ON BOUNDEDNESS AND COMPACTNESS OF TOEPLITZ OPERATORS IN WEIGHTED H^∞-SPACES.

JOSÉ BONET, WOLFGANG LUSKY, AND JARI TASKINEN

Abstract. We characterize the boundedness and compactness of Toeplitz operators T_a with radial symbols a in weighted H^∞-spaces H_v^∞ on the open unit disc of the complex plane. The weights v are also assumed radial and to satisfy the condition (B) introduced by the second named author. The main technique uses Taylor coefficient multipliers, and the results are first proved for them. We formulate a related sufficient condition for the boundedness and compactness of Toeplitz operators in reflexive weighted Bergman spaces on the disc.

We also construct a bounded harmonic symbol f such that T_f is not bounded in H_v^∞ for any v satisfying mild assumptions. As a corollary, the Bergman projection is never bounded with respect to the corresponding weighted sup-norms. However, we also show that, for normal weights v, all Toeplitz operators with a trigonometric polynomial as the symbol are bounded on H_v^∞.

1. Introduction and preliminaries on Toeplitz operators.

In this paper we consider the boundedness and compactness of Toeplitz operators on weighted sup-normed spaces of holomorphic functions H_v^∞ on the open unit disc $\mathbb{D}$ of the complex plane. By a weight v we mean here a continuous function $\mathbb{D} \to]0,\infty[$ which is radial, vanishing on the boundary and decreasing with the radius, i.e. there holds $v(z) = v(|z|)$ for all $z \in \mathbb{D}$, $\lim_{|z|\to 1} v(z) = 0$ and $v(r) \geq v(s)$ if $1 > s > r > 0$. Moreover let μ be the area measure dA on $\mathbb{D}$ multiplied with v as density, i.e. $d\mu(re^{i\varphi}) = vdA := v(r)rdrd\varphi$, where r, φ are the polar coordinates of the complex plane. For $1 \leq p < \infty$ consider first the spaces

$$L_v^p = \Big\{ g : \mathbb{D} \to \mathbb{C} \text{ measurable } : \|g\|_{p,v}^p := \int_{\mathbb{D}} |g|^p d\mu < \infty \Big\} \text{ and}$$

$$A_v^p = \{ h \in L_v^p : h \text{ holomorphic } \},$$

which are denoted by $L^p = (L^p, \|\cdot\|_p)$ and A^p, respectively, if $v \equiv 1$ is the constant weight. The *Bergman space* A_v^p is a closed subspace (see below) of L_v^p, and the *Bergman projection* P_v is defined as the orthogonal projection of L_v^2 onto A_v^2. In the case of the constant weight it has the integral representation

$$Pg(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{g(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta).$$

For an integrable function $f \in L^1$, the *Toeplitz operator* T_f with symbol f is defined in the space $A_v^p \ni h$ by

$$T_f h(z) = P_v(fh)(z), \tag{1.1}$$

if the expression on the right-hand side makes sense. (We will soon comment this in detail for our case.) If for example the projection P_v is also a bounded operator $L_v^p \to A_v^p$, $1 < p < \infty$, it follows that $T_f : A_v^p \to A_v^p$ is bounded, whenever f is a bounded measurable function. The question of the boundedness of T_f on A^p_v with unbounded symbols is a long-standing, still open problem. Examples of unbounded symbols inducing bounded Toeplitz operators can be easily constructed, since the behaviour of the symbol inside any compact subset of $\mathbb{D}$ is not important for the boundedness of the operator. We refer to the papers [6], [7], [8], [10], [11], [16], [17], [19], [21], [22], [23], [24], [25], [26], [27], [28], [29] for classical and recent results on the boundedness and compactness of Toeplitz operators on Bergman spaces. In particular a solution to the boundedness problem is known in the case of radial symbols, if $p = 2$, also in many weighted cases and higher dimensional domains in place of $\mathbb{D}$. Also, the case of a positive symbol $f(z) \geq 0 \, \forall \, z \in \mathbb{D}$ can be treated with the help of the Berezin transform. The references above mostly concern unweighted Bergman spaces or spaces A^p_v with standard weights like $v(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$.

In the present paper, Theorem 3.6, we will provide a characterization of the boundedness and compactness of T_f in the case $p = \infty$ and for a general class of weights. Thus, in the case of radial symbols this question remains unsolved only for $1 \leq p < 2$ and $2 < p < \infty$; however, our necessary and sufficient condition (3.7) seems quite a lot more complicated for $p = \infty$ than for $p = 2$ (see (5.1)). Anyway, both conditions involve the Hausdorff moments of the symbol on the unit interval, see (3.23), (3.24). (We do not have an idea how to interpolate between the two cases. The situation is somewhat analogous to the problem of describing the solid hulls and cores of the spaces A^p_v and H^∞_v, see [2]. Both of these objects can be described in the cases $p = 2$ and ∞, but for other p there are only partial results. Actually, this phenomenon is not completely unrelated with Toeplitz operators.)

We define the Banach spaces to be considered by

$$h^\infty_v = \{h : \mathbb{D} \to \mathbb{C} : h \text{ harmonic}, \ \|h\|_v := \sup_{z \in \mathbb{D}} |h(z)| v(|z|) < \infty\}$$

and

$$H^\infty_v = \{h \in h^\infty_v : h \text{ analytic } \};$$

we denote $H^\infty_v = H^\infty = (H^\infty, \| \cdot \|_\infty)$ in the case $v \equiv 1$. We need to comment the definition of Toeplitz operators in the case of H^∞_v. In the Hilbert spaces L^2_v and A^2_v we denote the inner product by

$$\langle f, g \rangle = \int\limits_{\mathbb{D}} f \overline{g} \, d\mu$$

Then, the functions $e_k(z) = \Gamma_{2k}^{-1/2} z^k$, where $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$ and

$$\Gamma_k = 2\pi \int_0^1 r^{k+1} v(r) dr \quad \text{for } k \in \mathbb{N}_0, \tag{1.2}$$

form an orthonormal basis of A^2_v. We remark that the numbers Γ_k satisfy for all $0 < \varrho < 1$ and some constant $C_{v,\varrho} > 0$ the following lower bound

$$\Gamma_k \geq C_{v,\varrho} \varrho^k \tag{1.3}$$

for every $k \in \mathbb{N}_0$. This follows from (1.2) by considering the integral e.g. over the interval $[\varrho, 1 - (1 - \varrho)/2]$ only.

Convergence in the space A^p_v, $1 < p < \infty$, with respect to the norm $\| \cdot \|_{p,v}$ implies pointwise convergence (hence A^p_v is a closed subspace of L^p_v), and thus the point evaluation functionals at any point of $\mathbb{D}$ are bounded functionals on A^p_v. Consequently, we find the reproducing kernel, i.e. a family of functions $K_z \in A_v^2$, $z \in \mathbb{D}$, such that

$$g(z) = \langle g, K_z \rangle = \int_{\mathbb{D}} g(w)\overline{K_z(w)}\, d\mu(w) \tag{1.4}$$

for all $g \in A_v^2$. The integral operator defined by the right hand side can be extended to L_v^2, and it actually defines the orthogonal projection from L_v^2 onto A_v^2, i.e. the Bergman projection P_v; see [4], [5]. Using the orthonormal basis we can write for all $z \in \mathbb{D}$

$$P_v g(z) = \sum_{k=0}^{\infty} \langle g, e_k \rangle e_k(z) = \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{\Gamma_k} g(w) d\mu(w). \tag{1.5}$$

Here, the order of the summation and the integral can be changed, because (1.3) leads for any fixed $z \in \mathbb{D}$ to the estimate

$$\Big| \frac{z^k \overline{w}^k}{\Gamma_k} \Big| \le c_{v,\varrho} \Big(\frac{|z|}{\varrho} \Big)^k, \tag{1.6}$$

and we can choose here $\varrho > |z|$ so that the sum on the right-hand side of (1.5) converges well enough. Moreover, the estimate (1.6) implies that for every $z \in \mathbb{D}$ the *Bergman kernel* K_z is a bounded function:

$$|K_z(w)| \le C_z \ \text{ for all } w \in \mathbb{D}. \tag{1.7}$$

Now let $f : \mathbb{D} \to \mathbb{C}$ be a function which belongs to L^1. We define the Toeplitz operator T_f with symbol f on H_v^∞ by

$$T_f(h) = \int_{\mathbb{D}} f(w) h(w) \overline{K_z(w)}\, d\mu(w). \tag{1.8}$$

It follows from (1.7) that the integral converges for all $z \in \mathbb{D}$ and for all $h \in H_v^\infty$, since by definition $hv \in L^\infty$. However, the resulting linear operator is not necessarily bounded $H_v^\infty \to H_v^\infty$ and does not necessarily even map H_v^∞ into itself, but this will be clarified by our Theorem 3.6. (We remark that the a priori assumption $f \in L^1$ is usual also in the theory of Toeplitz operators in the reflexive Bergman spaces, but in that case this assumption does not guarantee that the defining integral (1.8) converges for all $h \in A_v^p$ even for the constant weight $v \equiv 1$. From this point of view, the case $p = \infty$ is more simple.)

If $h \in H_v^\infty$ is such that $f \cdot h \in L_v^2$, we also have

$$\begin{aligned} (T_f h)(z) &= \sum_{n=0}^{\infty} \langle f \cdot h, e_n \rangle e_n(z) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{2n}} \int_{\mathbb{D}} f(w) h(w) \overline{w}^n v(w) dA\ , \end{aligned} \tag{1.9}$$

where the series converges in L_v^2. However, the formula also holds for all $h \in H_v^\infty$ (since we are assuming $f \in L^1$) and the product fhv thus belongs to L^1, and one can commute the summation and integration in (1.8), due to (1.6). In the latter case, the sum (1.9) converges uniformly for z in compact subsets of the disc.

As for the contents of this paper, we recall in Section 2 the known fact that a Toeplitz operator T_f on H_v^∞ with holomorphic symbol is a bounded operator $H_v^\infty \to H_v^\infty$ if and only if f is an element of H^∞. The situation is completely different if harmonic symbols are considered instead of holomorphic ones. In particular we construct in Theorem 2.3 a bounded harmonic function $f : \mathbb{D} \to \mathbb{C}$ such that T_f is unbounded on H_v^∞ for every weight v. This result also has the consequence, Corollary 2.4, that the Bergman projection P_v is never bounded on L_v^∞.

Moreover in Section 3 we give our characterization of the continuity and compactness of Toeplitz operators with radial symbols in H_v^∞, see Theorem 3.6. Toeplitz operators with radial symbols are nothing but Taylor coefficient multipliers. They are studied at first in Section 3. Our main result for multipliers, Theorem 3.4, is a generalization of a result in [13].

The negative result of Theorem 2.3 motivates the further studies of Toeplitz operators with bounded symbols in Section 4. We show, among other things that any Toeplitz operator with a trigonometric polynomial as the symbol is bounded, at least if the weight is normal. In Section 5 we put the condition (3.7) of Theorem 3.6 into a form which is natural for the Bergman spaces A_v^p, $1 < p < \infty$, and show that the condition is sufficient for the boundedness of T_f in that case, see Proposition 5.1.

As for the notation on analytic function spaces and operators in them, we refer to [28]. All function spaces are defined over the domain $\mathbb{D}$ unless otherwise stated. In addition we only remark that c, C, C_r etc. denote generic positive constants the exact value may change from place to place. We sometimes denote the pointwise multiplication by $f \cdot h$ for clarity.

2. Toeplitz operators with holomorphic and harmonic symbols

Let us consider a symbol $f : \mathbb{D} \to \mathbb{C}$ which is holomorphic and integrable over the disc, i.e. $f \in A^1$. If $h \in H_v^\infty$, then $f \cdot h$ is holomorphic on $\mathbb{D}$ so that there are numbers $a_n \in \mathbb{C}$ with

$$(f \cdot h)(re^{i\varphi}) = \sum_{n=0}^{\infty} a_n r^n e^{in\varphi}.$$

This implies in view of (1.8), (1.4) that $T_f(h) = f \cdot h$. Hence T_f is just the pointwise multiplier with symbol f; we denote this operator by S_f as the notation M_f will be reserved for the coefficient multiplier, see Section 2.

The following result for multiplication operators is known, see [1], and by the above explanation it can also be interpreted as a result for T_f, $f \in A^1$. This seemingly simple result should be compared with Theorem 2.3, below.

Proposition 2.1. *Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. Then S_f is a bounded operator $H_v^\infty \to H_v^\infty$ if and only if $f \in H^\infty$. Assuming in addition $f \in A^1$, the operator T_f is bounded $H_v^\infty \to H_v^\infty$, if and only if $f \in H^\infty$.*

Since the reference does not contain a proof and since our weights are pretty general, we prove the necessity statement for the multiplier; the other parts are quite trivial. Indeed, if S_f is continuous on H_v^∞, then its transpose map $S_f^* : (H_v^\infty)^* \to (H_v^\infty)^*$ is continuous in the dual space $\big((H_v^\infty)^*, \|\cdot\|_*\big)$. Clearly, given $z \in \mathbb{D}$, the point evaluation functional $\delta_z : f \mapsto f(z)$ belongs to the dual, and we have $S_f^*(\delta_z) = f(z)\delta_z$ for each $z \in \mathbb{D}$. Therefore

$$|f(z)| = \|S_f^*(\delta_z)\|_* / \|\delta_z\|_* \le \|S_f^*\|_{\rm op} = \|S_f\|_{\rm op}$$

for all $z \in \mathbb{D}$, where we denoted by $\|\cdot\|_{\rm op}$ the operator norm in the relevant spaces. We get $f \in H^\infty$.

We have the following corollary.

Corollary 2.2. *For any weight v there is an element $f \in H_v^\infty \cap L^1$ such that T_f is unbounded on H_v^∞.*

For the same reason as above, let us sketch the proof that the set $(H_v^\infty \cap L^1) \setminus H^\infty$ is non-empty. First, the usual argument based on Montel's theorem and the assumption on the vanishing of the weight v on the boundary imply that the embedding $H^\infty \hookrightarrow H_v^\infty$ is compact. The sequence of monomials $(z^n)_{n=1}^\infty$ is bounded in H^∞ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, hence $\|z^n\|_v \to 0$ as $n \to \infty$ (see [20], Section 2.4). Also $\|z^n\|_1 \to 0$ and $\|z^n\|_\infty = 1$ for all n, by direct calculations.

If the space H^∞ were equal to $H_v^\infty \cap L^1$, the closed graph theorem would yield a constant $C > 0$ such that

$$\|h\|_\infty \le C \max(\|h\|_v, \|h\|_1) \tag{2.1}$$

for all $h \in H^\infty$ (since the converse of the inequality (2.1) holds trivially). We get a contradiction from the above norm estimates for the monomials z^n.

We proceed to study the case of harmonic symbols, which is much more complicated. Since the Bergman projection is known to be unbounded with respect to the norm $\|\cdot\|_v$ for many weights v, one may expect that there are examples of bounded symbols $f \in L^\infty$ so that T_f is not a bounded operator from H_v^∞ into itself. While we do not exactly know such examples in the literature, let us mention Section 5 of [18], where possible pathologies of Toeplitz operators with bounded symbols were considered in the case of reflexive Bergman spaces on polygonal domains. In the following theorem we find a very strong negative example; cf. also Proposition 2.1.

Theorem 2.3. *There is a bounded harmonic function $f : \mathbb{D} \to \mathbb{C}$ such that T_f is not a bounded operator $H_v^\infty \to H_v^\infty$ for any weight v on $\mathbb{D}$.*

Since the pointwise multiplication with a bounded function f is always a bounded operator $H_v^\infty \to L_v^\infty$, this result immediately implies the following conclusion.

Corollary 2.4. *The Bergman projection P_v is never (for any weight under consideration) a bounded mapping $L_v^\infty \to L_v^\infty$.*

Namely, if P_v were bounded, this would imply $T_f : H_v^\infty \to H_v^\infty$ is bounded for every $f \in L^\infty$, which would contradict Theorem 2.3.

To prove Theorem 2.3 we need some lemmas. Fix a weight v on $\mathbb{D}$. Let $\tilde{f}_0 : \partial\mathbb{D} \to \mathbb{C}$ be the map with

$$\tilde{f}_0(e^{i\varphi}) = \begin{cases} 1\,, & \text{if } -\pi/2 \le \varphi \le \pi/2 \\ 0 & \text{else}\,. \end{cases}$$

Then, the following is true.

Lemma 2.5. *Let f_0 be the harmonic extension of $\tilde{f}_0$ on $\mathbb{D}$. We have*

$$f_0(z) = \frac{1}{2} + \frac{1}{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{2k+1} \left(z^{2k+1} + \bar{z}^{2k+1} \right), \qquad z \in \mathbb{D}.$$

Clearly, f_0 is bounded on the disc due to the maximum principle.

Proof. Let a_k, $k \in \mathbb{Z}$, be the Fourier coefficients of $\tilde{f}_0$. Then we have

$$a_k = \frac{1}{2\pi} \int\limits_{-\pi/2}^{\pi/2} e^{-ikt} dt = \frac{e^{ik\pi/2} - e^{-ik\pi/2}}{2k\pi i} = \frac{e^{i|k|\pi/2} - e^{-i|k|\pi/2}}{2|k|\pi i}$$
$$= \begin{cases} \frac{(-1)^j}{(2j+1)\pi}, & \text{if } |k| = 2j+1 \\ 0 & \text{else,} \end{cases}$$

provided that $k \neq 0$. Moreover, $a_0 = 1/2$. This proves the lemma. □

Lemma 2.6. *Let*

$$f(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}, \qquad z \in \mathbb{D},$$

for some a_k. Put $(Sf)(z) = (f(z) - if(iz))/2$. Then

$$(Sf)(z) = \sum_{k=0}^{\infty} a_{4k+1} z^{4k+1} \qquad \textit{and} \qquad \sup_{|z|=r} |(Sf)(z)| \le \sup_{|z|=r} |f(z)|$$

for all r.

Proof. The first assertion follows from

$$1 - i \cdot i^{2k+1} = 1 + (-1)^k = \begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

The second assertion is trivial. □

Consider $m > 0$ and let r_m be a point where the function $r^m v(r)$ attains its absolute maximum on $[0,1]$. We easily see that $r_n \geq r_m$ if $n \geq m$ and $\lim_{m\to\infty} r_m = 1$; see for example [12] for details.

Let us set for all $m \in \mathbb{N}_0$

$$g_m(re^{i\varphi}) = \frac{r^m e^{im\varphi}}{r_m^m v(r_m)}, \quad re^{i\varphi} \in \mathbb{D}.$$

Then $\|g_m\|_v = 1$. Recalling the notation (1.2) for Γ_k we state the following result.

Lemma 2.7. *Let $f : \mathbb{D} \to \mathbb{C}$ be harmonic, say $f(re^{i\varphi}) = \sum_{k=-\infty}^{\infty} b_k r^{|k|} e^{ik\varphi}$. For all $m \in \mathbb{N}_0$ we have*

$$T_f(g_m)(re^{i\varphi}) = \sum_{k=0}^{m} b_{k-m} \frac{\Gamma_{2m}}{\Gamma_{2k}} \frac{r^k e^{ik\varphi}}{r_m^m v(r_m)} + \sum_{k=m+1}^{\infty} b_{k-m} \frac{r^k e^{ik\varphi}}{r_m^m v(r_m)} \tag{2.2}$$

Proof. This follows from

$$f(re^{i\varphi}) \cdot g_m(re^{i\varphi}) = \sum_{j\in\mathbb{Z}} b_j \frac{r^{m+|j|} e^{i(j+m)\varphi}}{r_m^m v(r_m)}$$
$$= \sum_{k=m+1}^{\infty} b_{k-m} \frac{r^k e^{ik\varphi}}{r_m^m v(r_m)} + \sum_{k=-\infty}^{m} b_{k-m} \frac{r^{2m-k} e^{ik\varphi}}{r_m^m v(r_m)}$$

and (1.9). □

Proof of Theorem 2.3. We take f_0 of Lemma 2.5 and show that T_{f_0} is unbounded on H_v^∞. Put

$$f_1(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \left(z^{2j+1} + \bar{z}^{2j+1} \right).$$

It suffices to show that T_{f_1} is unbounded since $T_{f_0} = T_{1/2} + \pi^{-1} T_{f_1}$ and $T_{1/2}$ is bounded. Fix a positive integer m, say $m = 4m_0$ for $m_0 \in \mathbb{N}$. Then

$$k - m \text{ is } \begin{cases} \text{odd} & \text{if } k \text{ is odd} \\ \text{even} & \text{if } k \text{ is even} \end{cases} \quad \text{and} \quad j - 2m_0 \text{ is } \begin{cases} \text{odd} & \text{if } j \text{ is odd} \\ \text{even} & \text{if } j \text{ is even.} \end{cases}$$

Lemma 2.7 yields with $b_k = 0$, if k is even, and with $b_k = (-1)^k / |2k+1|$ if k is odd

$$T_{f_1}(g_m)(re^{i\varphi}) = \sum_{\substack{k=0, \\ k \text{ odd}}}^{m} b_{k-m} \frac{\Gamma_{2m}}{\Gamma_{2k}} \frac{r^k e^{ik\varphi}}{r_m^m v(r_m)} + \sum_{\substack{k=m+1, \\ k \text{ odd}}}^{\infty} b_{k-m} \frac{r^k e^{ik\varphi}}{r_m^m v(r_m)}.$$

Using Lemma 2.6 we obtain

$$\begin{aligned} S(T_{f_1}(g_m))(re^{i\varphi}) &= \sum_{0 \le 4j+1 \le m} b_{4j+1-m} \frac{\Gamma_{2m}}{\Gamma_{8j+2}} \frac{r^{4j+1} e^{i(4j+1)\varphi}}{r_m^m v(r_m)} \\ &\quad + \sum_{m+1 \le 4j+1 < \infty} b_{4j+1-m} \frac{r^{4j+1} e^{i(4j+1)\varphi}}{r_m^m v(r_m)}. \end{aligned}$$

Recall that $b_{4j+1-m} = 1/|4(j-m_0)+1|$. So if we take $\varphi = 0$ then all summands in the preceding sum are non-negative. Hence

$$\begin{aligned} \frac{r_m}{5} \log\left(\frac{1}{1 - r_m^4} \right) &= \frac{r_m}{5} \sum_{j=1}^{\infty} \frac{(r_m^4)^j}{j} \le \sum_{j=0}^{\infty} \frac{r_m^{4j+1}}{4j+1} \\ = \sum_{m+1 \le 4j+1 < \infty} & b_{4j+1-m} \frac{r_m^{4j+1} v(r_m)}{r_m^m v(r_m)} \le S(T_{f_1}(g_m))(r_m) v(r_m) \\ \le \|S(T_{f_1}(g_m))\|_v & \le \|T_{f_1}(g_m)\|_v. \end{aligned}$$

Since $\lim_{m\to\infty} r_m = 1$, the left-hand side of the preceding estimate grows to the infinity, when $m \to \infty$. Hence T_{f_1} and also T_{f_0} cannot be bounded. $\square$

3. Multipliers from h_v^∞ into H_v^∞ and Toeplitz operators

Toeplitz operators with radial (thus in general non-harmonic) symbols on the disc correspond to Taylor coefficient multipliers so we proceed to study them. At first we mention some general results concerning the Banach space h_v^∞. These are collected from the references [12], [14] and [15]. We recall that the numbers $r_m \in]0,1[$ were defined above Lemma 2.7.

Definition 3.1. (i) The weight v satisfies the condition (B), if

$$\forall b_1 > 1 \ \exists b_2 > 1 \ \exists c > 0 \ \forall m, n > 0$$
$$\left(\frac{r_m}{r_n} \right)^m \frac{v(r_m)}{v(r_n)} \le b_1 \ \text{ and } \ m, n, |m-n| \ge c \ \Rightarrow \left(\frac{r_n}{r_m} \right)^n \frac{v(r_n)}{v(r_m)} \le b_2.$$

(ii) Also, v is called *normal* if

$$\sup_{n\in\mathbb{N}} \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty \quad \text{and} \quad \inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1.$$

Note that in (i), m and n need not be integers. Condition (B) is crucial for the structure, in particular for the isomorphic character of H_v^∞. Actually it is equivalent to the fact that H_v^∞ is isomorphic to the Banach space ℓ^∞ of bounded sequences (Theorem 1.1 of [14]). Examples of weights satisfying (B) are all normal weights, in particular the standard weights $v(r) = (1-r)^\alpha$ (or $v(r) = (1-r^2)^\alpha$) where $\alpha > 0$. Moreover, for $\beta > 0$ and $\gamma > 0$ the weight $v(r) = \exp(-\gamma/(1-r)^\beta)$ satisfies (B) but is not normal; see [14].

Fix $b > 2$. We define by induction the indices $0 \le m_1 < m_2 < \dots$ such that

$$b = \min\left(\left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})}\right).$$

This is always possible according to Lemma 5.1. of [14]. (Actually it suffices to choose the indices such that the preceding minimum lies between b and some constant $b_1 > b$.)

Now let the numbers $b_k \in \mathbb{C}$, $k \in \mathbb{Z}$, be given and denote by $h(\varphi) = \sum_{k\in\mathbb{Z}} b_k e^{ik\varphi}$ a series which may or may not converge. We take the preceding numbers m_n and put for all $n \in \mathbb{N}$

$$\begin{aligned}(W_n h)(\varphi) &= \sum_{m_{n-1}<|k|\le m_n} \frac{|k| - [m_{n-1}]}{[m_n] - [m_{n-1}]} b_k e^{ik\varphi} + \sum_{m_n<|k|\le m_{n+1}} \frac{[m_{n+1}] - |k|}{[m_{n+1}] - [m_n]} b_k e^{ik\varphi} \\ &=: \sum_{k\in\mathbb{Z}} w_{nk} b_k e^{ik\varphi}\end{aligned}$$

where $m_0 = 0$. Here $[r]$ is the largest integer not greater than r. The operators W_n are also considered as acting on the harmonic functions by

$$W_n : \sum_{k=-\infty}^{\infty} b_k r^{|k|} e^{ik\varphi} \mapsto \sum_{k=-\infty}^{\infty} w_{nk} b_k r^{|k|} e^{ik\varphi}$$

For any function $g : \mathbb{D} \to \mathbb{C}$ and radius $0 \le r \le 1$ we denote

$$M_\infty(g, r) = \sup_{|z|=r} |g(z)|.$$

The Riesz projection P is defined by

$$P\Big(\sum_{k\in\mathbb{Z}} a_k r^{|k|} e^{ik\varphi}\Big) = \sum_{k=0}^{\infty} a_k r^k e^{ik\varphi}. \tag{3.1}$$

Theorem 3.2. *Let v satisfy (B). Then there are constants $c_1, c_2 > 0$ such that, for all $g \in h_v^\infty$,*

$$c_1 \sup_{n\in\mathbb{N}} M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \le \|g\|_v \le c_2 \sup_{n\in\mathbb{N}} M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \tag{3.2}$$

and

$$c_1 M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \le \|W_n g\|_v \le c_2 M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \tag{3.3}$$

for all n. Moreover,

$$\sup_n (m_{n+1} - m_n)/(m_n - m_{n-1}) < \infty. \tag{3.4}$$

Finally, the Riesz projection $P : h_v^\infty \to H_v^\infty$ *is bounded.*

This is Theorem 1 of [15]. See also Propositions 4.1. and 5.2. of [14]. One can even show that the boundedness of the Riesz projection in h_v^∞ is equivalent to (B) (for details, see [14]).

Remark 3.3. If a sequence $(b_k)_{k=-\infty}^\infty$ of complex numbers is given such that

$$\sup_{n\in\mathbb{N}} M_\infty\Big(\sum_{k=-\infty}^{\infty} w_{nk} b_k r^{|k|} e^{ik\varphi}, r_{m_n}\Big) v(r_{m_n}) \tag{3.5}$$
$$= \sup_{n\in\mathbb{N}} \sup_{\varphi\in[0,2\pi]} \Big| \sum_{m_{n-1}<|k|\le m_{n+1}} w_{nk} b_k r_{m_n}^k e^{ik\varphi} \Big| v(r_{m_n}) < \infty,$$

then the series defining the harmonic function $g(re^{i\varphi}) = \sum_{k=-\infty}^\infty b_k r^{|k|} e^{ik\varphi}$ converges in the compact-open topology, and g belongs to h_v^∞ and $\|g\|_v$ is bounded by a constant times the expression in (3.5). For this statement, see Remark 1, *(iii)* of [15].

Examples. If v is normal then one can take $m_n = 2^{kn}$ for suitable fixed $k > 0$ (see [14], Example 2.4., and [12]). For $v(r) = \exp(-\gamma/(1-r)^\beta)$ one can take $m_n = \beta(\beta/\gamma)^{1/\beta} n^{2+2/\beta} - \beta n^2$, see [2].

We next turn to a theorem which was proven for a more restricted class of weights in Theorem 4.1 of [13]. In the theorem we assume that a sequence $(\gamma_k)_{k=0}^\infty$ of complex numbers is given, and consider the formal series $f(\varphi) = \sum_{k=0}^\infty \gamma_k e^{ik\varphi}$ and the multiplier M_f with

$$(M_f h)(re^{i\varphi}) = \sum_{k=0}^{\infty} \gamma_k b_k r^k e^{ik\varphi} \tag{3.6}$$

for harmonic functions $h(re^{i\varphi}) = \sum_{k=-\infty}^\infty b_k r^{|k|} e^{ik\varphi}$. By definition, $M_f h$ is holomorphic, if the series (3.6) converges.

Theorem 3.4. *Let the weight* v *satisfy condition* (B)*. Then* M_f *maps* h_v^∞ *into* H_v^∞ *and is bounded, if and only if*

$$\sup_{n\in\mathbb{N}} \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi < \infty. \tag{3.7}$$

Moreover, assume (3.7) *holds. Then* $M_f : h_v^\infty \to H_v^\infty$ *is compact, if and only if*

$$\int_0^{2\pi} |(W_n f)(\varphi)| d\varphi \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

Proof. Assume (3.7) holds. We first remark that M_f is a convolution operator, i.e. at least in the case of only finitely many non-zero entries γ_k, the expression (3.6) can be written as

$$(M_f h)(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi - \psi) h(re^{i\psi}) d\psi.$$

So, if $h \in h_v^\infty$, then we have for all $re^{i\varphi} \in \mathbb{D}$

$$|(M_{W_n f} h)(re^{i\varphi})| v(r) \le \frac{1}{2\pi} \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi \, \|h\|_v. \tag{3.9}$$

Hence,

$$M_\infty(M_{W_n f} h, r) v(r) \le C \|h\|_v$$

for all n and r, where the constant $C > 0$ is the supremum on the left- hand side of (3.7). This bound and Remark 3.3 imply that the series on the right-hand side of (3.6) converges in the compact-open topology, defines an element of H_v^∞ and is bounded by $\|h\|_v$. In other words, M_f maps h_v^∞ continuously into H_v^∞.

As for compactness of the operator M_f, let $(h_j)_{j=1}^\infty$ be sequence which converges to 0 with respect uniformly on compact subsets of $\mathbb{D}$ and which is contained in the closed unit ball of h_v^∞. It suffices to show that M_f maps such a sequence into a one converging to 0 with respect to the norm; see for example [20], Section 2.4. Let $\varepsilon > 0$. If (3.8) is assumed, we can fix $N \in \mathbb{N}$ such that

$$\int_0^{2\pi} |(W_n f)(\varphi)| d\varphi < \varepsilon \tag{3.10}$$

for $n > N$. Moreover, we note that for every $n \in \mathbb{N}$, the operator

$$W_n M_f : \sum_{k=-\infty}^{\infty} b_k r^{|k|} e^{ik\varphi} = \sum_{k=0}^{\infty} w_{nk} \gamma_k b_k r^k e^{ik\varphi}$$

is bounded in the space h_v^∞ when this space is endowed with the norm

$$\sup_{|z| \le r_{m_n}} |h(z)|; \tag{3.11}$$

to see this notice that every functional

$$g \mapsto r_{m_n}^{-k} \int_0^{2\pi} g\big(r_{m_n} e^{ik\varphi}\big) e^{-ik\varphi} d\varphi \ , \quad g \in h_v^\infty,$$

is bounded with respect to the norm (3.11) on h_v^∞, and $W_n M_f$ is a finite linear combination of these functionals. Consequently, due to the uniform convergence on compact sets, we can can choose a large enough $J = J(N) \in \mathbb{N}$ such that

$$\sup_{|z| \le r_{m_n}} |W_n M_f h_j(z)| v(z) < \varepsilon \tag{3.12}$$

for all $n \le N$, all $j \ge J$. For such j we obtain by Theorem 3.2

$$\begin{aligned} & c_2^{-1} \|M_f h_j\|_v \\ \le & \sup_{n \le N} M_\infty(W_n M_f h_j, r_{m_n}) v(r_{m_n}) + \sup_{n > N} M_\infty(W_n M_f h_j, r_{m_n}) v(r_{m_n}) \end{aligned} \tag{3.13}$$

The first term on the right-hand side of (3.13) is bounded by ε due to (3.12), and the second one can be estimated in the same way as in (3.9), and (3.10) implies that this term is bounded by ε. Thus, M_f is compact.

To prove the necessity of (3.7) for the boundedness, we fix an arbitrary $0 < \varepsilon < 1$, and $n \in \mathbb{N}$ and $\psi \in [0, 2\pi]$ and find, by for example the Fejer approximation theorem, a trigonometric polynomial h, depending on n, ψ and ε,

$$h(re^{i\varphi}) = \sum_{k \in \mathbb{Z}} h_k r^{|k|} e^{ik\varphi}$$

such that

$$\Big| h(r_{m_n} e^{i\varphi}) - \frac{\overline{W_n f(\psi - \varphi)}}{|W_n(\psi - \varphi)| v(r_{m_n})} \Big| < \frac{\varepsilon}{v(r_{m_n})} \tag{3.14}$$

for all $\varphi \in [0, 2\pi|$, in particular

$$M_\infty(h, r_{m_n}) v(r_{m_n}) \leq 2. \tag{3.15}$$

As a consequence,

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi &= \frac{1}{2\pi} \int_0^{2\pi} |(W_n f)(\psi - \varphi)| d\varphi \\
&\leq \frac{1}{2\pi} \Big| \int_0^{2\pi} (W_n f)(\psi - \varphi) h(r_{m_n} e^{i\varphi}) d\varphi \Big| v(r_{m_n}) + \varepsilon \\
&= \frac{1}{2\pi} \Big| \int_0^{2\pi} f(\psi - \varphi)(W_n h)(r_{m_n} e^{i\varphi}) d\varphi \Big| v(r_{m_n}) + \varepsilon \\
&= |M_f W_n h(r_{m_n} e^{i\psi})| \, v(r_{m_n}) + \varepsilon. \qquad (3.16)
\end{aligned}$$

We obtain

$$\frac{1}{2\pi} \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi \leq \|M_f\| \cdot \|W_n h\|_v + \varepsilon. \tag{3.17}$$

For any $r > 0$, Lemma 3.3. of [14] implies

$$M_\infty(W_n h, r) \leq 4 \Big(\frac{[m_{n+1}] - [m_{n-1}]}{[m_n] - [m_{n-1}]} \Big) \Big(3 + 4 \frac{[m_{n+1}] - [m_{n-1}]}{[m_{n+1}] - [m_n]} \Big) M_\infty(h, r). \tag{3.18}$$

Due to Theorem 3.2 (in particular (3.4)) and (3.15) we find a universal constant $d > 0$ such that

$$\|W_n h\|_v \leq c_2 M_\infty(W_n h, r_{m_n}) v(r_{m_n}) \leq c_2 d M_\infty(h, r_{m_n}) v(r_{m_n}) \leq 2c_2 d.$$

Hence $\sup_n \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi < \infty$.

Finally, to prove the necessity of the condition (3.8) for the compactness of M_f, we first observe that given any $k \in \mathbb{N}$ we have, for all $r \leq r_k$,

$$\Big(\frac{r}{r_{m_n}} \Big)^{m_n} \frac{v(r)}{v(r_{m_n})} \to 0 \quad \text{as } n \to \infty. \tag{3.19}$$

To see this, fix k for a moment and denote for all $m \in \mathbb{N}$ and $r \in [0, 1[$

$$G_m(r) = \Big(\frac{r}{r_m} \Big)^m \frac{v(r)}{v(r_m)} \quad \text{and} \quad c_k = \sup_{r \leq r_k} G_k(r) < \infty.$$

For all $r \le r_k$ we get

$$\frac{G_{m_n}(r)}{G_{k+1}(r)} = \Big(\frac{r}{r_{m_n}}\Big)^{m_n}\Big(\frac{r_{k+1}}{r}\Big)^{k+1}\frac{v(r)}{v(r_{m_n})}\frac{v(r_{k+1})}{v(r)}$$
$$= \Big(\frac{r}{r_{k+1}}\Big)^{m_n-(k+1)}\frac{r_{k+1}^{m_n}v(r_{k+1})}{r_{m_n}^{m_n}v(r_{m_n})} \le \Big(\frac{r}{r_{k+1}}\Big)^{m_n-k-1},$$

where the last inequality follows from the definition that r_{m_n} is the maximum point of the function $r^{m_n}v(r)$. We see that (3.19) holds, since

$$\sup_{r\le r_k} G_{m_n}(r) \le c_k \sup_{r\le r_k}\frac{G_{m_n}(r)}{G_{k+1}(r)} \le c_k \sup_{r\le r_k}\Big(\frac{r}{r_{k+1}}\Big)^{m_n-k} \to 0 \ \text{ as } n\to\infty.$$

We next choose for every n and ψ the trigonometric polynomial $h =: h_{n,\psi}$ with $\varepsilon = 1/n$, as in (3.14). As a consequence of Theorem 3.2, (3.15) and Lemma 3.3. of [14] (cf. (3.18)),

$$\begin{aligned} &M_\infty(W_n h_{n,\psi}, r_{m_{n-1}})v(r_{m_{n-1}}) \le \|W_n h_{n,\psi}\|_v \\ &\le c_2 M_\infty(W_n h_{n,\psi}, r_{m_n})v(r_{m_n}) \le C M_\infty(h_{n,\psi}, r_{m_n})v(r_{m_n}) \le C'. \end{aligned} \tag{3.20}$$

Let us again fix $k\in\mathbb{N}$. We claim that for every $\delta>0$ there exists N such that

$$\sup_{|z|\le r_k} |W_n h_{n,\psi}|v(z) < \delta \tag{3.21}$$

for all $n \ge N$ and ψ. To see this, notice that the smallest power of r in the trigonometric polynomial $W_n h_{n,\psi}$ is m_{n-1}, hence, assuming n is so large that $m_{n-1} > k$, Lemma 3.1.(b) of [14] yields for all $|z| \le r_k$

$$|W_n h_{n,\psi,\varepsilon}(z)| \le 2\Big(\frac{|z|}{r_{m_{n-1}}}\Big)^{m_{n-1}} M_\infty(W_n h_{n,\psi}, r_{m_{n-1}}). \tag{3.22}$$

We obtain (3.21) for large enough N by using (3.22), (3.19), (3.20), since

$$\begin{aligned} &\sup_{|z|\le r_k} |W_n h_{n,\psi,\varepsilon}(z)|v(z) \\ &\le 2M_\infty(W_n h_{n,\psi}, r_{m_{n-1}})v(r_{m_n-1}) \sup_{|z|\le r_k}\Big(\frac{|z|}{r_{m_{n-1}}}\Big)^{m_{n-1}}\frac{v(z)}{v(r_{m_n-1})}. \end{aligned}$$

In other words, the functions $W_n h_{n,\psi}$ form a sequence converging to zero uniformly on compact subset of the open disc (and also uniformly with respect to ψ). Fixing $\psi\in[0,2\pi]$, the compact operator M_f maps the sequence $(W_n h_{n,\psi})_{n=1}^\infty$ into a sequence converging to 0 in the norm. Taking this into account in the estimate (3.16)–(3.17), we get (3.8). $\square$

Corollary 3.5. *Let the weight v satisfy condition (B). Then M_f maps H_v^∞ continuously into H_v^∞ if and only if*

$$\sup_n \int_0^{2\pi} |(W_n f)(\varphi)|d\varphi < \infty.$$

Moreover, a bounded $M_f : H_v^\infty \to H_v^\infty$ is compact, if and only if

$$\sup_{n\to\infty} \int_0^{2\pi} |(W_n f)(\varphi)|d\varphi = 0.$$

Proof. The sufficiency follows directly from Theorem 3.4. As for necessity, note that the Riesz projection P is bounded, by the assumption and Theorem 3.2. Thus, if $M_f|_{H_v^\infty}$ is bounded then M_f is also bounded on h_v^∞, and the necessary condition follows from Theorem 3.4. The statements concerning compactness can be proven by analogous arguments. □

Now we go back to Toeplitz operators. Let T_a be a Toeplitz operator on H_v^∞ with a given radial symbol $a \in L^1$, i.e. where $a(z) = a(|z|)$ for all (almost every) z. Then, with $h(z) = \sum_{n=0}^\infty h_n z^n \in H_v^\infty$, (1.9) reduces to

$$T_a h(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma_{2n}} \int_0^1 \int_0^{2\pi} a(r) h(re^{i\varphi}) r^{n+1} e^{-in\varphi} v(r)\, d\varphi dr$$

$$= \sum_{n=0}^\infty \frac{z^n}{\Gamma_{2n}} \int_0^1 a(r) r^{2n+1} v(r) h_n dr = \sum_{n=0}^\infty \gamma_n h_n z^n = M_{f_a} h(z) \tag{3.23}$$

where

$$\gamma_n = \frac{1}{\Gamma_{2n}} \int_0^1 r^{2n+1} v(r) a(r) dr \quad \text{and} \quad f_a(\varphi) = \sum_{k=0}^\infty \gamma_k e^{ik\varphi}. \tag{3.24}$$

We obtain by Corollary 3.5

Theorem 3.6. *Let the weight satisfy (B). If $a \in L^1$ is radial then T_a is bounded as operator $H_v^\infty \to H_v^\infty$ if and only if*

$$\sup_n \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi < \infty,$$

and T_a is a compact operator $H_v^\infty \to H_v^\infty$, if and only if

$$\lim_{n\to\infty} \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi = 0.$$

4. More on Toeplitz operators.

Since it was observed above that the boundedness of a symbol is not enough to guarantee the boundedness of the Toeplitz operator, we present in this section some complementary results and examples on this topic; see also the remark at the end of this section. In the following we denote by Q_m, $m \in \mathbb{N}_0$, the projection

$$Q_m\Big(\sum_{k=0}^\infty b_k e^{ik\varphi}\Big) = \sum_{k=0}^m b_k e^{ik\varphi} \quad \text{or} \quad Q_m\Big(\sum_{k=0}^\infty b_k z^k\Big) = \sum_{k=0}^m b_k z^k \tag{4.1}$$

It is well-known that

$$\Big|Q_m\Big(\sum_{k=0}^\infty c_k e^{ik\varphi}\Big)\Big| \le d \log m \sup_{0\le\psi\le 2\pi} \Big|\sum_{k=0}^\infty c_k e^{ik\psi}\Big|$$

where $d > 0$ is a universal constant independent of m and c_k.

At first we show

Theorem 4.1. *Let $a_j \in L^1$,$j = -n, \dots, n$, be radial functions and define*

$$f(z) = \sum_{j=-n}^{n} a_j(|z|)z^j, \qquad z \in \mathbb{D} \setminus \{0\},$$

and $f_j(z) = a_j(|z|)z^j$. Then the following are equivalent:
(i) T_f is bounded on H_v^∞.
(ii) T_{f_j} are bounded on H_v^∞ for all j.
(iii) T_{a_j} are bounded on H_v^∞ for $j = -n, \dots, 0$ and $T_{a_j} \circ (id - Q_{j-1})$ are bounded on H_v^∞ for $j = 1, \dots, n$.
(iv) The multipliers M_{g_j} are bounded on H_v^∞ for all j where

$$g_j(\varphi) = \sum_{k=\max(j,0)}^{\infty} \frac{1}{\Gamma_{2k}} \int_0^1 r^{2k+1} a_j(r)v(r)dr e^{ik\varphi}.$$

We prove Theorem 4.1 below. Notice that any $f \in L^1$ can be expanded as follows:

$$f(z) \sim \sum_{j=-\infty}^{\infty} a_j(|z|)z^j,$$

for some radial functions a_j. (Expand $f(re^{i\varphi})$ into a Fourier series for each fixed $r \in [0, 1[$.)

Example. Let $f(z) = 1/z$. Then $f \in L^1$ and T_f is bounded according to Theorem 4.1. since $f(z) = 1 \cdot z^{-1}$ and $T_1 = id$ is bounded but f is unbounded. This is no contradiction to Theorem 2.1 since f is not holomorphic in 0.

Let $h(z) = \sum_{k=0}^{\infty} b_k z^k$ for given constant coefficients b_k. Notice that we have

$$|b_k|r^k v(r) = \frac{1}{2\pi}\left| \int_0^{2\pi} e^{-ik\varphi} h(re^{i\varphi + i\psi}) d\psi \right| v(r) \le |h(re^{i\varphi})|v(r) \le \|h\|_v \tag{4.2}$$

for each r. For $j \in \mathbb{Z}$ we introduce the shift

$$S_j(h)(z) = \sum_{k=-\min(j,0)}^{\infty} b_k z^{k+j}.$$

For $\psi \in \mathbb{R}$ let R_ψ be the translation

$$R_\psi(h)(z) = h(e^{i\psi}z) = \sum_{k=0}^{\infty} b_k z^k e^{ik\psi}.$$

Lemma 4.2. *R_ψ and S_j are bounded operators on H_v^∞. Moreover, we have*

$$S_j S_{-j} = \begin{cases} \mathrm{id}_{H_v^\infty} , & \text{if } j \le 0 \\ \left(\mathrm{id}_{H_v^\infty} - Q_{j-1}\right) , & \text{if } j > 0. \end{cases}$$

Proof. The boundedness of R_ψ is a direct consequence of the definition. If $j \ge 0$ then $S_j(h)(z) = z^j h(z)$ and hence S_j is bounded.

Now let $j < 0$. Put $h(z) = \sum_{k=0}^{\infty} b_k z^k$. Then

$$S_j(h)(z) = \sum_{k=|j|}^{\infty} b_k z^{k-|j|} = z^j \big((\mathrm{id}_{H_v^\infty} - Q_{|j|-1})h \big)(z).$$

Hence, if $|z| > 1/2$ then $|S_j(h)(z)|v(|z|) \leq 2d_1(1 + \log|j|)\|h\|_v$ for some universal constant d_1. By the preceding and (4.2), where $r = 3/4$, we have for $|z| \leq 1/2$,

$$|S_j(h)(z)|v(|z|) \leq \sum_{k=|j|}^{\infty} |b_k| \frac{1}{2^{k-|j|}} v(0)$$

$$\leq \sum_{k=|j|}^{\infty} \frac{\|h\|_v}{v(3/4)} \left(\frac{4}{3}\right)^k \left(\frac{1}{2}\right)^{k-|j|} v(0) = 3\left(\frac{4}{3}\right)^{|j|} \frac{v(0)}{v(3/4)} \|h\|_v.$$

Thus S_j is bounded. The last identities of Lemma 4.2 follow from the definition. $\square$

Proof of Theorem 4.1 The implication $(ii) \Rightarrow (i)$ follows from the fact that $T_f = \sum_{j=-n}^{n} T_{f_j}$.

Using (3.23) we see that, with $h(z) = \sum_{k=0}^{\infty} b_k z^k$, we have

$$T_{f_j}(h)(z) = \sum_{k=-\min(j,0)}^{\infty} \frac{1}{\Gamma_{k+j}} \int_0^1 a_j(r) r^{k+j+1} v(r) dr\, b_k z^{k+j} = T_{a_j} S_j(h)(z) \tag{4.3}$$

so that $(iii) \Rightarrow (ii)$ follows from (4.3) and Lemma 4.2.

For $(i) \Rightarrow (iii)$ we note that (1.9) implies

$$T_f(R_\psi h)(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi\Gamma_{2n}} \int_0^{2\pi}\int_0^1 f(re^{i\varphi}) h(re^{i\varphi+i\psi}) r^{n+1} e^{-in\varphi} v(r) dr d\varphi z^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi\Gamma_{2n}} \int_0^{2\pi}\int_0^1 f(re^{i\varphi-i\psi}) h(re^{i\varphi}) r^{n+1} e^{-in+in\psi\varphi} v(r) dr d\varphi z^n$$

$$= T_{R_{-\psi}f}(h)(e^{i\psi}z) = \sum_{j=-n}^{n} e^{-ij\psi} T_{f_j}(h)(e^{i\psi}z).$$

This yields

$$T_{f_j}(h)(z) = \frac{1}{2\pi} \int_0^{2\pi} R_{-\psi} T_f(R_\psi h)(z) e^{ij\psi} d\psi$$

and hence

$$\|T_{f_j}(h)\|_v \leq \|T_f\| \cdot \|R_\psi h\|_v = \|T_f\| \cdot \|h\|_v.$$

Therefore T_{f_j} is bounded for all j. Now (4.3) and Lemma 4.2 imply

$$T_{f_j} \circ S_{-j} = \begin{cases} T_{a_j} & \text{if } \; j \leq 0 \\ T_{a_j} \circ (id - Q_{j-1}) & \text{if } \; j > 0. \end{cases}$$

Finally, $(iii) \Leftrightarrow (iv)$ follows from $T_{a_j} = M_{g_j}$, if $j \leq 0$, and $T_{a_j} \circ (id - Q_{j-1}) = M_{g_j}$, if $j > 0$. $\square$

Theorem 4.3. *Assume that v is normal. Let a_j be polynomials in r, hence $f_j \in L^1$, where $f_j(z) = a_j(|z|)z^j$, $j = -n, \ldots, n$. Put*

$$f(z) = \sum_{k=-n}^{n} a_j(|z|) z^j.$$

Then T_f is bounded on H_v^∞.

We prove Theorem 4.3 at the end of this section. We immediately get, in contrast to Theorem 2.3,

Corollary 4.4. *Let v be normal. Then, for any trigonometric polynomial f, the Toeplitz operator T_f is bounded on H_v^∞.*

Proof. Let $f_j(re^{i\varphi}) = \alpha_j r^{|j|} e^{ij\varphi}$ and $f = \sum_{j=-n}^{n} f_j$. Then all $f_j \in L^1$. Put $a_j(r) = \alpha_j$ if $j \geq 0$ and $a_j(r) = \alpha_j r^{2|j|}$ if $j < 0$. Then $f_j(z) = a_j(|z|) z^j$ for all j and the corollary follows from Theorem 4.3 □

To prove Theorem 4.3 we need the following

Lemma 4.5. *Let v be normal. Then there is a universal constant $c > 0$ such that, for any k, m with $0 < k \leq m \leq 2k$, we have*

$$\frac{\Gamma_{k-1}}{\Gamma_{m-1}} \leq c.$$

Proof. It follows from the definition of normal weight that there is a constant $d > 0$ with $\sup_{0 \leq r < 1} v(r^2)/v(r) < d$. With the substitution $s^{(m+1)/(k+1)} = r$ we see that

$$\Gamma_{k-1} = \int_0^1 r^k v(r) dr = \frac{m+1}{k+1} \int_0^1 s^m v(s^{(m+1)/(k+1)}) ds \leq 2d \int_0^1 s^m v(s) ds = 2d\Gamma_{m-1}.$$

Here we used $s^{(m+1)/(s+1)} \geq s^2$ and hence $v(s^{(m+1)/(k+1)}) \leq v(s^2) \leq dv(s)$. Hence the lemma follows with $c = 2d$. □

Proposition 4.6. *Assume that v is normal. Let $a \in L^1$ be radial such there is a constant $d > 0$ with*

$$r^k |a(r)| \leq \frac{d}{k} \quad \text{for all } k \in \mathbb{N}. \tag{4.4}$$

or

$$r^k |a(r) - 1| \leq \frac{d}{k} \quad \text{for all } k \in \mathbb{N}. \tag{4.5}$$

Then T_a is bounded on H_v^∞.

Proof. Since v is normal it satisfies condition (B). Let m_n be the indices of Theorem 3.2 and let γ_n and f_a be as in (3.24). We have to study the boundedness of the multiplier $M_{f_a} = T_a$. At first assume that a satisfies (4.4). We obtain

$$|\gamma_k| \leq \frac{1}{\Gamma_{2k}} \int_0^1 r^{2k+1} v(r) |a(r)| dr \leq \frac{d}{k\Gamma_{2k}} \int_0^1 r^{2k+1} v(r) \frac{1}{r^k} dr = \frac{d}{k} \frac{\Gamma_k}{2\pi \Gamma_{2k}}.$$

Let $D > 0$ be the supremum in (3.4). We have

$$\int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi$$

$$\leq d \Big(\sum_{m_{n-1} < k \leq m_n} \frac{k - [m_{n-1}]}{[m_n] - [m_{n-1}]} \frac{\Gamma_k}{k\Gamma_{2k}} + \sum_{m_n < k \leq m_{n+1}} \frac{[m_{n+1}] - k}{[m_{n+1}] - [m_n]} \frac{\Gamma_k}{k\Gamma_{2k}} \Big)$$

$$\le Dd \sum_{m_{n-1}<k\le m_{n+1}} \frac{\Gamma_k}{k\Gamma_{2k}} \le cDd\frac{m_{n+1}-m_{n-1}}{m_{n-1}}$$

where c is the constant of Lemma 4.5. We can apply (3.4) again to conclude

$$\sup_n \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi < \infty.$$

According to Theorem 3.6 T_a is bounded.

If (4.5) holds then $\tilde{a} = a - 1$ satisfies (4.4). Hence $T_{\tilde{a}}$ is bounded. But $T_a = T_{\tilde{a}} + T_1$ and $T_1 = id$ which implies T_a is bounded. □

The idea of the proof of the last statement can clearly be generalized: if $b \in L^1$ is a symbol such that T_b is bounded in H_v^∞ and $\tilde{a} \in L^1$ is another symbol such that $a := b - \tilde{a}$ is a radial function satisfying (4.4), then $T_{\tilde{a}}$ is bounded in H_v^∞.

Proof of Theorem 4.3. In view of Theorem 4.2. it suffices to show that T_a is bounded when $a(r) = r^\ell$ for some $\ell > 0$. But this follows from Proposition 4.6 since a satisfies (4.5). Indeed, fix k and consider the polynomial $g(r) = r^k - r^{k+\ell}$, $0 \le r \le 1$. Clearly, g attains its supremum at $(k/(k+\ell))^{1/\ell}$ and we have

$$0 \le g(r) \le \left(\frac{k}{k+\ell}\right)^{k/\ell} \frac{\ell}{k+\ell} \le \frac{\ell}{k+\ell} \le \frac{\ell}{k} \text{ for all } r. \qquad \square$$

We finally remark that condition (4.4) holds for symbols $a(r) = (1-r)^\alpha$, if and only if $\alpha \ge 1$. This is not a precise condition for the boundedness of T_a, since for the normal weights $v(r) = (1-r)^\delta$, $0 < \delta < 1$, any symbol a with $|a(z)| \le C(1-r)^\delta$, produces a bounded Toeplitz operator T_a in H_v^∞. This so since the pointwise multiplier $S_a : h \mapsto a \cdot h$ maps H_v^∞ into the space L^∞, and the Bergman projection only causes at most logarithmic singularity on the boundary of the disc, i.e. it maps L^∞ into H_w^∞ with the weight $w(r) = 1/(|\log(1-r)|+1)$, and this space is of course continuously embedded into H_v^∞.

A more careful study of these growth estimates is postponed to a planned future work.

5. Remarks on operators on reflexive Bergman spaces.

For radial symbols, the boundedness of T_a as an operator from the Bergman-Hilbert space A_v^2 into itself is characterized by the condition

$$\sup_{n\in\mathbb{N}} |\gamma_n| < \infty, \tag{5.1}$$

where the numbers γ_n are as in (3.24). However, the conditions (3.7) and (5.1) seem not to "interpolate" easily in a way, which would characterize the boundedness and compactness of $T_a : A_v^p \to A_v^p$ for $2 < p < \infty$ (or $1 < p < 2$). Nevertheless we will still show that a condition analogous to (3.7) is sufficient for the boundedness of T_a in A_v^p. Let us remark that in [16] the authors used somewhat similar methods to show the connection of the boundedness problem for $T_a : A_v^p \to A_v^p$ to the boundedness problem for multipliers in Hardy spaces.

We need to introduce some more notation and definitions: for details of these, see [16]. For a holomorphic $g(z) = \sum_{k=0}^{\infty} g_k z^k$ and $0 < r < 1$ we define

$$M_p(g,r) = \Big(\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\varphi})|^p d\varphi \Big)^{1/p}$$

and recall the notation $Q_n g(z) = \sum_{k=0}^{n} g_k z^k$, see (4.1). It is well-known that, for $1 < p < \infty$, there are universal constants $c_p > 0$ with $M_p(Q_n g, r) \le c_p M_p(g,r)$ where c_p does not depend on g, n or r. Moreover, we fix a number $\beta > 16 \cdot 3^{p-1}(1+2^p)c_p^p + 2$ and use induction to obtain the increasing numerical sequences $0 = \ell_1 < \ell_2 < \ell_3 \ldots$ and $0 \le s_1 < s_2 \ldots < R$ such that

$$\int_0^{s_n} r^{\ell_n p} d\mu = \beta \int_{s_n}^{R} r^{\ell_n p} d\mu \quad \text{and} \quad \int_0^{s_n} r^{\ell_{n+1} p} d\mu = \frac{1}{\beta} \int_{s_n}^{R} r^{\ell_{n+1} p} d\mu. \tag{5.2}$$

(These numbers were calculated in some examples in the paper [3].) We define for all $n \in \mathbb{N}$

$$Z_n f = (Q_{[\ell_{n+1}]} - Q_{[\ell_n]}) f,$$

and

$$\omega_n = \Big(\int_0^{s_n} \Big(\frac{r}{s_n} \Big)^{\ell_n p} d\mu + \int_{s_n}^{R} \Big(\frac{r}{s_n} \Big)^{\ell_{n+1} p} d\mu \Big)^{1/p}.$$

We get for the norm of A_v^p a representation analogous to (3.2): there are constants $d_1, d_2 > 0$ such that, for every $f \in A_\mu^p$,

$$d_1 \|f\|_{p,v} \le \Big(\sum_{n=1}^{\infty} \omega_n^p M_p^p (Z_n f, s_n) \Big)^{1/p} \le d_2 \|f\|_{p,v}. \tag{5.3}$$

This was shown in [9] for $p = 1$ and in [15] for $1 < p < \infty$ and $R = 1$,

Proposition 5.1. *Let the weight satisfy (B), let $a \in L^1$ be a radial function and let $f_a(\varphi) = \sum_{k=0}^{\infty} \gamma_k e^{ik\varphi}$ be as in (3.24). Then the Toeplitz operator T_a is a well-defined, bounded operator from A_v^p into itself, if*

$$\sup_{n \in \mathbb{N}} \int_0^{2\pi} |(Z_n f_a)(\varphi)| d\varphi < \infty. \tag{5.4}$$

Moreover, $T_a : A_v^p \to A_v^p$ is compact, if

$$\int_0^{2\pi} |(Z_n f_a)(\varphi)| d\varphi \to 0 \quad \text{as } n \to \infty. \tag{5.5}$$

Proof. Let us denote by M_f the convolution operator, or the sequence space multiplier, corresponding to T_a, see (3.24). So, if $h \in A_v^p$ then for all $re^{i\varphi} \in \mathbb{D}$ we get by the usual orthogonality relations of functions $e^{ik\varphi}$,

$$(Z_n M_f h)(re^{i\varphi}) = (M_{Z_n f} h)(re^{i\varphi}) = \int_0^{2\pi} Z_n f(\varphi - \psi) h(re^{id\psi}) d\psi$$

$$= \int_0^{2\pi} Z_n f(\varphi - \psi) Z_n h(re^{id\psi}) d\psi.$$

We apply the Young inequality

$$\|a * b\|_{L^p(\partial\mathbb{D})} \le \|a\|_{L^1(\partial\mathbb{D})} \|b\|_{L^p(\partial\mathbb{D})}$$

to get

$$M_p(Z_n M_f h, r) \le \int_0^{2\pi} |(Z_n f)(\varphi)| d\varphi \, M_p(Z_n h, r) \tag{5.6}$$

The inequality $\|M_f h\|_{p,v} \le C\|h\|_{p,v}$ thus follows by applying (5.4) and (5.3) to both $\|M_f h\|_{p,v}$ and $\|h\|_{p,v}$. This shows that (5.4) is sufficient for T_a to map A_v^p continuously into itself.

If (5.5) holds, the proof for the compactness of T_a is similar to the corresponding proof in Theorem 3.4. We again let $(h_j)_{j=1}^\infty$ be a sequence which is contained in the unit ball of A_v^p and which converges to 0 uniformly on compact subsets of $\mathbb{D}$, and assume $\varepsilon > 0$ is given. We choose $N \in \mathbb{N}$ such that $\int_0^{2\pi} |(Z_n f)(\varphi)| d\varphi < \varepsilon$. Then, we use the convergence of our sequence in the compact-open topology and the argument in the proof of Theorem 3.4 to find a large enough $J \in \mathbb{N}$ such that

$$\sup_{|z| \le r_{m_n}} |Z_n M_f h_j(z)| v(z) < \frac{\varepsilon}{2\pi N \omega_n} \quad \Rightarrow \quad M_p(Z_n M_f h_j, r_{m_n}) < \frac{\varepsilon}{N \omega_n}$$

for all $n \le N$, all $j \ge J$. In view of (5.6) and (5.3) this implies

$$\begin{aligned} \|M_f h_j\|_{p,v}^p &\le \sum_{n=1}^N \omega_n^p M_p(Z_n M_f h_j, r_{m_n})^p + \sum_{n=N+1}^\infty \omega_n^p M_p(Z_n M_f h_j, r_{m_n})^p \\ &\le \varepsilon + \varepsilon \sum_{n=N+1}^\infty \omega_n^p M_p(Z_n h_j, r_{m_n})^p \le 2\varepsilon \|h_j\|_{p,v}^p \le 2\varepsilon. \quad \square \end{aligned}$$

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Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, E-46071 Valencia, Spain

E-mail address: jbonet@mat.upv.es

Institute of Mathematics, University of Paderborn, Warburger Str. 100, D-33098 Paderborn, Germany

E-mail address: lusky@uni-paderborn.de

Department of Mathematics and Statistics, P.O. Box 68, University of Helsinki, 00014 Helsinki, Finland

E-mail address: jari.taskinen@helsinki.fi