Rank-one perturbations of matrix pencils

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Abstract

We solve the problem of characterizing the Kronecker structure of a matrix pencil obtained by a rank-one perturbation of another matrix pencil. The results hold over arbitrary fields.

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1 Introduction

Given a matrix pencil $A(s) = A_0 + sA_1 \in \mathbb{F}^{n\times m}$, the rank perturbation problem consists in characterizing the Kronecker structure of $A(s) + P(s)$, where $P(s)$ is a matrix pencil of bounded rank.

The Kronecker structure of a matrix pencil is determined by the complete system of invariants for the strict equivalence of matrix pencils, i.e., the invariant factors, infinite elementary divisors, and row and column minimal indices. For regular matrix pencils the Kronecker structure is known as the Weierstrass structure and is determined only by the invariant factors and the infinite elementary divisors. In particular, the Jordan structure of a square matrix is defined by the Weierstrass structure of the associated characteristic pencil, which is a regular pencil without infinite elementary divisors. Analogously, the feedback invariants of a pair of matrices, i.e., the invariant factors and the column (or row) minimal indices are the Kronecker invariants of the associated characteristic pencil.

In the last decades rank perturbations of matrix pencils have been analyzed in many papers from different approaches. The problem has been studied generically, i.e., when the perturbation $P(s)$ belongs to an open and dense subset of

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the set of pencils of rank less than or equal to r, for a given integer r. In other cases, the pencil $P(s)$ is an arbitrary perturbation belonging to the whole set of pencils of rank less than or equal to r. In this paper we follow the second approach.

From a generic point of view, changes in the Jordan structure of a square constant matrix or in the Weierstrass structure of a regular pencil corresponding to a fixed eigenvalue after low rank perturbations have been studied, among others, in [3, 4, 6, 7, 16, 18, 19, 21, 22]. See also the references therein.

The case where a square matrix (regular pencil) is perturbed by an arbitrary perturbation matrix P (arbitrary matrix pencil $P(s)$) has also been studied by several authors. For square constant matrices and a constant perturbation of bounded rank r, a solution is given in [23] and [25]. For $r = 1$ the problem was already solved in [24]. The case where the perturbation has fixed rank has been solved in [23] over algebraically closed fields.

For regular pencils the problem has been studied for $r = 1$ in [15]. For arbitrary perturbations of bounded rank the problem has been solved in [2], and for perturbations of fixed rank in [1]. In both cases the solutions obtained do not involve any condition on the rank of the type "low-rank", and the results hold for fields having sufficient number of elements (fields requiring just the condition that at least one element of the field or the point at infinity is not included neither in the spectrum of the original pencil nor in the perturbed one).

There is less literature dealing with the case of singular pencils. The problem is more difficult, since the row and column minimal indices of the pencils are involved. For non full rank pencils the change of the four types of invariants under generic low rank perturbations has been characterized in [5]. For square singular pencils, in [17] the authors represent pencils via linear relations and obtain bounds for the number of Jordan chains which may change under an arbitrary rank-one perturbation. The problem of characterizing the feedback equivalence invariants of a pair of matrices, i.e., the Kronecker invariants of the associated characteristic pencil, under a constant perturbation of bounded rank is solved in [11]. Here, the authors find the solution relating the problem to a matrix pencil completion problem.

In this paper we study arbitrary rank-one perturbations of matrix pencils. We solve the problem transforming it into a matrix pencil completion problem. The solution obtained holds for arbitrary fields.

The paper is organized as follows. In Section 2 we introduce the notation, basic definitions and preliminary results. In Section 3 we establish the problem which we are going to study and relate it to a matrix pencil completion problem. Then, in Section 4 we introduce previous results about completion of matrix pencils which will be needed later. In Section 5, a solution to the rank-one perturbation problem is stated in Theorem 5.1. Several technical lemmas necessary to prove it are given in Subsection 5.1. Theorem 5.1 is proven is Subsection 5.2. Finally, in Section 6 we summarize the main contributions of the paper.

2 Preliminaries

Let $\mathbb{F}$ be a field. $\mathbb{F}[s]$ denotes the ring of polynomials in the indeterminate s with coefficients in $\mathbb{F}$ and $\mathbb{F}[s,t]$ the ring of polynomials in two variables s,t with coefficients in $\mathbb{F}$. We denote by $\mathbb{F}^{p\times q}$, $\mathbb{F}[s]^{p\times q}$ and $\mathbb{F}[s,t]^{p\times q}$ the vector spaces of $p\times q$ matrices with elements in $\mathbb{F}$, $\mathbb{F}[s]$ and $\mathbb{F}[s,t]$, respectively. $\mathrm{Gl}_p(\mathbb{F})$ will be the general linear group of invertible matrices in $\mathbb{F}^{p\times p}$.

Given a polynomial matrix $G(s)\in\mathbb{F}[s]^{p\times q}$, the *degree* of $G(s)$, denoted by $\deg(G(s))$, is the maximum of the degrees of its entries. The *normal rank* of $G(s)$, denoted by $\mathrm{rank}(G(s))$, is the order of the largest nonidentically zero minor of $G(s)$, i.e., it is the rank of $G(s)$ considered as a matrix on the field of fractions of $\mathbb{F}[s]$.

A *matrix pencil* is a polynomial matrix $G(s)\in\mathbb{F}[s]^{p\times q}$ such that $\deg(G(s))\leq 1$. The pencil is *regular* if $p=q$ and $\det(G(s))$ is not the zero polynomial. Otherwise it is *singular*.

Two matrix pencils $G(s)=G_0+sG_1, H(s)=H_0+sH_1\in\mathbb{F}[s]^{p\times q}$ are *strictly equivalent* ($G(s)\overset{s.e.}{\sim}H(s)$) if there exist invertible matrices $Q\in\mathrm{Gl}_p(\mathbb{F})$, $R\in\mathrm{Gl}_q(\mathbb{F})$ such that $G(s)=QH(s)R$.

Given the pencil $G(s)=G_0+sG_1\in\mathbb{F}[s]^{p\times q}$ of rank $G(s)=n$, a complete system of invariants for the strict equivalence of matrix pencils is formed by a chain of homogeneous polynomials $\Gamma_1(s,t)\mid\cdots\mid\Gamma_n(s,t),\ \Gamma_i(s,t)\in\mathbb{F}[s,t],\ 1\leq i\leq n$, called the *homogeneous invariant factors*, and two collections of nonnegative integers $c_1\geq\cdots\geq c_{q-n}$ and $u_1\geq\cdots\geq u_{p-n}$, called the *column and row minimal indices* of the pencil, respectively. In turn, the homogeneous invariant factors are determined by a chain of polynomials $\gamma_1(s)\mid\ldots\mid\gamma_n(s)$ in $\mathbb{F}[s]$, called the invariant factors, and a chain of polynomials $t^{k_1}\mid\ldots\mid t^{k_n}$ in $\mathbb{F}[t]$, called the infinite elementary divisors. In fact, we can write

$$\Gamma_i(s,t)=t^{k_i}t^{\deg(\gamma_i(s))}\gamma_i\left(\frac{s}{t}\right),\ 1\leq i\leq n.$$

The associated canonical form is the Kronecker canonical form. For details see [13, Ch. 2] or [14, Ch. 12] for infinite fields, and [20, Ch. 2] for arbitrary fields. In what follows we will work with the homogeneous invariant factors. We will take $\Gamma_i(s,t)=1$ ($\gamma_i(s)=1$) whenever $i<1$ and $\Gamma_i(s,t)=0$ ($\gamma_i(s)=0$) when $i>n$. The sum of the degrees of the homogeneous invariant factors plus the sum of the minimal indices is equal to the rank of the pencil. Also, if $T(s)=G(s)^T$, then $G(s)$ and $T(s)$ share the homogeneous invariant factors and have interchanged minimal indices, i.e., the column (row) minimal indices of $T(s)$ are the row (column) minimal indices of $G(s)$.

Observe that if $G(s)\in\mathbb{F}[s]^{p\times q}$ and $\mathrm{rank}(G(s))=p$ ($\mathrm{rank}(G(s))=q$), then $G(s)$ does not have row (column) minimal indices. As a consequence, the invariants for the strict equivalence of regular matrix pencils are reduced to the homogeneous invariant factors.

In this paper we study the Kronecker structure of arbitrary pencils perturbed by pencils of rank one. A matrix pencil of rank one allows a very simple decomposition (see [15] for $\mathbb{F} = \mathbb{C}$). In the next proposition we analyze this decomposition for arbitrary fields, depending on the Kronecker structure of the pencil.

Proposition 2.1 *Let $P(s) \in \mathbb{F}[s]^{p\times q}$ be a matrix pencil of* $\operatorname{rank} P(s) = 1$.

1. *If $P(s)$ has a nontrivial invariant factor, then there exist nonzero vectors $u \in \mathbb{F}^p$, $\bar{v} \in \mathbb{F}^q$ and nonzero pencils $\bar{u}(s) \in \mathbb{F}[s]^p$, $v(s) \in \mathbb{F}[s]^q$ such that*
$$P(s) = uv(s)^T = \bar{u}(s)\bar{v}^T.$$
2. *If $P(s)$ has an infinite elementary divisor, then there exist nonzero vectors $u \in \mathbb{F}^p$, $v \in \mathbb{F}^q$ such that*
$$P(s) = uv^T.$$
3. *If $P(s)$ has a positive column minimal index, then there exist a nonzero vector $u \in \mathbb{F}^p$ and a nonzero pencil $v(s) \in \mathbb{F}[s]^q$ such that*
$$P(s) = uv(s)^T.$$
4. *If $P(s)$ has a positive row minimal index, then there exist a nonzero vector $v \in \mathbb{F}^q$ and a nonzero pencil $u(s) \in \mathbb{F}[s]^p$ such that*
$$P(s) = u(s)v^T.$$

Proof. Let $P_c(s)$ be the Kronecker canonical form of $P(s)$. Then, there exist $Q = \begin{bmatrix} q_1 & \dots & q_p \end{bmatrix} \in \mathrm{Gl}_p(\mathbb{F})$ and $R = \begin{bmatrix} r_1^T \\ \vdots \\ r_q^T \end{bmatrix} \in \mathrm{Gl}_q(\mathbb{F})$ such that $P(s) = QP_c(s)R$.

1. If $P(s)$ has a nontrivial invariant factor $s+\lambda$, $\lambda \in \mathbb{F}$, then $P_c(s) = \begin{bmatrix} s+\lambda & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{F}[s]^{p\times q}$. Hence, $P(s) = ((s+\lambda)q_1)r_1^T = q_1((s+\lambda)r_1)^T$.
2. If $P(s)$ has an infinite elementary divisor, then $P_c(s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{F}[s]^{p\times q}$. Therefore, $P(s) = q_1 r_1^T$.
3. If $P(s)$ has a positive column minimal index, then $P_c(s) = \begin{bmatrix} s & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{F}[s]^{p\times q}$. Therefore, $P(s) = q_1(sr_1^T + r_2^T)$.
4. If $P(s)$ has a positive row minimal index, then $P(s)^T$ has a positive column minimal index. Therefore, $P(s)^T = q_1(sr_1^T + r_2^T)$, i.e., $P(s) = (sr_1 + r_2)q_1^T$, as desired.

□

3 Statement of the problem

The problem we deal with in this paper is the following:

Problem 3.1 (Rank-one perturbation of matrix pencils) *Given two matrix pencils $A(s), B(s) \in \mathbb{F}[s]^{p\times q}$, find necessary and sufficient conditions for the existence of a matrix pencil $P(s) \in \mathbb{F}[s]^{p\times q}$ of $\operatorname{rank}(P(s)) = 1$ such that $A(s) + P(s) \overset{s.e.}{\sim} B(s)$.*

First of all we analyze two particular cases.

- $p = 1$ or $q = 1$, and $A(s) \neq 0$ or $B(s) \neq 0$. If $\mathbb{F} \neq \{0, 1\}$, there always exists $P(s) = P_0 + sP_1 \in \mathbb{F}[s]^{p\times q}$ of $\operatorname{rank}(P(s)) = 1$ such that $A(s) + P(s) \overset{s.e.}{\sim} B(s)$. For example, if $p = 1$, let $c \in \mathbb{F} \setminus \{0\}$ be such that $A(s) \neq cB(s)$. Then $A(s) + (cB(s) - A(s)) \overset{s.e.}{\sim} B(s)$. If $\mathbb{F} = \{0, 1\}$, then there exists $P(s) \in \mathbb{F}[s]^{p\times q}$ such that $\operatorname{rank}(P(s)) = 1$ and $A(s) + P(s) \overset{s.e.}{\sim} B(s)$ if and only if $A(s) \neq B(s)$.

- $p > 1$ or $q > 1$, and $A(s), B(s) \in \mathbb{F}[s]^{p\times q}$ are such that $A(s) \overset{s.e.}{\sim} B(s)$ and $A(s) \neq 0$. Then there always exists $P(s) \in \mathbb{F}[s]^{p\times q}$ of $\operatorname{rank}(P(s)) = 1$ such that $A(s) + P(s) \overset{s.e.}{\sim} B(s)$. For example, let $q = 2$ and let $a_1(s) \neq 0$, $a_2(s)$ be the columns of $A(s)$. Then, $B(s) \overset{s.e.}{\sim} A(s) \overset{s.e.}{\sim} A(s)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A(s) + \begin{bmatrix} 0 & a_1(s) \end{bmatrix}$.

The next lemma shows that in order to solve Problem 3.1 the pencil $A(s)$ can be substituted by any other pencil strictly equivalent to $A(s)$. It was proven in [2, Lemma 3.2] for $p = q$. The proof for the general case is completely analogous.

Lemma 3.2 *Let $A(s), B(s), P(s) \in \mathbb{F}[s]^{p\times q}$ be matrix pencils. Let $Q \in \mathrm{Gl}_p(\mathbb{F})$, $R \in \mathrm{Gl}_q(\mathbb{F})$ and $A'(s) = QA(s)R$. If $A(s) + P(s) \overset{s.e.}{\sim} B(s)$ then $A'(s) + QP(s)R \overset{s.e.}{\sim} B(s)$.*

Problem 3.1 can be stated as a pencil completion problem, as we see next.

Lemma 3.3 *Let $A(s), B(s) \in \mathbb{F}[s]^{p\times q}$ be matrix pencils such that $A(s) \overset{s.e.}{\not\sim} B(s)$. Then there exists a matrix pencil $P(s) \in \mathbb{F}[s]^{p\times q}$ of $\operatorname{rank} P(s) = 1$ such that $A(s) + P(s) \overset{s.e.}{\sim} B(s)$ if and only if one of the following conditions holds:*

(i) There exist matrix pencils $a(s), b(s) \in \mathbb{F}[s]^{1\times q}$, $A_{21}(s) \in \mathbb{F}[s]^{(p-1)\times q}$ such that $A(s) \overset{s.e.}{\sim} \begin{bmatrix} a(s) \\ A_{21}(s) \end{bmatrix}$ and $B(s) \overset{s.e.}{\sim} \begin{bmatrix} b(s) \\ A_{21}(s) \end{bmatrix}$.

(ii) There exist matrix pencils $\bar{a}(s), \bar{b}(s) \in \mathbb{F}[s]^{p\times 1}$, $A_{12}(s) \in \mathbb{F}[s]^{p\times(q-1)}$ such that $A(s) \overset{s.e.}{\sim} \begin{bmatrix} \bar{a}(s) & A_{12}(s) \end{bmatrix}$ and $B(s) \overset{s.e.}{\sim} \begin{bmatrix} \bar{b}(s) & A_{12}(s) \end{bmatrix}$.

Proof. Assume that there exists a matrix pencil $P(s) \in \mathbb{F}[s]^{p\times q}$ of rank $P(s) = 1$ such that $A(s)+P(s) \overset{s.e.}{\sim} B(s)$. By Proposition 2.1, there exist nonzero pencils $u(s) \in \mathbb{F}[s]^p$, $v(s) \in \mathbb{F}[s]^q$ such that $P(s) = u(s)v(s)^T$ and $u(s) = u \in \mathbb{F}^p$ or $v(s) = v \in \mathbb{F}^q$.

If $u(s) = u \in \mathbb{F}^p$, let $R \in \mathrm{Gl}(p)$ be such that $Ru = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{F}^{(1+(p-1))}$ and let $RA(s) = \begin{bmatrix} a(s) \\ A_{21}(s) \end{bmatrix} \in \mathbb{F}[s]^{(1+(p-1))\times q}$ and $b(s) = a(s) + v(s)^T$. Then $A(s) \overset{s.e.}{\sim} \begin{bmatrix} a(s) \\ A_{21}(s) \end{bmatrix}$ and $B(s) \overset{s.e.}{\sim} R(A(s) + P(s)) = \begin{bmatrix} a(s) \\ A_{21}(s) \end{bmatrix} + \begin{bmatrix} v(s)^T \\ 0 \end{bmatrix} = \begin{bmatrix} b(s) \\ A_{21}(s) \end{bmatrix}$. Therefore, (i) holds.

If $v(s) = v \in \mathbb{F}^q$, we can analogously obtain (ii).

Conversely, let us assume that (i) holds. As $A(s) \overset{s.e.}{\not\sim} B(s)$, we have $a(s) \neq b(s)$. Let $\bar{P}(s) = \begin{bmatrix} b(s) - a(s) \\ 0 \end{bmatrix} \in \mathbb{F}[s]^{(1+(p-1))\times q}$. Then $\operatorname{rank} \bar{P}(s) = 1$ and $\begin{bmatrix} a(s) \\ A_{21}(s) \end{bmatrix} + \bar{P}(s) = \begin{bmatrix} b(s) \\ A_{21}(s) \end{bmatrix}$. By Lemma 3.2, there exists a pencil $P(s)$ such that $\operatorname{rank} P(s) = 1$ and $A(s) + P(s) \overset{s.e.}{\sim} B(s)$.

If (ii) holds, the result holds applying the previous case to $A(s)^T$ and $B(s)^T$. □

Remark 3.4 *Observe that if the permutation $P(s)$ is of the form $P(s) = uv(s)^T$, $u \in \mathbb{F}^p$, then the perturbation problem can be transformed into a row completion problem. Analogously, if $P(s) = u(s)v^T$, $v \in \mathbb{F}^q$, then the perturbation problem can be transformed into a column completion problem.*

4 Matrix pencil completion theorems

According to Lemma 3.3, the Problem 3.1 can be approached as a matrix pencil completion problem. We introduce in this section some results in that area which will be used later. To state them we need some notation and definitions.

Given two integers n and m, whenever $n > m$ we take $\sum_{i=n}^{m} = 0$. In the same way, if a condition is stated for $n \leq i \leq m$ with $n > m$, we understand that the condition disappears.

Given a sequence of integers $a_1, \ldots, a_m$ such that $a_1 \geq \cdots \geq a_m$ we will write $\mathbf{a} = (a_1, \ldots, a_m)$ and we will take $a_i = \infty$ for $i < 1$ and $a_i = -\infty$ for $i > m$. If $a_m \geq 0$, the sequence $\mathbf{a} = (a_1, \ldots, a_m)$ is called a *partition*.

Given three sequences $\mathbf{d}$, $\mathbf{a}$ and $\mathbf{g}$, we introduce next the concept of generalized majorization.

Definition 4.1 (Generalized majorization [9, Definition 2]) *Given three sequences of integers $\mathbf{d} = (d_1, \ldots, d_m)$, $\mathbf{a} = (a_1, \ldots, a_s)$ and $\mathbf{g} = (g_1, \ldots, g_{m+s})$*

such that $d_1 \geq \cdots \geq d_m$, $a_1 \geq \cdots \geq a_s$, $g_1 \geq \cdots \geq g_{m+s}$, *we say that* $\mathbf{g}$ *is majorized by* $\mathbf{d}$ *and* $\mathbf{a}$ ($\mathbf{g} \prec' (\mathbf{d}, \mathbf{a})$) *if*

$$d_i \geq g_{i+s}, \quad 1 \leq i \leq m, \tag{1}$$

$$\sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j - j} d_i \leq \sum_{i=1}^{j} a_i, \quad 1 \leq j \leq s, \tag{2}$$

where $h_j = \min\{i \,:\, d_{i-j+1} < g_i\}$, $1 \leq j \leq s \quad (d_{m+1} = -\infty)$,

$$\sum_{i=1}^{m+s} g_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i. \tag{3}$$

In the case that $s = 0$, condition (2) disappears, and conditions (1) and (3) are equivalent to $\mathbf{d} = \mathbf{g}$.

In the case that $s = 1$, from condition (3) we observe that a_1 is completely determined by $\mathbf{d}$ and $\mathbf{g}$ ($a_1 = \sum_{i=1}^{m+1} g_i - \sum_{i=1}^{m} d_i$), therefore we will write $\mathbf{g} \prec' (\mathbf{d}, a_1)$ as $\mathbf{g} \prec' \mathbf{d}$ and we will refer to it as 1*step-generalized majorization*. Moreover, it is easy to see that $\mathbf{g} \prec' \mathbf{d}$ if and only if

$$d_i = g_{i+1}, \quad h \leq i \leq m,$$

where $h = \min\{i : d_i < g_i\}$.

In [9], the 1step-generalized majorization is called *elementary generalized majorization* and it is denoted by $\mathbf{g} \prec'_1 (\mathbf{d}, a_1)$.

Remark 4.2

1. *If* $\mathbf{g}$ *and* $\mathbf{d}$ *satisfy that* $g_i \leq d_i$ *for* $1 \leq i \leq m$*, then* $h = m + 1$ *and* $\mathbf{g} \prec' \mathbf{d}$.
2. *Notice that if* $\mathbf{g} \prec' \mathbf{d}$ *and for some index* $1 \leq i \leq m$ *we have* $d_i > g_{i+1}$*, then* $i < h$.

Given two pencils $H_1(s) \in \mathbb{F}[s]^{(n+p)\times(n+m)}$ and $H(s) \in \mathbb{F}[s]^{(n+p+x+y)\times(n+m)}$, of $\operatorname{rank}(H_1(s)) = n$ and $\operatorname{rank}(H(s)) = n + x$, in [12, Theorem 4.3] (see also [8, Theorem 2]), necessary and sufficient conditions are given for the existence of a pencil $Y(s) \in \mathbb{F}[s]^{(x+y)\times(n+m)}$ such that $H(s) \overset{s.e.}{\sim} \begin{bmatrix} H_1(s) \\ Y(s) \end{bmatrix}$. The two following lemmas are particular cases of [12, Theorem 4.3] for $x + y = 1$. First, we state the result when $x = 0$, $y = 1$.

Lemma 4.3 ([12, Particular case of Theorem 4.3]) *Given two matrix pencils* $H_1(s) \in \mathbb{F}[s]^{(n+p)\times(n+m)}$, $H(s) \in \mathbb{F}[s]^{(n+p+1)\times(n+m)}$ *of* $\operatorname{rank}(H_1(s)) = \operatorname{rank}(H(s)) = n$*, let* $\pi_1^1(s,t) \mid \cdots \mid \pi_n^1(s,t)$, $g_1 \geq \cdots \geq g_m \geq 0$ *and* $w_1 \geq \cdots \geq w_\theta > 0 = w_{\theta+1} \geq \cdots \geq w_p$ *be the homogeneous invariant factors, the column and the row minimal indices of* $H_1(s)$*, respectively, and let* $\pi_1(s,t) \mid \cdots \mid \pi_n(s,t)$,

$c_1 \geq \cdots \geq c_m \geq 0$ *and* $u_1 \geq \cdots \geq u_{\bar{\theta}} > 0 = u_{\bar{\theta}+1} \geq \cdots \geq u_{p+1}$ *be the homogeneous invariant factors, the column and the row minimal indices indices of* $H(s)$*, respectively.*

Let $\mathbf{g} = (g_1, \ldots, g_m)$, $\mathbf{w} = (w_1, \ldots, w_p)$, $\mathbf{c} = (c_1, \ldots, c_m)$, $\mathbf{u} = (u_1, \ldots, u_{p+1})$.

There exists a pencil $h(s) \in \mathbb{F}[s]^{1\times(n+m)}$ *such that* $H(s) \overset{s.e.}{\sim} \begin{bmatrix} h(s) \\ H_1(s) \end{bmatrix}$ *if and only if*

$$\bar{\theta} \geq \theta, \tag{4}$$

$$\pi_i(s,t) \mid \pi_i^1(s,t) \mid \pi_{i+1}(s,t), \quad 1 \leq i \leq n, \tag{5}$$

$$\mathbf{u} \prec' \mathbf{w}, \tag{6}$$

$$\mathbf{g} = \mathbf{c}. \tag{7}$$

Next, we state the result when $x = 1$, $y = 0$.

Lemma 4.4 ([12, Particular case of Theorem 4.3]) *Given two matrix pencils* $H_1(s) \in \mathbb{F}[s]^{(n+p)\times(n+m)}$, $H(s) \in \mathbb{F}[s]^{(n+p+1)\times(n+m)}$ *of* $\operatorname{rank}(H_1(s)) = n$ *and* $\operatorname{rank}(H(s)) = n+1$*, let* $\pi_1^1(s,t) \mid \cdots \mid \pi_n^1(s,t)$, $g_1 \geq \cdots \geq g_m \geq 0$ *and* $w_1 \geq \cdots \geq w_p$ *be the homogeneous invariant factors, the column and the row minimal indices of* $H_1(s)$*, respectively, and let* $\pi_1(s,t) \mid \cdots \mid \pi_{n+1}(s,t)$, $c_1 \geq \cdots \geq c_{m-1} \geq 0$ *and* $u_1 \geq \cdots \geq u_p$ *be the homogeneous invariant factors, the column and the row minimal indices of* $H(s)$*, respectively.*

Let $\mathbf{g} = (g_1, \ldots, g_m)$, $\mathbf{w} = (w_1, \ldots, w_p)$, $\mathbf{c} = (c_1, \ldots, c_{m-1})$*, and* $\mathbf{u} = (u_1, \ldots, u_p)$.

There exists a pencil $h(s) \in \mathbb{F}[s]^{1\times(n+m)}$ *such that* $H(s) \overset{s.e.}{\sim} \begin{bmatrix} h(s) \\ H_1(s) \end{bmatrix}$ *if and only if (5),*

$$\mathbf{g} \prec' \mathbf{c}, \tag{8}$$

$$\mathbf{w} = \mathbf{u}. \tag{9}$$

5 Main theorem

In the following theorem we give a solution to Problem 3.1.

Theorem 5.1 *Let* $A(s), B(s) \in \mathbb{F}[s]^{p\times q}$ *be matrix pencils such that* $A(s) \overset{s.e.}{\not\sim} B(s)$*. Let* $\operatorname{rank} A(s) = n_1$, $\operatorname{rank} B(s) = n_2$*, let* $\phi_1(s,t) \mid \cdots \mid \phi_{n_1}(s,t)$, $c_1 \geq \cdots \geq c_{q-n_1} \geq 0$ *and* $u_1 \geq \cdots \geq u_{p-n_1} \geq 0$ *be, respectively, the homogeneous invariant factors, column minimal indices and row minimal indices of* $A(s)$ *and let* $\psi_1(s,t) \mid \cdots \mid \psi_{n_2}(s,t)$, $d_1 \geq \cdots \geq d_{q-n_2} \geq 0$ *and* $v_1 \geq \cdots \geq v_{p-n_2} \geq 0$ *be, respectively, the homogeneous invariant factors, column minimal indices and row minimal indices of* $B(s)$*.*

Let $\mathbf{c} = (c_1, \ldots, c_{q-n_1})$, $\mathbf{d} = (d_1, \ldots, d_{q-n_2})$, $\mathbf{u} = (u_1, \ldots, u_{p-n_1})$, $\mathbf{v} = (v_1, \ldots, v_{p-n_2})$ *and* $n = \min\{n_1, n_2\}$*.*

1. *If* $\mathbf{c}=\mathbf{d}$, $\mathbf{u}=\mathbf{v}$, *then there exists a pencil* $P(s)\in\mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s))=1$ *such that* $A(s)+P(s)\overset{s.e.}{\sim}B(s)$ *if and only if*

$$\psi_{i-1}(s,t)\mid\phi_i(s,t)\mid\psi_{i+1}(s,t),\quad 1\leq i\leq n. \tag{10}$$

2. *If* $\mathbf{c}\neq\mathbf{d}$, $\mathbf{u}=\mathbf{v}$, *let*

$$\begin{aligned}\ell&=\max\{i\ :\ c_i\neq d_i\},\\ f&=\max\{i\in\{1,\dots,\ell\}\ :\ c_i<d_{i-1}\}\quad(d_0=+\infty),\\ f'&=\max\{i\in\{1,\dots,\ell\}\ :\ d_i<c_{i-1}\}\quad(c_0=+\infty),\\ G&=n-1-\sum_{i=1}^{n-1}\deg(\gcd(\phi_{i+1}(s,t),\psi_{i+1}(s,t)))-\sum_{i=1}^{p-n}u_i.\end{aligned}$$

Then there exists a pencil $P(s)\in\mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s))=1$ *such that* $A(s)+P(s)\overset{s.e.}{\sim}B(s)$ *if and only if (10) and*

$$G\leq\sum_{i=1}^{q-n}\min\{c_i,d_i\}+\max\{c_f,d_{f'}\}. \tag{11}$$

3. *If* $\mathbf{c}=\mathbf{d}$, $\mathbf{u}\neq\mathbf{v}$, *let*

$$\begin{aligned}\bar\ell&=\max\{i\ :\ u_i\neq v_i\},\\ \bar f&=\max\{i\in\{1,\dots,\bar\ell\}\ :\ u_i<v_{i-1}\}\quad(v_0=+\infty),\\ \bar f'&=\max\{i\in\{1,\dots,\bar\ell\}\ :\ v_i<u_{i-1}\}\quad(u_0=+\infty),\\ \bar G&=n-1-\sum_{i=1}^{n-1}\deg(\gcd(\phi_{i+1}(s,t),\psi_{i+1}(s,t)))-\sum_{i=1}^{q-n}c_i.\end{aligned}$$

Then there exists a pencil $P(s)\in\mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s))=1$ *such that* $A(s)+P(s)\overset{s.e.}{\sim}B(s)$ *if and only if (10) and*

$$\bar G\leq\sum_{i=1}^{p-n}\min\{u_i,v_i\}+\max\{u_{\bar f},v_{\bar f'}\}. \tag{12}$$

4. *If* $\mathbf{c}\neq\mathbf{d}$, $\mathbf{u}\neq\mathbf{v}$, *then there exists a pencil* $P(s)\in\mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s))=1$ *such that* $A(s)+P(s)\overset{s.e.}{\sim}B(s)$ *if and only if there exist homogeneous polynomials* $\pi_1^1(s,t)\mid\cdots\mid\pi_n^1(s,t)$ *such that*

$$\operatorname{lcm}(\phi_i(s,t),\psi_i(s,t))\mid\pi_i^1(s,t)\mid\gcd(\phi_{i+1}(s,t),\psi_{i+1}(s,t)),\quad 1\leq i\leq n, \tag{13}$$

and one of the four following conditions holds:

(a)

$$\mathbf{c}\prec'\mathbf{d},\quad \mathbf{u}\prec'\mathbf{v}, \tag{14}$$

$$\sum_{i=1}^{n}\deg(\pi_i^1(s,t))=n-\sum_{i=1}^{q-n_1}c_i-\sum_{i=1}^{p-n_2}v_i. \tag{15}$$

(b)

$$\mathbf{d} \prec' \mathbf{c}, \quad \mathbf{v} \prec' \mathbf{u}, \tag{16}$$

$$\sum_{i=1}^{n} \deg(\pi_i^1(s,t)) = n - \sum_{i=1}^{q-n_2} d_i - \sum_{i=1}^{p-n_1} u_i. \tag{17}$$

(c) (14) and (17).

(d) (16) and (15).

For the proof of the theorem we need some technical lemmas on 1step-generalized majorizations, which are presented in Subsection 5.1. Afterwards, the theorem will be proven in Subsection 5.2.

If $\mathbb{F}$ is algebraically closed, the conditions of the case $\mathbf{c} \neq \mathbf{d}$, $\mathbf{u} \neq \mathbf{v}$ can be written in terms of inequalities, as stated in the next lemma. The proof is inspired by that of [26, Corollary 4.3].

Lemma 5.2 *Let $\Omega_1(s,t), \dots, \Omega_n(s,t), \Psi_1(s,t), \dots, \Psi_{n+1}(s,t) \in \mathbb{F}[s,t]$ be homogeneous polynomials such that $\Omega_1(s,t) \mid \cdots \mid \Omega_n(s,t)$, $\Psi_1(s,t) \mid \cdots \mid \Psi_{n+1}(s,t)$, and*

$$\Psi_{i-1}(s,t) \mid \Omega_i(s,t) \mid \Psi_{i+1}(s,t), \quad 1 \leq i \leq n. \tag{18}$$

Let x be a nonnegative integer.

If $\mathbb{F}$ is an algebraically closed field, then there exist homogeneous polynomials $\pi_1^1(s,t) \mid \cdots \mid \pi_n^1(s,t)$ satisfying

$$\operatorname{lcm}(\Omega_i(s,t), \Psi_i(s,t)) \mid \pi_i^1(s,t) \mid \gcd(\Omega_{i+1}(s,t), \Psi_{i+1}(s,t)), \quad 1 \leq i \leq n, \tag{19}$$

and

$$\sum_{i=1}^{n} \deg(\pi_i^1(s,t)) = x \tag{20}$$

if and only if

$$\sum_{i=1}^{n} \deg(\operatorname{lcm}(\Omega_i(s,t), \Psi_i(s,t))) \leq x \leq \sum_{i=1}^{n} \deg(\gcd(\Omega_{i+1}(s,t), \Psi_{i+1}(s,t))). \tag{21}$$

Proof. From (19) and (20), clearly we deduce (21).

Conversely, assume that (21) holds. Condition (18) implies that

$$\operatorname{lcm}(\Omega_i(s,t), \Psi_i(s,t)) \mid \gcd(\Omega_{i+1}(s,t), \Psi_{i+1}(s,t)), \quad 1 \leq i \leq n,$$

hence, $\Delta_i(s,t) = \frac{\gcd(\Omega_{i+1}(s,t), \Psi_{i+1}(s,t))}{\operatorname{lcm}(\Omega_i(s,t), \Psi_i(s,t))}$ are homogeneous polynomials.

Let $\delta_i = \deg(\operatorname{lcm}(\Omega_i(s,t), \Psi_i(s,t))$, $\delta_i' = \deg(\gcd(\Omega_{i+1}(s,t), \Psi_{i+1}(s,t)))$, $1 \leq i \leq n$. From (21) we have

$$0 \leq x - \sum_{i=1}^{n} \delta_i \leq \sum_{i=1}^{n} (\delta_i' - \delta_i).$$

Let $z_1, \ldots, z_n$ be integers such that $0 \leq z_i \leq \delta'_i - \delta_i = \deg(\Delta_i(s,t))$ and $\sum_{i=1}^n z_i = x - \sum_{i=1}^n \delta_i$. As $\mathbb{F}$ is algebraically closed, there exists homogeneous polynomials $\gamma_i(s,t)$ such that $\deg(\gamma_i(s,t)) = z_i$ and $\gamma_i(s,t) \mid \Delta_i(s,t)$, for $1 \leq i \leq n$.

Let $\pi_i^1(s,t) = \operatorname{lcm}(\Omega_i(s,t), \Psi_i(s,t))\gamma_i(s,t)$, $1 \leq i \leq n$. Then, $\pi_i^1(s,t) \mid \pi_{i+1}^1(s,t)$ for $1 \leq i \leq n-1$, and they satisfy (19) and (20). □

Example 5.3 *Let* $\mathbb{F} = \mathbb{C}$, $n = 6$, $x = 3$. $\Omega_1(s,t) = \cdots = \Omega_5(s,t) = 1, \Omega_6(s,t) = s^2 + t^2$, $\Psi_1(s,t) = \ldots \Psi_5(s,t) = 1, \Psi_6(s,t) = \Psi_7(s,t) = s^2 + t^2$. *Then*

$$\operatorname{lcm}(\Omega_i(s,t), \Psi_i(s,t)) = 1,\ 1 \leq i \leq 5;\ \operatorname{lcm}(\Omega_6(s,t), \Psi_6(s,t)) = s^2 + t^2,$$

$$\gcd(\Omega_{i+1}(s,t), \Psi_{i+1}(s,t)) = 1,\ 1 \leq i \leq 4,$$
$$\gcd(\Omega_{i+1}(s,t), \Psi_{i+1}(s,t)) = s^2 + t^2,\ 5 \leq i \leq 6,$$

and (21) holds. The homogeneous polynomials

$$\pi_1^1(s,t) = \cdots = \pi_4^1(s,t) = 1, \quad \pi_5^1(s,t) \mid s + it, \quad \pi_6^1(s,t) = s^2 + t^2$$

satisfy (19) and (20).

Corollary 5.4 *Under the conditions of Theorem 5.1, if* $\mathbb{F}$ *is an algebraically closed field and* $\mathbf{c} \neq \mathbf{d}$, $\mathbf{u} \neq \mathbf{v}$, *there exists a matrix pencil* $P(s) \in \mathbb{F}[s]^{p \times q}$ *of* $\operatorname{rank}(P(s)) = 1$ *such that* $A(s) + P(s) \overset{s.e.}{\sim} B(s)$ *if and only if (10), and one of the four following conditions hold*

(a) (14) and

$$\sum_{i=1}^n \deg(\operatorname{lcm}(\phi_i(s,t), \psi_i(s,t))) \leq x \leq \sum_{i=1}^n \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))), \tag{22}$$

where $x = n - \sum_{i=1}^{q-n_1} c_i - \sum_{i=1}^{p-n_2} v_i$.

(b) (16) and

$$\sum_{i=1}^n \deg(\operatorname{lcm}(\phi_i(s,t), \psi_i(s,t))) \leq y \leq \sum_{i=1}^n \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))), \tag{23}$$

where $y = n - \sum_{i=1}^{q-n_2} d_i - \sum_{i=1}^{p-n_1} u_i$.

(c) (14) and (23).

(d) (16) and (22).

5.1 Technical results

All of the sequences involved in this subsection are ordered partitions of non-negative integers.

Lemma 5.5 *Let $S \geq 0$ be a nonnegative integer and let $\mathbf{a} = (a_1, \ldots, a_m)$ be a partition of nonnegative integers. Then there exists a partition of nonnegative integers $\mathbf{g} = (g_1, \ldots, g_{m+1})$ such that $\sum_{i=1}^{m+1} g_i = S$ and $\mathbf{g} \prec' \mathbf{a}$.*

Proof. Put $a_0 = +\infty$, $a_{m+1} = -\infty$. Then,

$$\sum_{j=i}^{m} a_j + i a_{i-1} \geq \sum_{j=i+1}^{m} a_j + (i+1)a_i, \quad 1 \leq i \leq m+1,$$

and $S \geq (m+2)a_{m+1}$. Let $k = \min\{i \in \{1, \ldots, m+1\} \ : \ S \geq \sum_{j=i+1}^{m} a_j + (i+1)a_i\}$, i.e.,

$$\sum_{j=k}^{m} a_j + k a_{k-1} > S \geq \sum_{j=k+1}^{m} a_j + (k+1)a_k = \sum_{j=k}^{m} a_j + k a_k.$$

Let $S' = S - \sum_{j=k}^{m} a_j$. Then $k a_{k-1} > S' \geq k a_k$. Let q and r be the quotient and the remainder of the euclidian division of S' by k, i.e., $S' = kq + r$ with $0 \leq r < k$. Then $a_{k-1} > q \geq a_k$. Observe that if $k \leq m$, then $a_k \geq 0$, and if $k = m+1$, then $S' = S \geq 0$, hence $q \geq 0$.

Let us define

$$\begin{array}{ll} g_i = q+1, & 1 \leq i \leq r, \\ g_i = q, & r+1 \leq i \leq k, \\ g_i = a_{i-1}, & k+1 \leq i \leq m+1. \end{array}$$

Then $g_1 \geq \cdots \geq g_{m+1} \geq 0$, $\sum_{i=1}^{m+1} g_i = S$ and

$$\begin{array}{ll} g_i \leq q+1 \leq a_{k-1} \leq a_i, & 1 \leq i \leq k-1, \\ g_{i+1} = a_i, & k \leq i \leq m. \end{array} \tag{24}$$

Let $h = \min\{i \ : \ a_i < g_i\}$. From (24) we derive that $h \geq k$ and $a_i = g_{i+1}$, $h \leq i \leq m$. Therefore, $(g_1, \ldots, g_{m+1}) \prec' \mathbf{a}$.

□

Example 5.6 *Given the partition $\mathbf{a} = (8, 6, 5, 5, 5, 3, 1)$ ($m = 7$), we show some examples of the previous result for different values of S.*

1. *$S = 50$. Then $\sum_{j=1}^{7} a_j + a_0 > S \geq \sum_{j=2}^{7} a_j + 2a_1$. $k = 1$, $S' = S - \sum_{j=1}^{7} a_j = 17$, $q = 17$, $r = 0$. It results $\mathbf{g} = (17, 8, 6, 5, 5, 5, 3, 1)$.*

2. *$S = 34$. Then $\sum_{j=3}^{7} a_j + 3a_2 > S \geq \sum_{j=4}^{7} a_j + 4a_3$. $k = 3$, $S' = S - \sum_{j=3}^{7} a_j = 15$, $q = 5$, $r = 0$. It results $\mathbf{g} = (5, 5, 5, 5, 5, 5, 3, 1)$.*

3. $S = 5$. *Then* $\sum_{j=8}^{7} a_j + 8a_7 > S \geq \sum_{j=9}^{7} a_j + 9a_8$. $k = 8$, $S' = S - \sum_{j=8}^{7} a_j = S = 5$, $q = 0$, $r = 5$. *It results* $\mathbf{g} = (1, 1, 1, 1, 1, 0, 0, 0)$.

Lemma 5.7 *Let $E \geq 0$ be a nonnegative integer and let $\mathbf{a} = (a_1, \ldots, a_m)$ be a partition of nonnegative integers. Then there exists a partition of nonnegative integers $\mathbf{e} = (e_1, \ldots, e_{m-1})$ such that $\sum_{i=1}^{m-1} e_i = E$ and $\mathbf{a} \prec' \mathbf{e}$ if and only if*

$$E = \sum_{i=2}^{m} a_i \text{ or } E \geq a_1 + \sum_{i=3}^{m} a_i. \tag{25}$$

Proof. Assume that there exists $\mathbf{e} = (e_1, \ldots, e_{m-1})$ such that $\sum_{i=1}^{m-1} e_i = E$ and $\mathbf{a} \prec' \mathbf{e}$. Then $e_i \geq a_{i+1}$, $1 \leq i \leq m-1$, hence $E \geq \sum_{i=2}^{m} a_i$.

Let us suppose that $E > \sum_{i=2}^{m} a_i$. Then $\sum_{i=1}^{m-1}(e_i - a_{i+1}) > 0$, hence there exists $k \in \{1, \ldots, m-1\}$ such that $e_k > a_{k+1}$. Thus $k < h = \min\{i \ : e_i < a_i\}$, which means $e_1 \geq a_1$ and therefore

$$E = e_1 + \sum_{i=2}^{m-1} e_i \geq a_1 + \sum_{i=3}^{m} a_i.$$

Conversely, let us assume that (25) holds. We define $e_i = a_{i+1}$, $2 \leq i \leq m-1$ and $e_1 = E - \sum_{i=3}^{m} a_i$. Then $\sum_{i=1}^{m-1} e_i = E$. If $E = \sum_{i=2}^{m} a_i$, then $e_1 = a_2$ and $\mathbf{a} \prec' (e_1, \ldots, e_{m-1})$. If $E \geq a_1 + \sum_{i=3}^{m} a_i$, then $e_1 \geq a_1 \geq a_3 = e_2$. Thus, $e_1 \geq \cdots \geq e_{m-1} \geq 0$, $h = \min\{i \ : e_i < a_i\} > 1$, and we have $e_i = a_{i+1}$, $h \leq i \leq m-1$. Therefore, $\mathbf{a} \prec' (e_1, \ldots, e_{m-1})$.

□

Given two pairs of nonincreasing sequences of integers, $(\mathbf{d}, \mathbf{a})$ and $(\mathbf{c}, \mathbf{b})$, in [10, Theorem 5.1] the authors solved the problem of obtaining necessary and sufficient conditions for the existence of a sequence $\mathbf{g}$ that is majorized (in the sense of generalized majorization) by both pairs. The conditions are very involved. In the first item of the next Lemma we solve the same problem for the 1step-generalized majorization of partitions. The characterization obtained is much more simple in this case.

Lemma 5.8 *Let $S, E \geq 0$ be nonnegative integers and let $\mathbf{c} = (c_1, \ldots, c_m)$, $\mathbf{d} = (d_1, \ldots, d_m)$ be partitions of nonnegative integers such that $\mathbf{c} \neq \mathbf{d}$.*

Let $\ell = \max\{i \ : \ c_i \neq d_i\}$, $f = \max\{i \in \{1, \ldots, \ell\} \ : \ c_i < d_{i-1}\}$ $(d_0 = +\infty)$, and $f' = \max\{i \in \{1, \ldots, \ell\} \ : \ d_i < c_{i-1}\}$ $(c_0 = +\infty)$.

1. *There exists a partition $\mathbf{g} = (g_1, \ldots, g_{m+1})$ of nonnegative integers such that $\sum_{i=1}^{m+1} g_i = S$, $\mathbf{g} \prec' \mathbf{c}$ and $\mathbf{g} \prec' \mathbf{d}$ if and only if*

$$S \leq \sum_{i=1}^{m} \min\{c_i, d_i\} + \max\{c_f, d_{f'}\}. \tag{26}$$

2. *If $f > 1$ and $f' > 1$, there exists a partition of nonnegative integers $\mathbf{e} = (e_1, \dots, e_{m-1})$ such that $\sum_{i=1}^{m-1} e_i = E$, $\mathbf{c} \prec' \mathbf{e}$ and $\mathbf{d} \prec' \mathbf{e}$ if and only if*
$$E \geq \sum_{i=1}^{m} \max\{c_i, d_i\} - \max\{c_f, d_{f'}\}. \tag{27}$$

3. *If $f = 1$ or $f' = 1$, there exists a partition of nonnegative integers $\mathbf{e} = (e_1, \dots, e_{m-1})$ such that $\sum_{i=1}^{m-1} e_i = E$, $\mathbf{c} \prec' \mathbf{e}$ and $\mathbf{d} \prec' \mathbf{e}$ if and only if*
$$\begin{array}{l} E = \sum_{i=1}^{m} \max\{c_i, d_i\} - \max\{c_f, d_{f'}\}, \\ \text{or} \\ E \geq \sum_{i=1}^{m} \max\{c_i, d_i\} - \max\{c_{f+1}, d_{f'+1}\}. \end{array} \tag{28}$$
Equivalently,
$$E = \sum_{i=2}^{m} \max\{c_i, d_i\} \text{ or } E \geq \max\{c_1, d_1\} + \sum_{i=3}^{m} \max\{c_i, d_i\}.$$

Proof. Let us assume that $c_\ell > d_\ell$. If $d_\ell > c_\ell$ the proof is analogous.

We have $c_{\ell-1} \geq c_\ell > d_\ell$, hence $f' = \ell$. Moreover, $c_i \geq c_{i+1} \geq d_i$, $f \leq i \leq \ell - 1$. Then, $c_f \geq d_f \geq d_\ell = d_{f'}$. Hence, when $c_\ell > d_\ell$, conditions (26), (27) and (28) are respectively equivalent to
$$S \leq \sum_{i=1}^{f-1} \min\{c_i, d_i\} + c_f + \sum_{i=f}^{m} d_i, \tag{29}$$
$$E \geq \sum_{i=1}^{f-1} \max\{c_i, d_i\} + \sum_{i=f+1}^{m} c_i, \tag{30}$$
$$E = \sum_{i=2}^{m} c_i \text{ or } E \geq c_1 + \sum_{i=3}^{m} c_i. \tag{31}$$

Moreover, if $f' = 1$ then $\ell = 1$ and as a consequence, $f = 1$. Therefore, when $c_\ell > d_\ell$, the condition $f = 1$ or $f' = 1$ is equivalent to $f = 1$.

Let us prove the different cases.

1. Assume that there exists a partition $\mathbf{g} = (g_1, \dots, g_{m+1})$ such that $\sum_{i=1}^{m+1} g_i = S$, $\mathbf{g} \prec' \mathbf{c}$ and $\mathbf{g} \prec' \mathbf{d}$.

   Let $h = \min\{i : c_i < g_i\}$ and $h' = \min\{i : d_i < g_i\}$. As $g_{\ell+1} \leq d_\ell < c_\ell$, by Remark 4.2, item 2, we have $\ell < h$. In the same way, as $g_f \leq c_f < d_{f-1}$, $f - 1 < h'$.

   Therefore,
$$\begin{array}{ll} g_i \leq \min\{c_i, d_i\}, & 1 \leq i \leq f-1, \\ g_f \leq c_f, & \\ g_{i+1} \leq d_i, & f \leq i \leq m, \end{array}$$
   from where we obtain (29).

   Conversely, let us assume that (29) holds.

- If $S < \sum_{i=1}^{f-1} \min\{c_i, d_i\}$ then $f > 1$. Let

$$k = \min\{i \in \{1, \dots f-1\} \ : \ S < \sum_{i=1}^{k} \min\{c_i, d_i\}\},$$

i.e., $\sum_{i=1}^{k-1} \min\{c_i, d_i\} \leq S < \sum_{i=1}^{k} \min\{c_i, d_i\}$ and define

$$\begin{array}{ll} g_i = \min\{c_i, d_i\}, & 1 \leq i \leq k-1, \\ g_k = S - \sum_{i=1}^{k-1} \min\{c_i, d_i\}, & \\ g_i = 0, & k+1 \leq i \leq m+1. \end{array}$$

Then $\sum_{i=1}^{m+1} g_i = S$ and $g_k < \min\{c_k, d_k\}$. Therefore, $g_1 \geq \cdots \geq g_{k-1} > g_k \geq 0 = g_{k+1} = \cdots = g_{m+1}$. Thus, $\mathbf{g} = (g_1, \dots, g_{m+1})$ is a partition. As $g_i \leq \min\{c_i, d_i\}$, $1 \leq i \leq m$, by Remark 4.2, item 1, we have $\mathbf{g} \prec' \mathbf{c}$ and $\mathbf{g} \prec' \mathbf{d}$.

- If $S \geq \sum_{i=1}^{f-1} \min\{c_i, d_i\}$, let $\overline{S} = S - \sum_{i=1}^{f-1} \min\{c_i, d_i\} \geq 0$. Then $\overline{S} \leq c_f + \sum_{i=f}^{m} d_i$. We define $\overline{d}_i = d_{f-1+i}$, $1 \leq i \leq m-f+1$, and $\overline{\mathbf{d}} = (\overline{d}_1, \dots, \overline{d}_{m-f+1})$, i.e., $\overline{\mathbf{d}} = (d_f, \dots, d_m)$. By Lemma 5.5, there exists a partition $\overline{\mathbf{g}} = (\overline{g}_1, \dots, \overline{g}_{m-f+2})$ such that $\sum_{i=1}^{m-f+2} \overline{g}_i = \overline{S}$ and $\overline{\mathbf{g}} \prec' \overline{\mathbf{d}}$.

  Now we define

$$\begin{array}{ll} g_i = \min\{c_i, d_i\}, & 1 \leq i \leq f-1, \\ g_i = \overline{g}_{i-f+1}, & f \leq i \leq m+1. \end{array}$$

  Let us see that $g_f \leq g_{f-1}$, i.e., that $\overline{g}_1 \leq \min\{c_{f-1}, d_{f-1}\}$. If $\overline{g}_1 \leq \overline{d}_1$, then $\overline{g}_1 \leq d_f = \min\{c_f, d_f\} \leq \min\{c_{f-1}, d_{f-1}\}$. If $\overline{g}_1 > \overline{d}_1$, then $\overline{d}_i = \overline{g}_{i+1}$, $1 \leq i \leq m-f+1$, hence $\overline{S} = \overline{g}_1 + \sum_{i=1}^{m-f+1} \overline{d}_i$. As $\overline{S} \leq c_f + \sum_{i=f}^{m} d_i$, we obtain $\overline{g}_1 \leq c_f \leq \min\{c_{f-1}, d_{f-1}\}$. Therefore $\mathbf{g} = (g_1, \dots, g_{m+1})$ is a partition.

  Let $h' = \min\{i \ : \ d_i < g_i\}$ and $\overline{h}' = \min\{i \ : \ \overline{d}_i < \overline{g}_i\}$. Observe that $d_i \geq g_i$, $1 \leq i \leq f-1$, $d_i = \overline{d}_{i-f+1} \geq \overline{g}_{i-f+1} = g_i$ for $f \leq i < f + \overline{h}' - 1$, and $d_{f+\overline{h}'-1} = \overline{d}_{\overline{h}'} < \overline{g}_{\overline{h}'} = g_{f+\overline{h}'-1}$. Then, $h' = f + \overline{h}' - 1$. As $d_i = \overline{d}_{i-f+1} = \overline{g}_{i-f+2} = g_{i+1}$ for $h' \leq i \leq m$, we obtain that $\mathbf{g} \prec' \mathbf{d}$.

  Let $h = \min\{i \ : \ c_i < g_i\}$. Recall that $d_i \leq c_i$ for $f \leq i \leq m$. We have,

$$\begin{array}{ll} g_i \leq c_i, & 1 \leq i \leq f-1, \\ g_i \leq d_i \leq c_i, & f \leq i \leq h'-1, \\ g_i \leq d_{i-1} \leq c_i, & f+1 \leq i \leq \ell. \end{array}$$

  Thus, $h > \max\{h'-1, \ell\}$ and, as a consequence, $c_i = d_i = g_{i+1}$ for $h \leq i \leq m$. Therefore $\mathbf{g} \prec' \mathbf{c}$.

2. Assume that $f > 1$ (hence $f' > 1$) and there exists a partition $\mathbf{e} = (e_1, \ldots, e_{m-1})$ such that $\sum_{i=1}^{m-1} e_i = E$, $\mathbf{c} \prec' \mathbf{e}$ and $\mathbf{d} \prec' \mathbf{e}$. Then $e_i \geq c_{i+1}$, $1 \leq i \leq m-1$. Moreover, $e_{\ell-1} \geq c_\ell > d_\ell$, by Remark 4.2, item 2, $e_i \geq d_i$, $1 \leq i \leq \ell - 1$. Hence, $e_{f-1} \geq d_{f-1} > c_f$, and as before it means that $e_i \geq c_i$, $1 \leq i \leq f-1$. Thus,

$$\begin{array}{ll} e_i \geq \max\{c_i, d_i\} & 1 \leq i \leq f-1, \\ e_i \geq c_{i+1}, & f \leq i \leq m-1, \end{array}$$

and we obtain (30).

Conversely, let us assume that $f > 1$ and (30) holds. Let us define

$$\begin{array}{ll} e_i = \max\{c_i, d_i\}, & 2 \leq i \leq f-1, \\ e_i = c_{i+1}, & f \leq i \leq m-1, \\ e_1 = E - \sum_{i=2}^{m-1} e_i. & \end{array}$$

Then, $E = \sum_{i=1}^{m-1} e_i$, $e_2 \geq \cdots \geq e_{f-1} \geq c_{f-1} \geq c_{f+1} = e_f \geq \cdots \geq e_{m-1}$ and, from (30) we derive $e_1 \geq \max\{c_1, d_1\} \geq e_2$. Therefore $(e_1, \ldots, e_{m-1})$ is a partition. Let $h = \min\{i \ : e_i < c_i\}$ and $h' = \min\{i \ : e_i < d_i\}$. It is clear that $h \geq f$ and $h' \geq f$, hence $e_i = c_{i+1}$, $h \leq i \leq m-1$, which means that $\mathbf{c} \prec' \mathbf{e}$. Moreover, for $f \leq i \leq \ell - 1$, $e_i = c_{i+1} \geq d_i$, thus $h' \geq \ell$ and $e_i = c_{i+1} = d_{i+1}$, $h' \leq i \leq m-1$. Therefore $\mathbf{d} \prec' \mathbf{e}$.

3. Let us assume that $f = 1$ and there exists a partition $\mathbf{e} = (e_1, \ldots, e_{m-1})$ such that $\sum_{i=1}^{m-1} e_i = E$, $\mathbf{c} \prec' \mathbf{e}$ and $\mathbf{d} \prec' \mathbf{e}$. From Lemma 5.7 we obtain (31).

   Conversely, let us assume that $f = 1$ and (31) holds. By Lemma 5.7, there exists a partition $\mathbf{e} = (e_1, \ldots, e_{m-1})$ such that $\sum_{i=1}^{m-1} e_i = E$ and $\mathbf{c} \prec' \mathbf{e}$. Therefore, $e_i \geq c_{i+1}, \ 1 \leq i \leq m-1$.

   Let $h = \min\{i \ : e_i < c_i\}$ and $h' = \min\{i \ : e_i < d_i\}$. As $f = 1$, we have $c_i \geq d_{i-1} \geq d_i$, $2 \leq i \leq \ell$. Therefore, $e_i \geq c_{i+1} \geq d_i$, $1 \leq i < \ell$ and $h' \geq \ell$. Since $e_{h'} < d_{h'} = c_{h'}$, $h \leq h'$ and $e_i = c_{i+1} = d_{i+1}$, $h' \leq i \leq m-1$. Hence, $\mathbf{d} \prec' \mathbf{e}$.

□

Remark 5.9 *Observe that condition (28) implies condition (27).*

Lemma 5.10 *Let* $\mathbf{a} = (a_1, \ldots, a_m)$, $\mathbf{e} = (e_1, \ldots, e_{m-1})$ *be partitions of nonnegative integers such that* $\mathbf{a} \prec' \mathbf{e}$ *and* $\sum_{i=1}^{m-1} e_i \leq \sum_{i=1}^{m} a_i$. *Let* $\theta = \#\{i \ : \ e_i > 0\}$ *and* $\overline{\theta} = \#\{i \ : \ a_i > 0\}$. *Then* $\overline{\theta} \geq \theta$.

Proof. We have $\theta \leq m-1$, $\overline{\theta} \leq m$ and $\sum_{i=1}^{\theta} e_i \leq \sum_{i=1}^{\overline{\theta}} a_i$. Let $h = \min\{i \ : e_i < a_i\}$. Then $e_i = a_{i+1}$ for $h \leq i \leq m-1$.

Assume that $\theta > \overline{\theta}$. Then $0 < \sum_{i=\overline{\theta}+1}^{\theta} e_i \leq \sum_{i=1}^{\overline{\theta}} (a_i - e_i)$. It means that there exists $i \in \{1, \ldots, \overline{\theta}\}$ such that $a_i - e_i > 0$. Therefore, $h \leq \overline{\theta} < \theta < m$, from where we conclude that $e_{\overline{\theta}} = a_{\overline{\theta}+1} = 0$, which is a contradiction with $\theta > \overline{\theta}$.

□

5.2 Proof of Theorem 5.1

Necessity. Let us assume that there exists a pencil $P(s) \in \mathbb{F}[s]^{p\times q}$ of rank $P(s) = 1$ such that $A(s) + P(s) \overset{s.e.}{\sim} B(s)$. By Lemma 3.3, one of the two following conditions holds:

(i) There exist pencils $a(s), b(s) \in \mathbb{F}[s]^{1\times q}$ and $A_{21}(s) \in \mathbb{F}^{(p-1)\times q}$ such that $A(s) \overset{s.e.}{\sim} \begin{bmatrix} a(s) \\ A_{21}(s) \end{bmatrix}$ and $B(s) \overset{s.e.}{\sim} \begin{bmatrix} b(s) \\ A_{21}(s) \end{bmatrix}$.

(ii) There exist pencils $\bar{a}(s), \bar{b}(s) \in \mathbb{F}[s]^{p\times 1}$ and $A_{12}(s) \in \mathbb{F}^{p\times(q-1)}$ such that $A(s) \overset{s.e.}{\sim} \begin{bmatrix} \bar{a}(s) & A_{12}(s) \end{bmatrix}$ and $B(s) \overset{s.e.}{\sim} \begin{bmatrix} \bar{b}(s) & A_{12}(s) \end{bmatrix}$.

- Let us assume that (i) holds. Then $n \geq \operatorname{rank}(A_{21}(s)) \geq \max\{n_1, n_2\} - 1 \geq n-1$, hence $\operatorname{rank}(A_{21}(s)) = n - x$ with $x = 0$ or $x = 1$. Let $\pi^1_i(s,t) \mid \cdots \mid \pi^1_{n-x}(s,t)$, $\mathbf{g} = (g_1, \dots, g_{q-n+x})$ and $\mathbf{w} = (w_1, \dots w_{p-1-n+x})$ be, respectively, the homogeneous invariant factors, column minimal indices and row minimal indices of $A_{21}(s)$. By Lemmas 4.3 and 4.4,

$$\begin{array}{ll} \phi_i(s,t) \mid \pi^1_i(s,t) \mid \phi_{i+1}(s,t), & 1 \leq i \leq n-x, \\ \psi_i(s,t) \mid \pi^1_i(s,t) \mid \psi_{i+1}(s,t), & 1 \leq i \leq n-x. \end{array} \tag{32}$$

  Thus,

$$\begin{array}{ll} \psi_{i-1}(s,t) \mid \phi_i(s,t), & 1 \leq i \leq n, \\ \phi_i(s,t) \mid \psi_{i+1}(s,t), & 1 \leq i \leq n-x. \end{array}$$

  Notice that in the case that $x = 1$ we have $n_1 = n_2 = n$, then $\phi_n(s,t) \mid \psi_{n+1}(s,t) = 0$ is also satisfied. Therefore, (10) holds.

  1. Assume that $\mathbf{c} = \mathbf{d}$, $\mathbf{u} = \mathbf{v}$. As it has been seen, condition (10) is necessary.
  2. Assume that $\mathbf{c} \neq \mathbf{d}$, $\mathbf{u} = \mathbf{v}$. Then $n_1 = n_2 = n$. If $\operatorname{rank}(A_{21}(s)) = n$, from Lemma 4.3 we obtain $\mathbf{g} = \mathbf{c}$ and $\mathbf{g} = \mathbf{d}$, which is a contradiction with $\mathbf{c} \neq \mathbf{d}$. Therefore, $\operatorname{rank}(A_{21}(s)) = n - 1$, i.e., $x = 1$. Applying Lemma 4.4, we obtain

$$\mathbf{g} \prec' \mathbf{c}, \quad \mathbf{g} \prec' \mathbf{d}, \tag{33}$$

$$\mathbf{w} = \mathbf{u} = \mathbf{v}. \tag{34}$$

     From (32) and (34),

$$\begin{array}{rl} \sum_{i=1}^{q-n+1} g_i = & n - 1 - \sum_{i=1}^{n-1} \deg(\pi^1_i(s,t) - \sum_{i=1}^{p-n} w_i \\ \geq & n - 1 - \sum_{i=1}^{n-1} \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))) \\ & - \sum_{i=1}^{p-n} u_i = G. \end{array}$$

     By Lemma 5.8,

$$\sum_{i=1}^{q-n+1} g_i \leq \sum_{i=1}^{q-n} \min\{c_i, d_i\} + \max\{c_f, d_{f'}\}.$$

     Therefore, (11) holds.

3. Assume that $\mathbf{c} = \mathbf{d}$, $\mathbf{u} \neq \mathbf{v}$. Then $n_1 = n_2 = n$. If $\operatorname{rank}(A_{21}(s)) = n-1$, from Lemma 4.4 we obtain $\mathbf{w} = \mathbf{u}$ and $\mathbf{w} = \mathbf{v}$, which is a contradiction with $\mathbf{u} \neq \mathbf{v}$. Therefore, $\operatorname{rank}(A_{21}(s)) = n$, i.e., $x = 0$. Applying Lemma 4.3, we obtain

$$\mathbf{u} \prec' \mathbf{w}, \quad \mathbf{v} \prec' \mathbf{w}, \tag{35}$$

$$\mathbf{g} = \mathbf{c} = \mathbf{d}. \tag{36}$$

From (32) and (36),

$$\begin{aligned} \sum_{i=1}^{p-n-1} w_i = \; & n - \sum_{i=1}^{n} \deg(\pi_i^1(s,t)) - \sum_{i=1}^{q-n} g_i \\ \leq \; & n - \sum_{i=1}^{n} \deg(\operatorname{lcm}(\phi_i(s,t), \psi_i(s,t))) - \sum_{i=1}^{q-n} c_i \\ = \; & \sum_{i=1}^{p-n} u_i - \sum_{i=1}^{n} \deg(\psi_i(s,t)) \\ & + \sum_{i=1}^{n} \deg(\gcd(\phi_i(s,t), \psi_i(s,t))). \end{aligned}$$

Observe that since $\mathbf{u} \neq \mathbf{v}$, we have $\gcd(\phi_1(s,t), \psi_1(s,t)) = 1$, hence

$$\sum_{i=1}^{n} \deg(\gcd(\phi_i(s,t), \psi_i(s,t))) = \sum_{i=1}^{n-1} \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))).$$

By Lemma 5.8 and Remark 5.9,

$$\begin{aligned} \sum_{i=1}^{p-n-1} w_i \geq \; & \sum_{i=1}^{p-n} \max\{u_i, v_i\} - \max\{u_{\bar{f}}, v_{\bar{f}'}\} \\ = \; & \sum_{i=1}^{p-n} u_i + \sum_{i=1}^{q-n} v_i - \sum_{i=1}^{p-n} \min\{u_i, v_i\} \\ & - \max\{u_{\bar{f}}, v_{\bar{f}'}\}, \end{aligned}$$

therefore,

$$\begin{aligned} \sum_{i=1}^{q-n} v_i + \; & \sum_{i=1}^{n} \deg(\psi_i(s,t)) - \sum_{i=1}^{n} \deg(\gcd(\phi_i(s,t), \psi_i(s,t))) \\ \leq \; & \sum_{i=1}^{p-n} \min\{u_i, v_i\} + \max\{u_{\bar{f}}, v_{\bar{f}'}\} \end{aligned}$$

which implies that

$$\begin{aligned} \bar{G} < \; & n - \sum_{i=1}^{n} \deg(\gcd(\phi_i(s,t), \psi_i(s,t))) - \sum_{i=1}^{p-n} c_i \\ \leq \; & \sum_{i=1}^{p-n} \max\{u_i, v_i\} + \max\{u_{\bar{f}}, v_{\bar{f}'}\}. \end{aligned}$$

Therefore, (12) holds.

4. Assume that $\mathbf{c} \neq \mathbf{d}$, $\mathbf{u} \neq \mathbf{v}$. If $\operatorname{rank}(A(s)) = \operatorname{rank}(B(s))$, then applying Lemmas 4.3 and 4.4, we obtain $\mathbf{g} = \mathbf{c} = \mathbf{d}$ or $\mathbf{w} = \mathbf{u} = \mathbf{v}$, which is a contradiction. Therefore, $\operatorname{rank}(A(s)) \neq \operatorname{rank}(B(s))$. Then $n \geq \operatorname{rank}(A_{21}(s)) \geq \max\{n_1, n_2\} - 1 = n$, i.e., $\operatorname{rank}(A_{21}(s)) = n$ ($x = 0$). From (32) we derive (13).

   If $\operatorname{rank}(A(s)) < \operatorname{rank}(B(s))$, then $\operatorname{rank}(A(s)) = n$, $\operatorname{rank}(B(s)) = n+1$. Applying Lemmas 4.3 and 4.4 we obtain

$$\mathbf{g} = \mathbf{c}, \quad \mathbf{u} \prec' \mathbf{w}, \tag{37}$$

$$\mathbf{g} \prec' \mathbf{d}, \quad \mathbf{w} = \mathbf{v}. \tag{38}$$

   From (37) and (38) we derive (14) and (15).

   Analogously, if $\operatorname{rank}(B(s)) < \operatorname{rank}(A(s))$ we obtain (16) and (17).

- Let us assume that (ii) holds. Then

$$A(s)^T \overset{s.e.}{\sim} \begin{bmatrix} \bar{a}(s)^T \\ A_{12}(s)^T \end{bmatrix}, \quad B(s)^T \overset{s.e.}{\sim} \begin{bmatrix} \bar{b}(s)^T \\ A_{12}(s)^T \end{bmatrix}.$$

  Recall that the column and row minimal indices of a pencil are, respectively, the row and column minimal indices of its transposed.

  Applying the results of the previous case, the interlacing condition (10) is satisfied and

  - If $\mathbf{c} = \mathbf{d}$, $\mathbf{u} \neq \mathbf{v}$, we obtain (12).
  - If $\mathbf{c} \neq \mathbf{d}$, $\mathbf{u} = \mathbf{v}$, we obtain (11).
  - If $\mathbf{c} \neq \mathbf{d}$, $\mathbf{u} \neq \mathbf{v}$ and $\operatorname{rank}(A(s)) < \operatorname{rank}(B(s))$, we obtain (14) and (17).
  - If $\mathbf{c} \neq \mathbf{d}$, $\mathbf{u} \neq \mathbf{v}$ and $\operatorname{rank}(B(s)) < \operatorname{rank}(A(s))$, we obtain (16) and (15).

Sufficiency.

Case $\mathbf{u} = \mathbf{v}$. In this case, $n = n_1 = n_2$.

Assume that $\mathbf{c} = \mathbf{d}$ and (10) holds or that $\mathbf{c} \neq \mathbf{d}$ and (10) and (11) hold. By Lemma 3.3, it is enough to prove the existence of matrix pencils $a(s), b(s) \in \mathbb{F}[s]^{1\times q}$, $A_{21}(s) \in \mathbb{F}[s]^{(p-1)\times q}$ such that $A(s) \overset{s.e.}{\sim} \begin{bmatrix} a(s) \\ A_{21}(s) \end{bmatrix}$ and $B(s) \overset{s.e.}{\sim} \begin{bmatrix} b(s) \\ A_{21}(s) \end{bmatrix}$.

Let

$$\pi_i^1(s,t) = \gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t)), \quad 1 \leq i \leq n-1.$$

Then $\pi_1^1(s,t) \mid \cdots \mid \pi_{n-1}^1(s,t)$ and (10) implies that

$$\begin{array}{ll} \phi_i(s,t) \mid \pi_i^1(s,t) \mid \phi_{i+1}(s,t), & 1 \leq i \leq n-1, \\ \psi_i(s,t) \mid \pi_i^1(s,t) \mid \psi_{i+1}(s,t), & 1 \leq i \leq n-1. \end{array} \tag{39}$$

Let $S = n - 1 - \sum_{i=1}^{n-1} \deg(\pi_i^1(s,t)) - \sum_{i=1}^{p-n} u_i$ and let us see that $S \geq 0$. We have

$$\begin{array}{rl} \sum_{i=1}^{n-1} \deg(\pi_i^1(s,t)) + \sum_{i=1}^{p-n} u_i \leq & \sum_{i=2}^{n} \deg(\phi_i(s,t)) + \sum_{i=1}^{p-n} u_i \\ = & n - \deg(\phi_1(s,t)) - \sum_{i=1}^{q-n} c_i, \end{array}$$

and

$$\begin{array}{rl} \sum_{i=1}^{n-1} \deg(\pi_i^1(s,t)) + \sum_{i=1}^{p-n} u_i \leq & \sum_{i=2}^{n} \deg(\psi_i(s,t)) + \sum_{i=1}^{p-n} u_i \\ = & n - \deg(\psi_1(s,t)) - \sum_{i=1}^{q-n} d_i. \end{array}$$

If $\sum_{i=1}^{n-1} \deg(\pi_i^1(s,t)) + \sum_{i=1}^{p-n} u_i = n$, then $\phi_1(s,t) = \psi_1(s,t) = 1$, $\sum_{i=1}^{q-n} c_i = \sum_{i=1}^{q-n} d_i = 0$, and

$$\sum_{i=1}^{n-1}(\deg(\pi_i^1(s,t)) - \deg(\phi_{i+1}(s,t))) = \sum_{i=1}^{n-1}(\deg(\pi_i^1(s,t)) - \deg(\psi_{i+1}(s,t))) = 0,$$

therefore, $\mathbf{c} = \mathbf{d}$, $\pi_i^1(s,t) = \phi_{i+1}(s,t) = \psi_{i+1}(s,t)$, $1 \leq i \leq n-1$ and $A(s) \overset{s.e.}{\sim} B(s)$. As $A(s) \overset{s.e.}{\not\sim} B(s)$, we derive $S \geq 0$.

Notice that in the case that $\mathbf{c} \neq \mathbf{d}$, because of condition (11), $S = G \leq \sum_{i=1}^{q-n} \min\{c_i, d_i\} + \max\{c_f, d_{f'}\}$. Then, by Lemma 5.5 (in the case $\mathbf{c} = \mathbf{d}$) or by Lemma 5.8 (in the case $\mathbf{c} \neq \mathbf{d}$), there exists a partition of nonnegative integers $\mathbf{g} = (g_1, \dots, g_{q-n+1})$ satisfying $\sum_i^{q-n+1} g_i = S$ and (33).

As $\sum_{i=1}^{n-1} \deg(\pi_i^1(s,t)) + \sum_{i=1}^{p-n} u_i + \sum_i^{q-n-1} g_i = n-1$, there exists a pencil $A_{21}(s) \in \mathbb{F}^{(p-1)\times q}$ of $\operatorname{rank}(A_{21}(s)) = n-1$, homogeneous invariant factors $\pi_i^1(s,t) \mid \cdots \mid \pi_{n-1}^1(s,t)$, column minimal indices $g_1 \geq \cdots \geq g_{q-n+1}$ and row minimal indices $u_1 \geq \cdots \geq u_{p-n}$.

From (39), (33) and Lemma 4.4, there exist pencils $a(s), b(s) \in \mathbb{F}[s]^{1\times q}$, such that $A(s) \overset{s.e.}{\sim} \begin{bmatrix} a(s) \\ A_{21}(s) \end{bmatrix}$ and $B(s) \overset{s.e.}{\sim} \begin{bmatrix} b(s) \\ A_{21}(s) \end{bmatrix}$.

Case $\mathbf{c} = \mathbf{d}$. The conclusion follows applying the previous result of the case $\mathbf{u} = \mathbf{v}$ to the pencils $A(s)^T$ and $B(s)^T$.

Case $\mathbf{c} \neq \mathbf{d}$, $\mathbf{u} \neq \mathbf{v}$. Assume that there exist homogeneous polynomials $\pi_1^1(s,t) \mid \cdots \mid \pi_n^1(s,t)$ satisfying (13).

(a) If (14) and (15) hold, then $q - n_1 = q - n_2 + 1$, i.e., $n_1 = n_2 - 1$, hence $n_1 = n$ and $n_2 = n+1$. From (15), there exists a pencil $A_{21}(s) \in \mathbb{F}^{(p-1)\times q}$ of $\operatorname{rank}(A_{21}(s)) = n$, homogeneous invariant factors $\pi_i^1(s,t) \mid \cdots \mid \pi_n^1(s,t)$, column minimal indices $c_1 \geq \cdots \geq c_{q-n}$ and row minimal indices $v_1 \geq \cdots \geq v_{p-n-1}$. Moreover, because of (13),

$$\begin{aligned} \sum_{i=1}^{p-n-1} v_i &= n - \sum_{i=1}^{q-n} c_i - \sum_{i=1}^{n} \deg(\pi_i^1(s,t)) \\ &\leq n - \sum_{i=1}^{q-n} c_i - \sum_{i=1}^{n} \deg(\phi_i(s,t)) = \sum_{i=1}^{p-n} u_i. \end{aligned}$$

From Lemma 5.10, we obtain $\#\{i : v_i > 0\} \leq \#\{i : u_i > 0\}$. Applying Lemmas 4.3 and 4.4, there exist pencils $a(s), b(s) \in \mathbb{F}[s]^{1\times q}$ such that $A(s) \overset{s.e.}{\sim} \begin{bmatrix} a(s) \\ A_{21}(s) \end{bmatrix}$ and $B(s) \overset{s.e.}{\sim} \begin{bmatrix} b(s) \\ A_{21}(s) \end{bmatrix}$. The sufficiency follows from Lemma 3.3.

The cases (b), (c) and (d) are similar.

□

Remark 5.11 *From the proof of Theorem 5.1 and Remark 3.4, when* $\mathbf{u} = \mathbf{v}$ *we can conclude that if* $A(s) + P(s) \overset{s.e.}{\sim} B(s)$ *where* $P(s)$ *is a pencil of* $\operatorname{rank}(P(s)) = 1$ *with a positive row minimal index, then there also exists* $\bar{P}(s) = \bar{u}\bar{v}(s)^T$ *(i.e.,* $\bar{P}(s)$ *has not a positive row minimal index) such that* $A(s) + \bar{P}(s) \overset{s.e.}{\sim} B(s)$.

Analogously, when $\mathbf{c} = \mathbf{d}$*, if* $A(s) + P(s) \overset{s.e.}{\sim} B(s)$ *where* $P(s)$ *is a pencil of* $\operatorname{rank}(P(s)) = 1$ *with a positive column minimal index, then there also exists* $\bar{P}(s) = \bar{u}(s)\bar{v}^T$ *(i.e.,* $\bar{P}(s)$ *has not a positive column minimal index) such that* $A(s) + \bar{P}(s) \overset{s.e.}{\sim} B(s)$.

6 Conclusions

Given a matrix pencil, regular or singular, we have completely characterized the Kronecker structure of a pencil obtained from it by a perturbation of rank one. The result holds over arbitrary fields.

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References

[1] Baragaña, I., and Roca, A. Fixed rank perturbations of regular matrix pencils. *arXiv e-prints* (Jul 2019), arXiv:1907.10657.

[2] Baragaña, I., and Roca, A. Weierstrass structure and eigenvalue placement of regular matrix pencils under low rank perturbation. *SIAM Journal on Matrix Analysis and Applications 40*, 2 (2019), 440–453.

[3] Batzke, L. Generic rank-one perturbations of structured regular matrix pencils. *Linear Algebra Appl 458* (2014), 638–670.

[4] Batzke, L., Mehl, C., Ran, A., and Rodman, L. Generik rank-k Perturbations of Structured Matrices. *Operator Theory 255* (2016), 27–48.

[5] De Terán, F., and Dopico, F. Low rank perturbation of Kronecker structures without full rank. *SIAM Journal on Matrix Analysis and Applications 29*, 2 (2007), 496–529.

[6] De Terán, F., and Dopico, F. Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations. *SIAM Journal on Matrix Analysis and Applications 37*, 3 (2016), 823–835.

[7] De Terán, F., Dopico, F., and Moro, J. Low rank perturbation of Weierstrass structure. *SIAM Journal on Matrix Analysis and Applications 30*, 2 (2008), 538–547.

[8] Dodig, M. Completion up to a matrix pencil with column minimal indices as the only nontrivial Kronecker invariants. *Linear Algebra and its Applications 438* (2013), 3155–3173.

[9] Dodig, M., and Stošić, M. On convexity of polynomial paths and generalized majorizations. *Electronic Journal of Combinatorics 17*, 1 (2010), 61.

[10] Dodig, M., and Stošić, M. On properties of the generalized majorization. *Electronic Journal of Linear Algebra 26* (2013), 471–509.

[11] Dodig, M., and Stošić, M. The rank distance problem for pairs of matrices and a completion of quasi-regular matrix pencils. *Linear Algebra and its Applications 457* (2014), 313–347.

[12] Dodig, M., and Stošić, M. The general matrix completion problem: a minimal case. *SIAM Journal on Matrix Analysis and Applications 40*, 1 (2019), 347–369.

[13] Friedland, S. *Matrices: algebra, analysis and applications.* World Scientific, Singapore, 2016.

[14] Gantmacher, F. *Matrix Theory, Vols I, II.* Chelsea, New York, 1974.

[15] Gernandt, H., and Trunk, C. Eigenvalue placement for regular matrix pencils with rank one perturbations. *SIAM Journal on Matrix Analysis and Applications 38*, 1 (2017), 134–154.

[16] Hörmander, L., and Melin, A. A remark on perturbations of compact operators. *Math. Scand. 75* (1994), 255–262.

[17] Leben, L., Martínez-Pería, F., Philipp, F., Trunk, C., and Winkler, H. Finite rank perturbations of linear relations and singular matrix pencils. *arXiv e-prints* (Jun 2018), arXiv:1806.07513.

[18] Mehl, C., Mehrmann, V., Ran, A., and Rodman, L. Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations. *Linear Algebra and its Applications 435* (2011), 687–716.

[19] Moro, J., and Dopico, F. Low rank perturbation of Jordan structure. *SIAM Journal on Matrix Analysis and Applications 25*, 2 (2003), 495–506.

[20] Roca, A. *Asignación de Invariantes en Sistemas de Control.* PhD thesis, Universitat Politècnica València, 2003.

[21] Savchenko, S. V. Typical changes in spectral properties under perturbations by a rank-one operator. *Mathematical Notes 74*, 4 (2003), 557–568.

[22] Savchenko, S. V. On the change in the spectral properties of a matrix under perturbations of sufficiently low rank. *Functional Analysis and Its Applications 38*, 1 (2004), 69–71.

[23] Silva, F. The Rank of the Difference of Matrices with Prescribed Similarity Classes. *Linear and Multilinear Algebra 24* (1988), 51–58.

[24] Thompson, R. Invariant factors under rank one perturbations. *Canad. J. Math 32* (1980), 240–245.

[25] Zaballa, I. Pole assignment and additive perturbations of fixed rank. *SIAM Journal on Matrix Analysis and Applications 12*, 1 (1991), 16–23.

[26] Zaballa, I. Controllability and hermite indices of matrix pairs. *International Journal of Control 68*, 1 (1997), 61–68.