{-1}B \cdots (\bar{M} + \Delta)^{n-1}Q^{-1}B\right),$$

and we have to analyze matrix products of the kind $(\bar{M})^i\Delta^jQ^{-1}B$ and $(\bar{M})^i\Delta^jQ^{-1}B$. In this way, we obtain

$$\begin{aligned} \Delta^iQ^{-1}B = &\quad Q^{-1}B(F_2B_2)^i, \\ \Delta(\bar{M})^iQ^{-1}B = &\quad Q^{-1}BF_2M_{22}^iB_2, \\ \Delta^i(\bar{M})^jQ^{-1}B = &\quad Q^{-1}B(F_2B_2)^iF_2M_{22}^jB_2, \\ (\bar{M})^i\Delta^jQ^{-1}B = &\quad (\bar{M})^iQ^{-1}B(F_2B_2)^j. \end{aligned}$$

So, we can assure that the blocks of the matrix $\mathrm{R}(\bar{M}_F, Q^{-1}B)$ satisfy

$$(\bar{M}_F)^iQ^{-1}B = (\bar{M})^iQ^{-1}B + V,$$

with $V = (v_1 \cdots v_m) \in \mathbb{R}^{n \times m}$ such that

$$v_i \in \mathrm{span}(Q^{-1}B,\ \bar{M}Q^{-1}B,\ \ldots,\ \bar{M}^{i-1}BQ^{-1}B),\ i = 1, \ldots, m,$$

where $\mathrm{span}(X,\ Y,\ \ldots)$ denotes the subspace of $\mathbb{R}^n$ generated by the column vectors of matrices X, Y, and so on. Thus,

$$\mathrm{span}\left(\mathrm{R}(\bar{M}, Q^{-1}B)\right) = \mathrm{span}\left(\mathrm{R}(\bar{M}_F, Q^{-1}B)\right).$$

□

5. Conclusions

In this work, we have introduced the minus partial order relation for control systems. This notion allows us to generalize in two senses the study done in [11]: (a) from autonomous systems to control systems and (b) from the sharp partial order (only defined for index-one matrices) to the minus partial order. In general, minus partially ordered matrices are not related under similarities, this fact allows us to do a more general study than that carried out in [11]. We have analyzed the reachability property for two ordered control systems under the minus partial order. Depending on the control coefficient matrix, this property is inherited by the successor of a system or we can get a reachable successor system from a non-reachable one. Moreover, we have studied feedbacks in compartmental systems to get related systems with the same structure and ordered under the minus partial order.

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