Analytic solution to the generalized delay diffusion equation with uncertain inputs in the random Lebesgue sense

Juan Carlos Cortés[a], **Marc Jornet**[a,*]

In this paper we deal with the randomized generalized diffusion equation with delay: $u_t(t,x) = a^2 u_{xx}(t,x) + b^2 u_{xx}(t-\tau,x)$**,** $t > \tau$**,** $0 \leq x \leq l$**;** $u(t,0) = u(t,l) = 0$**,** $t \geq 0$**;** $u(t,x) = \varphi(t,x)$**,** $0 \leq t \leq \tau$**,** $0 \leq x \leq l$**. Here** $\tau > 0$ **and** $l > 0$ **are constant. The coefficients** a^2 **and** b^2 **are non-negative random variables, and the initial condition** $\varphi(t,x)$ **and the solution** $u(t,x)$ **are random fields. The separation of variables method develops a formal series solution. We prove that the series satisfies the delay diffusion problem in the random Lebesgue sense rigorously. By truncating the series, the expectation and the variance of the random field solution can be approximated.**

Keywords: random generalized diffusion equation with delay; random Lebesgue calculus; series solution; expectation and variance approximation; uncertainty quantification

1. Introduction

Given a physical system, if its future state is independent of the past states, a mathematical model based on ordinary or partial differential equations may be formulated. Under this principle, many phenomena from Physics, Epidemiology, Ecology, Engineering, Finance, etc., have been successfully modeled. However, when the future state of a system explicitly depends on past states due to hereditary characteristics, such as aftereffects or time lags, a time-delay term must be included into the differential equation. The delay may be discrete, when a specific past information is used, or continuous, when complete past history is relevant for the future. From a theoretical viewpoint, the analysis of delay differential equations requires alternative techniques to those for classical differential equations, see general references [1, 2, 3, 4, 5, 6, 7, 8]. In practice, delay differential equations allow for more complex modeling and more realistic description of the phenomenon under study. They play a key role in different scientific and technical fields [9, 10, 11, 12, 13, 14, 15].

Given a mathematical model, the input coefficients are usually set from data, by employing their modeling interpretation or optimization techniques. But uncertainty is inherent to data: limited knowledge of the process, lack of information, bad calibration machines, variability of the system, etc. Hence the coefficients should be regarded as random quantities on a probability space. The uncertainty is thus propagated from the data to the system output, which becomes a stochastic process or a random field. Its specific realizations are not the main concern; uncertainty quantification must be conducted to clearly understand its statistical content (mean, variance, or any other statistic) [16, 17, 18].

From a theoretical point of view, differential equations with random parameters may be studied in a Lebesgue sense [19, 20, 21], [8, Ch. 8]. Given a stochastic process, its continuity, differentiability, Riemann integrability, etc., may be defined by considering the limits in Lebesgue spaces. Recall that the random Lebesgue spaces are the Banach spaces $(\mathrm{L}^p, \|\cdot\|_p)$, where $\|U\|_p = (\mathbb{E}[|U|^p])^{1/p} < \infty$ (finite p-th absolute moment), for $1 \leq p < \infty$, and $\|U\|_\infty = \inf\{C \geq 0 : |U| \leq C \text{ almost surely}\} < \infty$ (essential boundedness), for any random variable U. In particular, $(\mathrm{L}^2, \langle\cdot,\cdot\rangle)$ is the Hilbert space of random variables with finite variance, endowed with the inner product $\langle U, V\rangle = \mathbb{E}[UV]$. Here $\mathbb{E}[\cdot]$ is the expectation operator. A key aspect of $(\mathrm{L}^p, \|\cdot\|_p)$ is that convergence preserves the convergence of statistical moments up to order p. When $p \geq 2$, the convergence of the mean and the variance is preserved. This is important for uncertainty quantification [22, 23, 24, 25, 26].

Recently, random differential equations with discrete delay have been studied by employing random Lebesgue calculus. In [27], general delay random differential equations in L^p were analyzed, with the goal of extending some of the existing results on

[a] *Instituto Universitario de Matemática Multidisciplinar,*
Universitat Politècnica de València,
Camino de Vera s/n, 46022, Valencia, Spain
[*] *Correspondence to: Marc Jornet, Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, 46022, Valencia, Spain*
[†] E-mail: jccortes@imm.upv.es; marjorsa@doctor.upv.es

random ordinary differential equations with no delay from the book [19]. In [28], the study on delay random differential equations was started with the basic autonomous and homogeneous linear equation, by proving the existence and uniqueness of L^p-solution under certain conditions.

In this paper, we study the randomized generalized diffusion equation with delay:

$$\begin{cases} u_t(t,x) = a^2 u_{xx}(t,x) + b^2 u_{xx}(t-\tau,x), & t>\tau,\ 0\le x\le l, \\ u(t,0) = u(t,l) = 0, & t\ge 0, \\ u(t,x) = \varphi(t,x), & 0\le t\le \tau,\ 0\le x\le l. \end{cases} \tag{1}$$

A complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is implicitly assumed. The coefficients a^2 and b^2 are non-negative random variables, and the initial condition $\varphi(t,x)$ and the solution $u(t,x)$ are random fields. All these terms depend on the random outcomes $\omega \in \Omega$. The delay term τ and l are positive constants.

The deterministic version of (1) was studied in [29]. When $b=0$, the randomized problem was solved in [30], in the mean square (Lebesgue with exponent $p=2$) sense. These two references used the method of separation of variables (also called Fourier method) to derive the candidate series solution. The latter reference achieved approximations to the expectation and the variance of the series solution.

In this work, we aim at solving (1) in the random Lebesgue L^p sense, by employing the method of separation of variables. In the separate equation for the time variable, a particular linear random differential equation with delay arises, which is explicitly solved by utilizing [28]. Once the L^p-convergence of the series solution is established, statistical moments up to order p can be approximated by truncating the series. In particular, when $p\ge 2$, the expectation and the variance of the solution are estimated.

The organization of the present paper is as follows. In Section 2, we show the main definitions and results on random Lebesgue calculus that will be required to solve (1). In Section 3, we summarize the main findings obtained in [28] for the linear random differential equation with delay. After these preliminaries, Section 4 is devoted to solving (1) in the random L^p sense, by employing the method of separation of variables and addressing the L^p-convergence of the series solution. In Section 5, the series solution is truncated to approximate the first statistical moments. Some numerical computations are included. Finally, Section 6 draws the main conclusions of the paper and discusses potential avenues of research.

2. Preliminaries on random Lebesgue calculus

Fixed $1\le p<\infty$, we work on the Lebesgue space $(\mathrm{L}^p, \|\cdot\|_p)$ of random variables from Ω to $\mathbb{R}$ with finite p-th absolute moment. Given a stochastic process $u(t)$ or a random field $u(t,x)$, their continuity, differentiability, Riemann integrability, etc., may be defined in L^p. For instance, $u(t)$ is L^p-continuous at t_0 if $\lim_{t\to t_0}\|u(t)-u(t_0)\|_p = 0$; $u(t)$ is L^p-differentiable at t_0 if the limit of $\frac{u(t_0+h)-u(t_0)}{h}$ exists in L^p when $h\to 0$; and $u(t)$ is L^p-Riemann integrable on a time domain $[t_1,t_2]$ if there exists a random variable A and a sequence of partitions $\{P_n\}_{n=1}^\infty$ with mesh tending to 0, $P_n = \{t_1 = t_0^n < t_1^n < \ldots < t_{r_n}^n = t_2\}$, such that, for any choice of points $s_i^n \in [t_{i-1}^n, t_i^n]$, $i=1,\ldots,r_n$, $\lim_{n\to\infty}\sum_{i=1}^{r_n} u(s_i^n)(t_i^n - t_{i-1}^n) = A$ in L^p, where A is then denoted as $\int_{t_1}^{t_2} u(t)\,\mathrm{d}t$.

According to [8, Th. 8-20], if $u(t)$ is continuously differentiable on $I\subseteq\mathbb{R}$ in the L^p sense with derivative u', then there exists an equivalent stochastic process $\varphi(t,\omega)$ on $I\times\Omega$ (i.e., $\mathbb{P}[\varphi(t)=u(t)]=1$ for all t), product measurable, such that its sample paths are absolutely continuous, $\varphi'(t,\omega)$ exists almost everywhere on $I\times\Omega$, and $\varphi'(t,\cdot) = u'(t)$ almost surely for almost all $t\in I$.

By [8, p. 441], if $u(t)$ is L^p-continuous on $I=[t_1,t_2]$, then there exists an equivalent stochastic process $\varphi(t,\omega)$ on $I\times\Omega$, product measurable, such that $[\int_{t_1}^{t_2} u(s)\,\mathrm{d}s](\omega) = \int_{t_1}^{t_2}\varphi(s,\omega)\,\mathrm{d}s$, where the integral on the right is an ordinary Lebesgue integral for sample paths.

If $u(t)$ is L^p-bounded and L^p-almost everywhere continuous on I, then u is L^p-Riemann integrable on I. The converse is not true, since L^p does not have the Lebesgue Property [31].

Apart from these concepts and results, we will need the following findings related to random L^p operational calculus.

Lemma 2.1 *[30, Th. 3.1] Let $\{u_n(t)\}_{n=1}^\infty$ be a sequence of stochastic processes. Suppose that each $u_n(t)$ is continuously differentiable on an interval I in the L^p sense, $u(t)=\sum_{n=1}^\infty u_n(t)$ exists in L^p for each $t\in I$, and $\sum_{n=1}^\infty u_n'(t)$ is uniformly convergent on I in the L^p sense. Then u is L^p-differentiable on I and $u'(t)=\sum_{n=1}^\infty u_n'(t)$ in L^p, $t\in I$.*

Lemma 2.2 (Leibniz's integral rule) *[28, Prop. 2.2] Let $u(t,s)$ be a stochastic process on $[a,b]\times[c,d]$. Suppose that $u(t,\cdot)$ is L^p-continuous on $[c,d]$, for each $t\in[a,b]$, and that there exists the L^p-partial derivative $\frac{\partial u}{\partial t}(t,s)$ for all $(t,s)\in[a,b]\times[c,d]$, which is L^p-continuous on $[a,b]\times[c,d]$. Let $v(t)=\int_c^d u(t,s)\,\mathrm{d}s$ (the integral is understood as an L^p-Riemann integral). Then v is L^p-differentiable on $[a,b]$ and $v'(t)=\int_c^d \frac{\partial u}{\partial t}(t,s)\,\mathrm{d}s$.*

Lemma 2.3 (Integration by parts formula) *[19, p. 104] Let $u(t)$ be a stochastic process on $[a,b]$ and let $f(t)$ be a deterministic real function on $[a,b]$. Suppose that u is continuously differentiable in the L^p sense and that f is continuously differentiable. Then $\int_a^b f(t)u'(t)\,\mathrm{d}t = f(b)u(b) - f(a)u(a) - \int_a^b f'(t)u(t)\,\mathrm{d}t$, where the Riemann integrals are considered in the L^p sense.*

3. On the linear random differential equation with delay

In this section we show some results from [28]. The autonomous and homogeneous linear random differential equation with delay is

$$\begin{cases} u'(t) = au(t) + bu(t-\tau), & t \geq 0, \\ u(t) = g(t), & -\tau \leq t \leq 0, \end{cases} \tag{2}$$

where a and b are random variables, and $g(t)$ and $u(t)$ are stochastic process. By randomizing the deterministic solution obtained with the method of steps [32, Th. 1], the formal solution to (2) is

$$u(t) = \mathrm{e}^{a(t+\tau)} \mathrm{e}_\tau^{b_1,t} g(-\tau) + \int_{-\tau}^{0} \mathrm{e}^{a(t-s)} \mathrm{e}_\tau^{b_1,t-\tau-s} (g'(s) - ag(s))\,\mathrm{d}s, \tag{3}$$

where $b_1 = \mathrm{e}^{-a\tau} b$,

$$\mathrm{e}_\tau^{c,t} = \exp_\tau(c,t) = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \leq t < 0, \\ 1 + c\dfrac{t}{1!}, & 0 \leq t < \tau, \\ 1 + c\dfrac{t}{1!} + c^2\dfrac{(t-\tau)^2}{2!}, & \tau \leq t < 2\tau, \\ \vdots & \vdots \\ \displaystyle\sum_{k=0}^{n} c^k \frac{(t-(k-1)\tau)^k}{k!}, & (n-1)\tau \leq t < n\tau, \end{cases}$$

is the delayed exponential function [32, Def. 1], $c, t \in \mathbb{R}$, $\tau > 0$ and $n = \lfloor t/\tau \rfloor + 1$ (here $\lfloor\cdot\rfloor$ denotes the integer part defined by the so-called floor function). The integral from (3) is considered as an L^p-Riemann integral.

In [28], two results for existence and uniqueness of L^p-solution to (2) were stated and proved. In this paper we will only need the second result.

Lemma 3.1 (Existence and uniqueness) *[28, Th. 3.2] Fix $1 \leq p < \infty$. Suppose that $\mathbb{E}[\mathrm{e}^{a\zeta}] < \infty$ for all $\zeta \in \mathbb{R}$, b has absolute moments of any order, and g belongs to $C^1([-\tau, 0])$ in the $\mathrm{L}^{p+\eta}$ sense, for certain $\eta > 0$. Then the stochastic process $u(t)$ defined by (3) is the unique L^p-solution to (2).*

Lemma 3.2 (Existence and uniqueness) *[28, Th. 3.4] Fix $1 \leq p < \infty$. Suppose that a and b are bounded random variables, and g belongs to $C^1([-\tau, 0])$ in the L^p sense. Then the stochastic process $u(t)$ defined by (3) is the unique L^p-solution to (2).*

4. Solution to the random partial differential equation problem with delay

In this section, we solve (1) in the random L^p sense rigorously, for non-negative random variables a^2 and b^2 and random field $\varphi(t,x)$. We consider the formal series solution obtained by linear superposition via the method of separation of variables, and we prove that it converges in L^p and that it satisfies the delay partial differential equation from (1) by taking L^p-partial derivatives.

Theorem 4.1 *Fix $1 \leq p < \infty$. Suppose that a^2 and b^2 are non-negative and bounded random variables: $\|a^2\|_\infty < \infty$, $\|b^2\|_\infty < \infty$. Suppose also that $\varphi(t,0) = \varphi(t,l) = 0$ almost surely, $\varphi(t,\cdot) \in C^\infty[0,l]$ in L^p for each $t \in [0,\tau]$, $\frac{\partial\varphi}{\partial t}$ exists in the L^p sense and is L^p-continuous on $[0,\tau] \times [0,l]$. Let the series*

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{l}\right), \quad t \geq 0,\ x \in [0,l], \tag{4}$$

where

$$\begin{aligned} T_n(t) &= \exp\left(-\left(\frac{n\pi}{l}\right)^2 a^2 t\right) \exp_\tau\left(-\mathrm{e}^{\left(\frac{n\pi}{l}\right)^2 a^2\tau} \left(\frac{n\pi}{l}\right)^2 b^2, t-\tau\right) B_n(0) \\ &\quad + \int_0^\tau \exp\left(-\left(\frac{n\pi}{l}\right)^2 a^2 (t-s)\right) \exp_\tau\left(-\mathrm{e}^{\left(\frac{n\pi}{l}\right)^2 a^2\tau} \left(\frac{n\pi}{l}\right)^2 b^2, t-s-\tau\right) \\ &\quad \times \left(B_n'(s) + \left(\frac{n\pi}{l}\right)^2 a^2 B_n(s)\right) \mathrm{d}s, \end{aligned} \tag{5}$$

$$B_n(t) = \frac{2}{l} \int_0^l \varphi(t,x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x. \tag{6}$$

All the integrals and the derivatives here are considered in the L^p sense. Then (4) converges in L^p for all $t \geq 0$ and $x \in [0,l]$, $u(t,x)$ is L^p-continuous on $[0,\infty) \times [0,l]$, and it satisfies (1) with random L^p calculus. Moreover, it is the unique L^p-solution to the problem.

Proof. First, notice that $\varphi(t,x)=\sum_{n=1}^\infty B_n(t)\sin(n\pi x/l)$ on $[0,\tau]\times[0,l]$ in L^p. Indeed, given $B_n(t)$, integration by parts in L^p (Lemma 2.3) and the condition $\varphi(t,0)=\varphi(t,l)=0$ almost surely give

$$B_n(t)=-\frac{2l}{n^2\pi^2}\int_0^l \varphi_{xx}(t,x)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x.$$

Then

$$\|B_n(t)\|_p\leq \frac{2l}{n^2\pi^2}\int_0^l\|\varphi_{xx}(t,x)\|_p\,\mathrm{d}x\leq\frac{2l^2}{n^2\pi^2}\max_{[0,\tau]\times[0,l]}\|\varphi_{xx}\|_p.$$

By Weierstrass M-test, $\sum_{n=1}^\infty B_n(t)\sin(n\pi x/l)$ converges uniformly on $[0,\tau]\times[0,l]$ in L^p. On the other hand, the regularity of $\varphi(t,\cdot)$ in the L^p sense implies that the sample paths of $\varphi(t,\cdot)$ are absolutely continuous real functions on $[0,l]$, see Section 2. Therefore, for each $t\in[0,\tau]$, the Fourier series of $\varphi(t,\cdot)$ converges pointwise: $\varphi(t,x)=\sum_{n=1}^\infty B_n(t)\sin(n\pi x/l)$ on $[0,l]$ almost surely. Since the series converges in the random space L^p for each t and x, necessarily the equality holds in L^p, as wanted.

As shown in [29], the method of separation of variables provides a candidate solution of the form (4), where

$$\begin{cases}T_n'(t)=-\left(\frac{n\pi}{l}\right)^2a^2T_n(t)-\left(\frac{n\pi}{l}\right)^2b^2T_n(t-\tau), & t>\tau,\\ T_n(t)=B_n(t), & 0\leq t\leq\tau.\end{cases}\tag{7}$$

This problem (7) may be translated into

$$\begin{cases}Y_n'(t)=-\left(\frac{n\pi}{l}\right)^2a^2Y_n(t)-\left(\frac{n\pi}{l}\right)^2b^2Y_n(t-\tau), & t>0,\\ Y_n(t)=B_n(t+\tau), & -\tau\leq t\leq 0,\end{cases}$$

where $Y_n(t)=T_n(t+\tau)$, $t\geq-\tau$. Let us apply Lemma 3.2. First, a^2 and b^2 are bounded by assumption, so $-(n\pi/l)^2a^2$ and $-(n\pi/l)b^2$ are bounded. Second, $B_n(t)$ is continuously differentiable on $[0,\tau]$ in the L^p sense, by Lemma 2.2: $B_n'(t)=\frac{2}{l}\int_0^l\varphi_t(t,x)\sin(n\pi x/l)\,\mathrm{d}x$. Thus,

$$\begin{aligned}Y_n(t)&=\exp\left(-\left(\frac{n\pi}{l}\right)^2a^2(t+\tau)\right)\exp_\tau\left(-\mathrm{e}^{\left(\frac{n\pi}{l}\right)^2a^2\tau}\left(\frac{n\pi}{l}\right)^2b^2,t\right)B_n(0)\\&\quad+\int_{-\tau}^0\exp\left(-\left(\frac{n\pi}{l}\right)^2a^2(t-s)\right)\exp_\tau\left(-\mathrm{e}^{\left(\frac{n\pi}{l}\right)^2a^2\tau}\left(\frac{n\pi}{l}\right)^2b^2,t-s-\tau\right)\\&\quad\times\left(B_n'(s+\tau)+\left(\frac{n\pi}{l}\right)^2a^2B_n(s+\tau)\right)\mathrm{d}s.\end{aligned}$$

By using the relation $T_n(t)=Y_n(t-\tau)$ and by translating the L^p integral from $[-\tau,0]$ to $[0,\tau]$, we arrive at the required formula (5) for $T_n(t)$. With $T_n(t)$ given by (5), problem (7) is solved in the Lebesgue L^p sense rigorously.

Let us study the growth of $\|T_n(t)\|_p$. From $\varphi(t,\cdot)\in C^\infty[0,l]$ in L^p, we deduce $\|B_n(t)\|_p\leq C_k/n^k$, $k\geq1$, for constant $C_k>0$ independent of $t\in[0,\tau]$ and n (use integration by parts with Lemma 2.3). We prove by induction on the intervals $(\tau,2\tau]$, $(2\tau,3\tau]$, etc. that, for any t,

$$\|T_n(t)\|_p\leq\frac{R_k}{n^k},\tag{8}$$

$k\geq1$, where $R_k>0$ is a constant, independent of n and t within the interval of length τ but dependent on the interval. Suppose that (8) holds on $((m-1)\tau,m\tau]$ (induction hypothesis). Then, for $t\in(m\tau,(m+1)\tau]$, by (7)

$$\begin{aligned}T_n(t)&=T_n(m\tau)\exp\left(-\left(\frac{n\pi}{l}\right)^2a^2(t-m\tau)\right)\\&\quad-\int_{m\tau}^t\exp\left(-\left(\frac{n\pi}{l}\right)^2a^2(t-s)\right)\left(\frac{n\pi}{l}\right)^2b^2T_n(s-\tau)\,\mathrm{d}s,\end{aligned}$$

so, by bounding the exponential functions by 1 (because of their negative exponents),

$$\|T_n(t)\|_p\leq\|T_n(m\tau)\|_p+(t-m\tau)\left(\frac{n\pi}{l}\right)^2\|b\|_\infty^2\max_{((m-1)\tau,m\tau]}\|T_n\|_p.$$

By the induction hypothesis, (8) holds on $(m\tau,(m+1)\tau]$. This completes the proof of (8) by induction.

By Weierstrass M-test, the series that defines $u(t,x)$ in (4) converges in L^p, uniformly on $[0,t^*]\times[0,l]$, for all $t^*>0$.

It remains proving that (4) may be differentiated in L^p term-by-term so that the delay partial differential equation from (1) is satisfied. We apply Lemma 2.1. To differentiate (4) in L^p with respect to t, notice that, from (7) and (8),

$$\|T_n'(t)\|_p\leq\left(\frac{n\pi}{l}\right)^2\|a^2\|_\infty\|T_n(t)\|_p+\left(\frac{n\pi}{l}\right)^2\|b^2\|_\infty\|T_n(t-\tau)\|_p\leq\frac{S}{n^2},$$

where $S>0$, $t\in[0,t^*]$. By Weierstrass M-test, $\sum_{n=1}^{\infty} T_n'(t)\sin(n\pi x/l)$ converges uniformly on $[0,t^*]\times[0,l]$ in L^p, so that $u_t(t,x)=\sum_{n=1}^{\infty} T_n'(t)\sin(n\pi x/l)$ by Lemma 2.1. For differentiating twice with respect to x, we use (8), $n^2\|T_n(t)\|_p\leq R_4/n^2$, $t\in[0,t^*]$, so that $\sum_{n=1}^{\infty} T_n(t)\partial_{xx}^2\sin(n\pi x/l)$ converges uniformly on $[0,t^*]\times[0,l]$ in L^p, and $u_{xx}(t,x)=\sum_{n=1}^{\infty} T_n(t)\partial_{xx}^2\sin(n\pi x/l)$ by Lemma 2.1 again. Thus, (4) may be differentiated in L^p term-by-term, so we are done with the existence of L^p-solution.

For the uniqueness of L^p-solution, suppose that $u_1(t,x)$ and $u_2(t,x)$ are two L^p-solutions to (1). Let $v(t,x)=u_1(t,x)-u_2(t,x)$, which satisfies (1) in the L^p sense with $\varphi=0$. Since $v(t,\cdot)$ is continuously differentiable in the L^p sense, its sample paths are absolutely continuous (see Section 2). Then $v(t,x)=\sum_{n=1}^{\infty} R_n(t)\sin(n\pi x/l)$ in the sense of square integrable real functions with respect to the Lebesgue measure $\mathrm{d}x$ (the classical Fourier series), for each $t>\tau$. The Fourier coefficient is given by $R_n(t)=(2/l)\int_0^l v(t,x)\sin(n\pi x/l)\,\mathrm{d}x$, where this integral is an ordinary Lebesgue integral for sample paths. Now, since $v(t,\cdot)$ is L^p-Riemann integrable, that integral can also be considered in the L^p sense. Apply Lemma 2.2 to differentiate in the L^p sense: $R_n'(t)=(2/l)\int_0^l v_t(t,x)\sin(n\pi x/l)\,\mathrm{d}x=(2/l)\int_0^l(a^2v_{xx}(t,x)+b^2v_{xx}(t-\tau,x))\sin(n\pi x/l)\,\mathrm{d}x$. By applying Lemma 2.3, we derive that $R_n(t)$ satisfies (7) in L^p, with $B_n=0$. By Section 3, $R_n(t)=0$ almost surely, so $v(t,x)=0$ almost surely.

□

Remark 4.2 When $p\geq 2$, there is another proof of uniqueness. Notice that, by the method of steps, we just need uniqueness for $u_t(t,x)=a^2u_{xx}(t,x)$, $t\in(\tau,2\tau)$; $u(t,0)=u(t,l)=0$, $t\in[\tau,2\tau)$; $u(\tau,x)=0$. When $p\geq 2$, uniqueness of L^p-solution may be established by using an adapted energy method [33, Th. 3.1].

5. Uncertainty quantification

Uncertainty quantification consists in obtaining the statistical content of the solution $u(t,x)$ to (1). By truncating the series that defines $u(t,x)$, (4), we have an approximation $u_N(t,x)=\sum_{n=1}^{N} T_n(t)\sin(n\pi x/l)$ that converges to $u(t,x)$ in L^p when $N\to\infty$. The integral in the definition of $T_n(t)$ (5) (recall that L^p integrals are equal to sample path integrals, see Section 2) is approximated with Gauss-Legendre quadrature on $[0,\tau]$. The statistical moments of $u(t,x)$ can then be approximated by those of $u_N(t,x)$, up to order p. In particular, when $p\geq 2$, we have $\mathbb{E}[u_N(t,x)]\to\mathbb{E}[u(t,x)]$ and $\mathbb{V}[u_N(t,x)]\to\mathbb{V}[u(t,x)]$ when $N\to\infty$, where $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ are the expectation and the variance operators. These two statistics give information about the mean value and the dispersion.

The statistics of $u_N(t,x)$ may be estimated by using Monte Carlo simulation [16, pp. 53–54]. This is necessary when the dimension of the random space is not low, since integration via tensor quadratures becomes unfeasible. Variance reduction strategies for the Monte Carlo simulation, such as antithetic or control variates, may be conducted [34].

The exact expectation of $u_N(t,x)$ is given by $\sum_{n=1}^{N}\mathbb{E}[T_n(t)]\sin(n\pi x/l)$. Suppose that a^2 and b^2 have probability densities f_{a^2} and f_{b^2}, and that $\varphi(t,x)$ depends on a single random variable c with density f_c. Then

$$\begin{aligned}\mathbb{E}[T_n(t)] = & \iiint_{\mathbb{R}^3} \exp\left(-\left(\frac{n\pi}{l}\right)^2 a^2 t\right)\exp_\tau\left(-\mathrm{e}^{\left(\frac{n\pi}{l}\right)^2 a^2\tau}\left(\frac{n\pi}{l}\right)^2 b^2, t-\tau\right) B_n(0)\\ & \times f_{a^2}(a^2)f_{b^2}(b^2)f_c(c)\,\mathrm{d}a^2\,\mathrm{d}b^2\,\mathrm{d}c\\ & +\iiint_{\mathbb{R}^3}\int_0^\tau \exp\left(-\left(\frac{n\pi}{l}\right)^2 a^2(t-s)\right)\exp_\tau\left(-\mathrm{e}^{\left(\frac{n\pi}{l}\right)^2 a^2\tau}\left(\frac{n\pi}{l}\right)^2 b^2, t-s-\tau\right)\\ & \times\left(B_n'(s)+\left(\frac{n\pi}{l}\right)^2 a^2 B_n(s)\right) f_{a^2}(a^2)f_{b^2}(b^2)f_c(c)\,\mathrm{d}s\,\mathrm{d}a^2\,\mathrm{d}b^2\,\mathrm{d}c. \end{aligned} \tag{9}$$

So a multi-dimensional integral in four variables needs to be solved. For a second order statistic (as the variance), the formula $\mathbb{E}[u_N(t,x)^2]=\sum_{n,m=1}^{N}\mathbb{E}[T_n(t)T_m(t)]\sin(n\pi x/l)\sin(m\pi x/l)$ may be used. Each term $\mathbb{E}[T_n(t)T_m(t)]$ may be computed similarly to (9), but with a multi-dimensional integral of five variables. For higher order statistics, the integration dimension increases more. Thus, it might be possible to compute the expectation, and even the variance for moderate N, via tensor quadratures, but for higher order statistics Monte Carlo simulation on $u_N(t,x)$ may be the only feasible choice.

Example 5.1 Let us fix $l=1$ and $\tau=0.7$. Suppose that a^2 has a triangular distribution with endpoints 0 and 0.2. That is, the probability density f_{a^2} of a^2 has a triangular shape with endpoints 0 and 0.2 and peak at 0.1. Let b^2 be uniformly distributed on $[0.1,0.2]$. Finally, let $\varphi(t,x)=ctx(1-x)$, where c is a random variable with exponential distribution of mean 0.1: $f_c(c)=10\mathrm{e}^{-10c}$, $c\geq 0$. The conditions of Theorem 4.1 hold, so we have the L^p-solution defined by (4), $1\leq p<\infty$.

Let us approximate the expectation of $u(t,x)$ for different time-space locations by means of $\mathbb{E}[u_N(t,x)]=\sum_{n=1}^{N}\mathbb{E}[T_n(t)]\sin(n\pi x/l)$, where $\mathbb{E}[T_n(t)]$ is calculated by means of (9) and tensor quadratures. In Table 1, the results are tabulated up to eight significant digits. Observe that, as N increases, convergence of the mean values of $u_N(t,x)$ is achieved. When t increases, the convergence rate seems penalized and larger N is required. We would like to highlight the accuracy of the method, since it is able to determine several significant figures of the exact mean value. This level of precision cannot be achieved by the Monte Carlo simulation, because its slow convergence rate as the reciprocal of the square root of the number of realizations restricts its accuracy to two or three decimal digits, in general.

(t,x)	$(1,0.2)$	$(1,0.5)$	$(2,0.2)$	$(2,0.5)$
$N=1$	0.0070345207	0.011967841	−0.0036484069	−0.0062070406
$N=5$	0.0068324207	0.012090423	−0.0028142115	−0.0066180353
$N=10$	0.0068802825	0.012109457	−0.0031220156	−0.0067200030
$N=15$	0.0068673367	0.012116829	−0.0030169121	−0.0067888327
$N=20$	0.0068703977	0.012116214	−0.0030580722	−0.0067813572
$N=25$	0.0068688167	0.012115582	−0.0030359888	−0.0067673909
$N=30$	0.0068692174	0.012115634	−0.0030473259	−0.0067689534
$N=35$	0.0068689591	0.012115730	−0.0030401608	−0.0067733800
$N=40$	0.0068690382	0.012115722	−0.0030446297	−0.0067729556
$N=45$	0.0068689812	0.012115686	−0.0030414851	−0.0067708973
$N=50$	0.0068690209	0.012115689	−0.0030437029	−0.0067710747
$N=55$	0.0068690060	0.012115699	−0.0030420247	−0.0067721237
$N=60$	0.0068690171	0.012115698	−0.0030432686	−0.0067720497
$N=65$	0.0068690082	0.012115692	−0.0030422893	−0.0067714280
$N=70$	0.0068690151	0.012115693	−0.0030430647	−0.0067714720
$N=75$	0.0068690111	0.012115697	−0.0030424301	−0.0067718831
$N=80$	0.0068690134	0.012115696	−0.0030429393	−0.0067718515

Table 1. Approximations of the expectation $\mathbb{E}[u(t,x)]$ for different time-space locations and truncation orders N as indicated. The formula $\mathbb{E}[u_N(t,x)] = \sum_{n=1}^{N} \mathbb{E}[T_n(t)] \sin(n\pi x/l)$ has been employed, where $\mathbb{E}[T_n(t)]$ is calculated with (9) and tensor quadratures. Results are reported up to eight significant digits.

For the variance of $u(t,x)$ we proceed similarly. We use the formula $\mathbb{E}[u_N(t,x)^2] = \sum_{n,m=1}^{N} \mathbb{E}[T_n(t)T_m(t)] \sin(n\pi x/l)\sin(m\pi x/l)$ and compute each term $\mathbb{E}[T_n(t)T_m(t)]$ with tensor quadratures. Compared to $\mathbb{E}[u(t,x)]$, the complexity increases severely and unfortunately only moderate values of N can be tackled at reasonable CPU time. In Table 2, the approximations to the standard deviation $\sqrt{\mathbb{V}[u(t,x)]}$ are reported up to eight significant digits.

(t,x)	$(1,0.2)$	$(1,0.5)$	$(2,0.2)$	$(2,0.5)$
$N=1$	0.0071600523	0.012181408	0.0040801184	0.0069415120
$N=5$	0.0069676520	0.012289005	0.0030782602	0.0074975704
$N=10$	0.0070208788	0.012309646	0.0036398075	0.0076106803
$N=15$	0.0070060814	0.012319425	0.0033540365	0.0076716996
$N=20$	0.0070110544	0.012318439	0.0034710237	0.0076635280
$N=25$	0.0070084860	0.012316856	0.0033951073	0.0076529273
$N=30$	0.0070097299	0.012317061	0.0034354854	0.0076538645

Table 2. Approximations of the standard deviation $\sqrt{\mathbb{V}[u(t,x)]}$ for different time-space locations and truncation orders N as indicated. The formula $\mathbb{E}[u_N(t,x)^2] = \sum_{n,m=1}^{N} \mathbb{E}[T_n(t)T_m(t)] \sin(n\pi x/l)\sin(m\pi x/l)$ has been employed, where $\mathbb{E}[T_n(t)T_m(t)]$ is calculated with tensor quadratures. Results are reported up to eight significant digits.

6. Conclusions

In this paper, we have dealt with the randomized generalized diffusion equation with delay. The inputs of the model, namely the diffusion coefficients and the initial history function, are assumed to be random quantities on a probability space. The solution becomes a random field. The classical method of separation of variables provides a candidate series solution. By using random Lebesgue calculus, the delay problem has been solved in a stochastic sense rigorously. This fact also allows for uncertainty quantification, by truncating the series solution and estimating its main statistics.

Of course, the methodology followed is not restricted to the particular problem treated here. We believe that most of the problems that are solvable via separation of variables may be addressed in a stochastic sense with random Lebesgue calculus. This is not only a theoretical concern, but also an important issue for uncertainty quantification.

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Conflict of Interest Statement

The author declares that there is no conflict of interests regarding the publication of this article.

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