A study of the local convergence of a fifth order iterative method

Sukjith Singh[a], Eulalia Martínez[b], P. Maroju[c], R.Behl[d]

[a]*Dr. B.R. Ambedkar Nationl Instutute of Technology Jalandhar*
[b]*Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Spain*
[c]*Amrita School of Engineering, Department of Mathematics, Amrita Vishwa Vidhyapeetham, Coimbatore, India*
[d]*Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Abstract

We present a local convergence study of a fifth order iterative method to approximate a locally unique root of nonlinear systems. The analysis is discussed under the assumption that first order Fréchet derivative satisfies the Lipschitz continuity condition. Moreover, we consider the derivative free method obtained by approximating the derivative with divided difference along with the local convergence study. Finally, we provide computable radii and error bounds based on the Lipschitz constant for both cases analyzed in the theoretical results. Some of numerical examples are worked out and compared these results with existing methods results.

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1. Introduction

In this paper, we concerned with the problem of approximating a locally unique solution α of the nonlinear system

$$F(x) = 0. \tag{1}$$

Solving nonlinear system of equations play an important role in many branches of nonlinear Functional analysis, Numerical Analysis, Chemical engineering, Kinetic theory of gases [1, 2, 3, 4, 5, 6], etc. Many nonlinear problems arise from discretization of nonlinear integral equations and nonlinear differential equations by method of finite difference. In the literature, we can find several real world problems described by nonlinear models which can be transformed into system of nonlinear equations. Such non-linear model are like variational inequalities, Bratu's problem, a shallow arch, etc. find in the paper [7]. However, most of the equations are phrased in terms of system of nonlinear equations of form (1).

The applicability of the nonlinear system on the problem of investigating coarse-grained dynamical properties of neuronal networks in kinetic theory discussed in [8]. In addition, Nejat and Ollivier [9] presented the problem to study the effect of discretization order on preconditioning and convergence of high-order Newton-Krylov unstructured flow solver in computational fluid dynamics. In [10], Grosan and Abraham showed the applicability of the system of nonlinear equations in neurophysiology, kinematics syntheses problem, chemical equilibrium problem, combustion problem and economics modeling

Email addresses: sukhjitmath@gmail.com (Sukjith Singh), eumarti@mat.up.es (Eulalia Martínez), maroju.prashanth@gmail.com (P. Maroju), ramanbehl87@yahoo.in (R.Behl)

problem. Very recently, Awawdeh and Tsoulos et al. [11, 12] solved the reactor and steering problems by phrasing them like systems of nonlinear equations.

There are many standard methods that can be used to approximate the solution of this kind of nonlinear equations. Main reason to develop iterative methods is that analytical solution often is not available for most type of nonlinear equations. These iterative methods divided into one point and multipoint methods, with memory and without memory methods are being studied nowadays. In fact, many higher order multipoint iterative methods for solving nonlinear equations are developed and published in various journals of applied and computer mathematics. We have well known quadratically convergent Newton's method for solving nonlinear equation (1), most of new methods use this one for first step to increase the order of convergence. Many authors have developed robust and efficient iterative methods with higher convergence order but it is very important to discuss the local and semilocal convergence analysis for them.

The study about local convergence of higher order iterative methods can be analyzed under different continuity conditions in Banach spaces (see, [13, 14]). Argyros and George [15] developed the local convergence analysis of third order Halley-like methods under Lipchitz continuity conditions and it is given for $k = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_k &= x_k - F^{'}(x_k)^{-1}F(x_k) \\ u_k &= y_k + (1-a)F^{'}(x_k)^{-1}F(x_k) \\ z_k &= y_k - \gamma A_{a,k}F^{'}(x_k)^{-1}F(x_k) \\ x_{k+1} &= z_k - \eta B_{a,k}F^{'}(x_k)^{-1}F(z_k), \end{aligned} \tag{2}$$

where, $\eta, \gamma, a \in (-\infty, \infty) - \{0\}$, $H_{a,k} = \frac{1}{a}F^{'}(x_k)^{-1}\Big(F^{'}(u_k) - F^{'}(x_k)\Big)$, $A_{a,k} = I - \frac{1}{2}H_{a,k}\Big(I - \frac{1}{2}H_{a,k}\Big)$, $B_{a,k} = I - H_{1,k} + H_{a,k}^2$. The local convergence of Chebyshev-Halley-type method discussed in [17] and it is given for $k = 0, 1, 2 \ldots$ by

$$\begin{aligned} y_k &= x_k - F^{'}(x_k)^{-1}F(x_k) \\ z_k &= x_k - \Big(1 + (F(x_k) - 2\eta F(y_k))^{-1}F(y_k)\Big)F^{'}(x_k)^{-1}F(x_k) \\ x_{k+1} &= z_k - \Big(F^{'}(x_k) + \bar{F}''(x_k)(z_k - x_k)\Big)^{-1}F(z_k), \quad k \geq 0, \end{aligned} \tag{3}$$

where, $\bar{F}''(x_k) = 2F(y_k)F^{'}(x_k)^2F(x_k)^{-2}$ and η is a parameter. The order of this family is at least five for any value of η and for $\eta = 1$, it is six.

In this paper, we analyze the local convergence of fifth order iterative method which is proposed in [16] under the condition that first order Fréchet derivative satisfying the Lipschitz continuity condition. The existence and uniqueness region of the solution is established. Numerical examples worked out and convergence balls for each of them are obtained. We compare these results with the convergence balls of existing methods (2) and (3). Also, we discuss the local convergence of derivative free iterative method obtained by approximating the derivative by divided differences. Some numerical examples worked out and the convergence regions computed.

This paper is divided into four sections and organized as follows. In Section 1, we form the introduction. The local convergence study is performed in Section 2. The existence and uniqueness region of the solution is derived along with some numerical examples. In Section 3, the local convergence of the derivative free iterative method is discussed and also the computation of existence and uniqueness region of solution with numerical examples. Finally, the conclusion forms Section 4.

2. Local convergence analysis

In this section, we consider a fifth order iterative method proposed in [16] and its local convergence analysis under Lipschitz conditions on $F^{'}$. It is given for $k = 0, 1, 2, \dots$ by

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k) \\ z_k &= x_k - 2(F'(x_k) + F'(y_k))^{-1}F(x_k) \\ x_{k+1} &= z_k - F'(y_k)^{-1}F(z_k), \end{aligned} \tag{4}$$

where x_0 is the starting point. In [16], Cordero et al. presented the fifth order of convergence using Taylor series on higher order Fréchet derivative without obtaining the convergence balls. They also assumed that the starting point x_0 is sufficiently close to the solution without estimating this closeness. Now, we have addressed these problems using only first order Fréchet derivative.

Suppose that $B(v, \rho)$ and $\bar{B}(v, \rho)$ denote the open and closed balls, respectively with center v and radius ρ. Let $F : D \subseteq X \to Y$ be a Fréchet differentiable operator defined on open domain D such that for $\alpha \in D$, $L_0 > 0$, $L > 0$ and for all $x, y \in D$, we have

$$F(\alpha) = 0, \quad F'(\alpha)^{-1} \in \mathbf{L}(Y, X), F'(\alpha)^{-1} \neq 0 \tag{5}$$

$$\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| \leq L_0\|x - \alpha\|, \ \ \forall\, x \in D \tag{6}$$

$$\|F'(\alpha)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|, \ \ \forall\, x, y \in B\left(\alpha, \frac{1}{L_0}\right) \subseteq D. \tag{7}$$

Lemma 1. *If the nonlinear operator F satisfies the above assumptions, then for all $x \in B(\alpha, \frac{1}{L_0})$, we have*

$$\begin{aligned} \|F'(\alpha)^{-1}F'(x)\| &\leq 1 + L_0\|x - \alpha\|, \\ \|F'(\alpha)^{-1}F'(\alpha + t(x - \alpha))\| &\leq 1 + L_0\|x - \alpha\| \quad \forall\, t \in]0, 1[, \\ \|F'(\alpha)^{-1}F(x)\| &\leq (1 + L_0\|x - \alpha\|)\|x - \alpha\|. \end{aligned}$$

Proof. The proof is trivial and can be seen in [13].
The following result describes the local convergence theorem for the iterative method (4)

Theorem 1. *Let F be a nonlinear operator satisfying assumptions* (5), (6) *and* (7). *Then, the sequence $\{x_{k+1}\}$ generated by* (4) *is well defined for $x_0 \in B(\alpha, r_3)$ and converges to α, where, r_3 is the smallest positive root of s_3. Also, we obtain the following inequalities for $k \geq 0$:*

$$\|y_k - \alpha\| \leq g_1(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < r_3, \tag{8}$$

$$\|z_k - \alpha\| \leq g_2(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < r_3, \tag{9}$$

$$\|x_{k+1} - \alpha\| \leq g_3(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < r_3, \tag{10}$$

where, s_3, g_1, g_2 and g_3 are auxiliary functions defined in the proof. If there exists a $R \in [r_3, \frac{2}{L_0})$ such that $\overline{B}(\alpha, R) \subseteq D$, then α is the unique solution in $\overline{B}(\alpha, R)$.

Proof. Since $x_0 \in D$ and using (6), we get

$$\|I - F^{'}(\alpha)^{-1}F^{'}(x_0)\| = \|F^{'}(\alpha)^{-1}(F^{'}(x_0) - F^{'}(\alpha))\| \ \leq \ L_0\|x_0 - \alpha\| < 1$$

for $\|x_0-\alpha\| < \frac{1}{L_0}$. Therefore, by Banach Lemma, $F^{'}(x_0)^{-1}$ exists and

$$\|F^{'}(x_0)^{-1}F^{'}(\alpha)\| \leq \frac{1}{1-L_0\|x_0-\alpha\|}. \tag{11}$$

Thus, y_0 is well defined. From the first equation of (4) for $k=0$, we get

$$\begin{aligned} y_0-\alpha &= x_0-\alpha-F^{'}(x_0)^{-1}F(x_0) \\ &= -F^{'}(x_0)^{-1}\left(F(x_0)-F^{'}(x_0)(x_0-\alpha)\right) \\ &= -F^{'}(x_0)^{-1}F^{'}(\alpha)\int_0^1 F^{'}(\alpha)^{-1}[F^{'}(\alpha+t(x_0-\alpha))-F^{'}(x_0)](x_0-\alpha)dt \end{aligned}$$

By using adequately Banach Lemma, the assumptions and denoting $e_0=\|x_0-\alpha\|$, we have

$$\|y_0-\alpha\| \leq \frac{L\|x_0-\alpha\|}{2\,(1-L_0\|x_0-\alpha\|)}\|x_0-\alpha\| \leq g_1(e_0)e_0, \tag{12}$$

where,

$$g_1(t)=\frac{Lt}{2(1-L_0t)}.$$

Obviously g_1 is an increasing function, and by taking $r_1=\frac{2}{L+2L_0}$ it follows:

$$0 \leq g_1(t) < 1,\ \forall\, t\in[0,r_1). \tag{13}$$

Using (12) and (13), we get

$$\|y_0-\alpha\| \leq g_1(\|x_0-\alpha\|)\|x_0-\alpha\| < \|x_0-\alpha\|.$$

Again using (4) for $k=0$, we get

$$\begin{aligned} z_0-\alpha &= x_0-\alpha-2\big(F'(y_0)+F'(x_0)\big)^{-1}F(x_0) \\ &= (F'(y_0)+F'(x_0))^{-1}\Big[(F'(y_0)+F'(x_0))(x_0-\alpha)-2F(x_0)\Big] \\ &= -(F'(y_0)+F'(x_0))^{-1}\big[F(x_0)-F'(x_0)(x_0-\alpha)+F(x_0)-F'(y_0)(x_0-\alpha)\big] \\ &= -(F'(y_0)+F'(x_0))^{-1}F'(\alpha)\Big[2F'(\alpha)^{-1}\int_0^1[F'(\alpha+t(x_0-\alpha))-F'(x_0)](x_0-\alpha)dt \\ &+ F'(\alpha)^{-1}(F'(x_0)-F'(y_0))(x_0-\alpha)\Big]. \end{aligned}$$

To follow with the study of existence and bound for the product $(F'(y_0)+F'(x_0))^{-1}F'(\alpha)$ we observe that

$$-\frac{1}{2}F'(\alpha)^{-1}[F'(x_0)+F'(y_0)-2F'(\alpha)] = I-\underbrace{\frac{1}{2}F'(\alpha)^{-1}[F'(x_0)+F'(y_0)]}_{A} = I-A,$$

so we try to apply Banach Lemma

$$\begin{aligned}\|I-A\| &= \left\|\frac{1}{2}F'(\alpha)^{-1}\left[F'(x_0)+F'(y_0)-2F'(\alpha)\right]\right\| \\ &= \left\|\frac{1}{2}\left[F'(\alpha)^{-1}(F'(x_0)-F'(\alpha))+F'(\alpha)^{-1}(F'(y_0)-F'(\alpha))\right]\right\| \\ &\le \frac{1}{2}\left(L_0\|x_0-\alpha\|+L_0\|y_0-\alpha\|\right) \\ &\le \frac{1}{2}\left(L_0e_0+L_0g_1(e_0)e_0\right)=p_1(e_0)\end{aligned}$$

where $p_1(t)=\frac{1}{2}L_0(1+g_1(t))t$ is an increasing function such that $p_1(0)=0$ and $p_1(r_1)=\frac{1}{2}L_0r_1\left(1+g_1(r_1)\right)=L_0r_1<1$ and so one has:

$$\|2(F'(x_0)+F'(y_0))^{-1}F'(\alpha)^{-1}\| \le \frac{1}{1-p_1(e_0)}.$$

Then, turning to the expression of $z_0-\alpha$ we have

$$\begin{aligned}\|z_0-\alpha\| &\le \frac{1}{2(1-p(\|x_0-\alpha\|))}\left[2L\int_0^1\|\alpha+t(x_0-\alpha)-x_0\|\|x_0-\alpha\|dt+L\|x_0-y_0\|\|x_0-\alpha\|\right] \\ &\le \frac{1}{2(1-p(\|x_0-\alpha\|))}\left[2L\frac{\|x_0-\alpha\|^2}{2}+L(\|x_0-\alpha\|+\|y_0-\alpha\|)\|x_0-\alpha\|\right] \\ &\le \frac{1}{2(1-p(e_0))}\left[Le_0+L(e_0+g_1(e_0)e_0)\right]e_0 \\ &\le \frac{Le_0(2+g_1(e_0))}{2(1-p(e_0))}e_0=g_2(e_0)e_0 \end{aligned} \tag{14}$$

where

$$g_2(t)=\frac{Lt(2+g_1(t))}{2(1-p_1(t))}.$$

So, we consider $s_2(t)=g_2(t)-1$ having that $s_2(0)=-1$ and $s_2(r_1)>0$. Therefore, $s_2(t)$ has at least one root in $(0,r_1)$ and let r_2 be the smallest one. Therefore, $0<r_2<r_1$ and

$$0\le g_2(t)\le 1\;\forall\, t\in[0,r_2), \tag{15}$$

then by using (14) and (15), we get

$$\|z_0-\alpha\| \quad \le g_2(\|x_0-\alpha\|)\|x_0-\alpha\| < \|x_0-\alpha\|.$$

Again using (4) for $k=0$, we get

$$x_1-\alpha=z_0-\alpha-F'(y_0)^{-1}F'(\alpha)F'(\alpha)^{-1}F(z_0)$$

Since $y_0\in D$ and using (6), we get

$$\begin{aligned}\|I-F'(\alpha)^{-1}F'(y_0)\| &\le \|F'(\alpha)^{-1}(F'(\alpha)-F'(y_0))\| \\ &\le L_0\|y_0-\alpha\| \le L_0g_1(e_0)e_0=p_2(e_0)<1\end{aligned}$$

where

$$p_2(t) = L_0 g_1(t)t.$$

Then $\exists\ (F'(\alpha)^{-1}F'(y_0))^{-1}$ and

$$\|F'(y_0)^{-1}F'(\alpha)\| \leq \frac{1}{1-p_2(e_0)}.$$

Therefore, by using Lemma 1, we get

$$\begin{aligned}\|x_1-\alpha\| &\leq \|z_0-\alpha\| + \frac{1}{1-p_2(e_0)}(1+L_0\|z_0-\alpha\|)\|z_0-\alpha\| \\ &= \left(1+\frac{1+L_0 g_2(e_0)e_0}{1-p_2(e_0)}\right) g_2(e_0)e_0 = g_3(e_0)e_0, \qquad (16)\end{aligned}$$

where

$$g_3(t) = \left(1+\frac{1+L_0 g_2(t)t}{1-p_2(t)}\right) g_2(t).$$

Consider $s_3(t) = g_3(t) - 1$. Then, $s_3(0) = -1$ and $s_3(r_2) > 0$. Therefore, $s_3(t)$ has at least one root in $(0, r_2)$ and let r_3 the smallest one. Therefore, $0 < r_3 < r_2$

$$0 \leq g_3(t) \leq 1\ \forall\ t \in [0, r_3), \qquad (17)$$

then by using (16) and (17), we get

$$\|x_1-\alpha\| \leq g_3(\|x_0-\alpha\|)\|x_0-\alpha\| < \|x_0-\alpha\| < \eta.$$

Thus, Theorem 1 holds for $k = 0$. Changing x_0, y_0, z_0 and x_1 by x_k, y_k, z_k, x_{k+1}, we get the inequalities (8)-(10) for all $k \geq 0$. Since, $\|x_{k+1}-\alpha\| \leq \|x_k-\alpha\| < r_3$, this gives $x_{k+1} \in B(\alpha, r_3)$. Also $g_3(t)$ is an increasing function in $[0, r_3)$, since $g_3^{'}(t) > 0$ for all $t \in [0, r_3)$. Thus, we get

$$\begin{aligned}\|x_{k+1}-\alpha\| &\leq g_3(e_0)\|x_k-\alpha\| \leq g_3(e_0)g_3(e_0)\|x_{k-1}-\alpha\| \\ &\leq g_3(e_0)^2 g_3(e_0)\|x_{k-2}-\alpha\| \leq \ldots \leq g_3(e_0)^{k+1}\|x_0-\alpha\|.\end{aligned}$$

Therefore, $\lim\limits_{k\to\infty} x_k = \alpha$ as $g_3(t) < 1$.

For getting the uniqueness ball for root α, let $\beta \in B(\alpha, R)$ be such that $F(\beta) = 0$ and $\beta \neq \alpha$. Consider $P = \int_0^1 F^{'}(\beta + t(\alpha-\beta))dt$. Using (6), we get

$$\|F^{'}(\alpha)^{-1}(P - F^{'}(\alpha))\| \leq \int_0^1 L_0\|\beta + t(\alpha-\beta) - \alpha\|dt \leq \frac{L_0}{2}\|\alpha-\beta\| = \frac{L_0}{2}R < 1,$$

therefore, by Banach Lemma, P^{-1} exists. Then,

$$0 = F(\alpha) - F(\beta) = P(\alpha-\beta),$$

we obtain $\alpha = \beta$. ■

2.1. *Numerical examples*

In this subsection, we consider numerical examples to demonstrate the applicability of our work. Moreover, we compare our results with the local convergence of a modified Halley-Like method (2) and Chebyshev-Halley-type methods (3) respectively.

Example 1. *Let* $X = Y = \mathbb{R}^3$, $D = \overline{U}(0, 1)$. *Define* F *on* D *for* $v = (x, y, z)$ *by*

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right).$$

Clearly, $\alpha = (0, 0, 0)$, $F'(\alpha) = F'(\alpha)^{-1} = diag\{1, 1, 1\}$, $L_0 = e - 1$, and $L = e$. Then, we have

$$r_3 = 0.13125 < r_2 = 0.21657 < r_1 = 0.32495.$$

Example 2. *Consider the system of nonlinear equations*

$$2x_1 - \frac{1}{9}x_1^2 - x_2 = 0,$$
$$-x_1 + 2x_2 - \frac{1}{9}x_2^2 = 0$$

The associated nonlinear operator $F : R^2 \to R^2$ *is given by*

$$F(x_1, x_2) = \begin{pmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{pmatrix}$$

where $F_1(x_1, x_2) = 2x_1 - \frac{1}{9}x_1^2 - x_2$ *and* $F_2(x_1, x_2) = -x_1 + 2x_2 - \frac{1}{9}x_2^2$.

Clearly $\alpha = (9, 9)^T$ is a solution of above nonlinear system and for all $(x, y) \in R^2$ we have:

$$\|F'(\alpha)^{-1}(F'(x) - F'(y))\| = \frac{2}{9}\|x - y\|$$
$$\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| = \frac{2}{9}\|x - \alpha\|.$$

Taking $L_0 = \frac{2}{9}$ and $L = \frac{2}{9}$, we get

$$r_3 = 1.44284 < r_2 = 2.25000 < r_1 = 3.00000.$$

Example 3. *Consider the nonlinear Hammerstein type integral equation given by*

$$F(x(s)) = x(s) - 5\int_0^1 s\,t\,x(t)^3 dt, \tag{18}$$

with $x(s)$ *in* $\mathcal{C}[0, 1]$.

Clearly $\alpha = 0$. Taking $L_0 = 7.5$ and $L = 15$, we get

$$r_3 = 0.02481 < r_2 = 0.04185 < r_1 = 0.06667.$$

Now, we compare our results with the local convergence of a modified Halley-Like method (2) and Chebyshev-Halley-type methods (3) respectively. The value of parameters used by these methods are listed in Table 1. The radius of a convergence ball of a fifth order method (4) is compared with method (2) and method (3) in Table 2. We can observe that the larger radius of convergence ball is obtained by our approach.

Table 1: Values of parameters

Examples	a	γ	η
1	1.0125	0.3	0.03
2	1	$1/9$	$2/9$
3	1	0.575	0.003

Table 2: Comparison of radius of a ball

Examples	Method (4)	Method (2)	Method (3)
1	0.13125	0.02726	0.00892
2	1.44284	0.55264	0.06989
3	0.02481	0.00709	0.00755

3. The derivative free method and its local convergence analysis

In this section our purpose is to complete the study of iterative method (4), when we use adequate approximation of the derivatives by divided differences. So, now the aim is to obtain the local convergence study in this case.

In order to obtain derivative free iterative methods we approximate derivatives by divided differences. That is an operator $[x, y; F]$ verifying

$$[x, y; F](x - y) = F(x) - F(y), forall x, y \in D$$

and if F is Fréchet differentiable at $a \in D$ then $[\alpha, \alpha; F] = F'(\alpha)$. One can see different approximations of divided differences in [18, 19].

We consider the derivative free iterative method given for $k = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_k &= x_k - [x_k, x_k + F(x_k); F]^{-1} F(x_k) \\ z_k &= x_k - 2([x_k, x_k + F(x_k); F] + [y_k, y_k + F(y_k); F])^{-1} F(x_k) \\ x_{k+1} &= z_k - [y_k, y_k + F(y_k); F]^{-1} F(z_k), \end{aligned} \tag{19}$$

where x_0 is the starting point.

We use the following assumptions for setting the local convergence study in this case. Let $K_0 > 0$, $K > 0$ and for all $x, y, u, v \in D$, we have $F(\alpha) = 0$, $F'(\alpha)^{-1} \neq 0$, in D, moreover

$$\|F'(\alpha)^{-1}([x, y; F] - [u, v; F])\| \leq K(\|x - u\| + \|y - v\|), \tag{20}$$

$$\|F'(\alpha)^{-1}([x, y; F] - [\alpha, \alpha; F])\| \leq K_0(\|x - \alpha\| + \|y - \alpha\|), \tag{21}$$

$$\|F(x) - F(\alpha)\| \leq L\|x - \alpha\|. \tag{22}$$

The next result describes the local convergence theorem for the derivative free iterative method (19)

Theorem 2. *Let F the nonlinear operator satisfying assumptions* (20), (21) *and* (22). *Then, the sequence $\{x_{k+1}\}$ generated by* (19) *is well defined for any starting point $x_0 \in B(\alpha, \rho_3)$ and converges to α, where ρ_3 is the smallest positive root of function q_3. Also, we obtain the following inequalities for $k \geq 0$:*

$$\begin{aligned}
\|y_k-\alpha\| &\leq h_1(\|x_k-\alpha\|)\|x_k-\alpha\| < \|x_k-\alpha\| < \rho_3, && (23)\\
\|z_k-\alpha\| &\leq h_2(\|x_k-\alpha\|)\|x_k-\alpha\| < \|x_k-\alpha\| < \rho_3, && (24)\\
\|x_{k+1}-\alpha\| &\leq h_3(\|x_k-\alpha\|)\|x_k-\alpha\| < \|x_k-\alpha\| < \rho_3, && (25)
\end{aligned}$$

where, h_1, h_2, h_3, q_3 are auxiliary functions defined in the proof and ρ_3 is the smallest root of $q_3(t)$. Moreover, if there exists a $R_1 \in [\rho_3, \frac{1}{K_0})$ such that $\overline{B}(\alpha, R_1) \subseteq D$, then α is the unique solution in $\overline{B}(\alpha, R_1)$.

Proof. Since $x_0 \in D$ and using (21), we get

$$\begin{aligned}
\|F^{'}(\alpha)^{-1}([x_0, x_0+F(x_0); F]-[\alpha,\alpha;F])\| &\leq K_0(\|x_0-\alpha\|+\|x_0+F(x_0)-\alpha\|\\
&\leq K_0(\|x_0-\alpha\|+(1+L)\|x_0-\alpha\|)\\
&= K_0(2+L)\|x_0-\alpha\| < 1,
\end{aligned}$$

for $\|x_0-\alpha\| < \frac{1}{K_0(2+L)}$. Therefore, by Banach Lemma, $[x_0, x_0+F(x_0); F]^{-1}$ exists and

$$\|[x_0, x_0+F(x_0); F]^{-1}F^{'}(\alpha)\| \leq \frac{1}{1-K_0(2+L)\|x_0-\alpha\|}. \tag{26}$$

Thus, y_0 is well defined. Using (19) for $k=0$, we get

$$\begin{aligned}
y_0-\alpha &= x_0-\alpha-[x_0, x_0+F(x_0); F]^{-1}F(x_0)\\
&= [x_0, x_0+F(x_0); F]^{-1}F^{'}(\alpha)F^{'}(\alpha)^{-1}\left([x_0, x_0+F(x_0); F]-[x_0,\alpha;F]\right)(x_0-\alpha).
\end{aligned}$$

Using (20), we have

$$\|F'(\alpha)^{-1}([x_0, x_0+F(x_0); F]-[x_0,\alpha;F])\| \leq K(\|x_0+F(x_0-\alpha\|) \leq K(1+L)\|x_0-\alpha\|.$$

Therefore,

$$\begin{aligned}
\|y_0-\alpha\| &\leq \Big(\frac{K(1+L)\|x_0-\alpha\|}{1-K_0(2+L)\|x_0-\alpha\|}\Big)\|x_0-\alpha\|\\
&= h_1(e_0)e_0, && (27)
\end{aligned}$$

where, $h_1(t) = \frac{K(1+L)t}{1-K_0(2+L)t}$ and $e_0 = \|x_0-\alpha\|$.
Consider the function $q_1(t) = h_1(t)-1$. Then, $q_1(0) = -1$ and $q_1(\frac{1}{K_0(2+L)}) \to +\infty$. Therefore, $q_1(t)$ has atleast one root in $\Big(0, \frac{1}{K_0(2+L)}\Big)$ and let ρ_1 be such a smallest root. Therefore, $0 < \rho_1 < \frac{1}{K_0(2+L)}$, and

$$0 \leq h_1(t) < 1 \,\forall\, t \in [0, \rho_1). \tag{28}$$

Using (27) and (28), we get

$$\|y_0-\alpha\| \leq h_1(\|x_0-\alpha\|)\|x_0-\alpha\| < \|x_0-\alpha\|.$$

Again using (19) for $k=0$, we get

$$\begin{aligned} z_0-\alpha &= x_0-\alpha-2\big([x_0,x_0+F(x_0);F]+[y_0,y_0+F(y_0);F]\big)^{-1}F(x_0)\\ &= \big([x_0,x_0+F(x_0);F]+[y_0,y_0+F(y_0);F]\big)^{-1}F'(\alpha)F'(\alpha)^{-1}\Big([x_0,x_0+F(x_0);F]\\ &+ [y_0,y_0+F(y_0);F]-2[x_0,\alpha;F]\Big)(x_0-\alpha). \end{aligned}$$

Now we study the existence and bound for the product $\big([x_0,x_0+F(x_0);F]+[y_0,y_0+F(y_0);F]\big)^{-1}F'(\alpha)$. Then we observe that

$$\begin{aligned} &\frac{1}{2}F'(\alpha)^{-1}\Big(\big([x_0,x_0+F(x_0);F]+[y_0,y_0+F(y_0);F]\big)-2F'(\alpha)\Big)\\ &= I-\underbrace{\frac{1}{2}F'(\alpha)^{-1}[\big([x_0,x_0+F(x_0);F]+[y_0,y_0+F(y_0);F]\big)]}_{C}=I-C. \end{aligned}$$

So we try to apply Banach Lemma

$$\begin{aligned} \|I-C\| &\le \frac{1}{2}\Big(\|F^{'}(\alpha)^{-1}([x_0,x_0+F(x_0);F]-F^{'}(\alpha))\|+\|F^{'}(\alpha)^{-1}([y_0,y_0+F(y_0);F]-F^{'}(\alpha))\|\Big)\\ &\le \frac{K_0}{2}\Big(\,(\|x_0-\alpha\|+\|x_0+F(x_0)-\alpha\|)+(\|y_0-\alpha\|+\|y_0+F(y_0)-\alpha\|)\,\Big)\\ &\le \frac{K_0}{2}\Big((2+L)\|x_0-\alpha\|+(2+L)\|y_0-\alpha\|\Big)\\ &\le \frac{K_0}{2}\Big((2+L)\|x_0-\alpha\|+(2+L)h_1(\|x_0-\alpha\|)\|x_0-\alpha\|\Big)\\ &= \phi_1(e_0), \end{aligned}$$

where $\phi_1(t)=\frac{K_0}{2}(2+L)(1+h_1(t))t$ is an increasing function such that $\phi_1(0)=0$ and $\phi_1(\rho_1)=\frac{K_0}{2}(2+L)\rho_1\,(1+h_1(\rho_1))=\frac{K_0}{2}(2+L)\rho_1<1$ and so we have

$$\|C^{-1}\|=\Big\|2\Big([x_0,x_0+F(x_0);F]+[y_0,y_0+F(y_0);F]\Big)^{-1}F'(\alpha)\Big\|\le\frac{1}{1-\phi_1(e_0)}.$$

Therefore,

$$\begin{aligned} \|z_0-\alpha\| &\le \tfrac{K}{2(1-\phi_1(\|x_0-\alpha\|))}\Big(\|x_0+F(x_0)-\alpha\|+\|y_0-x_0\|+\|y_0+F(y_0)-\alpha\|\Big)\|x_0-\alpha\|\\ &\le \tfrac{K}{2(1-\phi_1(e_0))}\Big((1+L)e_0+(1+h_1(e_0))e_0+(1+L)h_1(e_0)e_0\Big)e_0\\ &\le \tfrac{K}{2(1-\phi_1(e_0))}\Big((2+L)(1+h_1(e_0))e_0\Big)e_0\\ &= h_2(e_0)e_0, \end{aligned} \tag{29}$$

where

$$h_2(t)=\frac{K}{2(1-\phi_1(t))}\Big((2+L)(1+h_1(t))t\Big).$$

Consider $q_2(t)=h_2(t)-1$. Then, $q_2(0)=-1$ and $q_2(\rho_1)=\frac{K\rho_1(2+L)}{1-K_0(2+L)\rho_1}>0$. Therefore, $q_2(t)$ has atleast one root in $(0,\rho_1)$ and let ρ_2 be such a smallest root. Therefore, $0<\rho_2<\rho_1$ and

$$0\le h_2(t)\le 1\;\forall\, t\;\in\;[0,\rho_2). \tag{30}$$

Using (29) and (30), we get

$$\|z_0-\alpha\| \quad \le h_2(\|x_0-\alpha\|)\|x_0-\alpha\| < \|x_0-\alpha\|.$$

Taking $k=0$ in (19), we get

$$x_1-\alpha = z_0-\alpha-[y_0,y_0+F(y_0);F]^{-1}F'(\alpha)F'(\alpha)^{-1}F(z_0) \quad .$$

Since $y_0 \in D$, we have

$$\begin{aligned}\|F^{'}(\alpha)^{-1}([y_0,y_0+F(y_0);F]-[\alpha,\alpha;F])\| &\le K_0(\|y_0-\alpha\|+\|y_0+F(y_0)-\alpha\| \\ &\le K_0(\|y_0-\alpha\|+(1+L)\|y_0-\alpha\|) \\ &\le K_0(2+L)h_1(e_0)e_0=\phi_2(e_0)<1,\end{aligned}$$

where $\phi_2(t)=K_0(2+L)h_1(t)t$. Thus, by Banach Lemma, we have

$$\|[y_0,y_0+F(y_0);F]^{-1}F'(\alpha)\| \le \frac{1}{1-\phi_2(e_0)}.$$

Therefore,

$$\begin{aligned}\|x_1-\alpha\| &\le \|[y_0,y_0+F(y_0);F]^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}([y_0,y_0+F(y_0);F]-[z_0,\alpha;F])\|\|z_0-\alpha\| \\ &\le \frac{K(\|y_0-z_0\|+\|y_0+F(y_0)-\alpha\|)}{1-\phi_2(e_0)}\|z_0-\alpha\|.\end{aligned}$$

As

$$\|y_0-z_0\| \le \|y_0-\alpha\|+\|z_0-\alpha\| \le (h_1(t)+h_2(t))\|x_0-\alpha\|.$$

$$\begin{aligned}\|x_1-\alpha\| &= \frac{K\Big((h_1(e_0)+h_2(e_0))\,e_0+(1+L)h_1(e_0)e_0\Big)}{1-\phi_2(e_0)}h_2(e_0)\|x_0-\alpha\| \\ &= h_3(e_0)e_0. \qquad (31)\end{aligned}$$

where

$$h_3(t)=\Big(\frac{K\big((2+L)h_1(t)t+h_2(t)t\big)}{1-\phi_2(t)}\Big)h_2(t).$$

Consider $q_3(t)=h_3(t)-1$. Then, $q_3(0)=-1$ and $q_3(\rho_2)>0$. Therefore, $q_3(t)$ has at least one root in $(0,\rho_2)$ and let ρ_3 be such a smallest root. Therefore, $0<\rho_3<\rho_2$, and

$$0 \le h_3(t) \le 1\;\forall\, t\; \in\; [0,\rho_3). \qquad (32)$$

Using (31) and (32), we get

$$\|x_1-\alpha\| \quad \le h_3(\|x_0-\alpha\|)\|x_0-\alpha\| < \|x_0-\alpha\|$$

Thus, theorem holds for $k=0$. Changing x_0,y_0,z_0 and x_1 by x_k,y_k,z_k,x_{k+1}, we get the inequalities (23)-(25) for all $k \ge 0$. Since, $\|x_{k+1}-\alpha\| \le \|x_k-\alpha\| < r_3$, this gives $x_{k+1} \in B(\alpha,\rho_3)$. Also $h_3(t)$ is an increasing function in $[0,\rho_3)$, since $h_3^{'}(t)>0$ for all $t \in [0,\rho_3)$. Thus, we get

$$\begin{aligned}\|x_{k+1}-\alpha\| &\le h_3(e_0)\|x_k-\alpha\| \le h_3(e_0)h_3(e_0)\|x_{k-1}-\alpha\| \\ &\le h_3(e_0)^2h_3(e_0)\|x_{k-2}-\alpha\| \le \cdots \le h_3(e_0)^{k+1}\|x_0-\alpha\|.\end{aligned}$$

Therefore, $\lim_{k\to\infty} x_k = \alpha$ as $h_3(t) < 1$.
For uniqueness part, let $P_1 = [\alpha, \beta; F]$ where $F(\beta) = 0$ and $\beta \in \overline{B}(\alpha, R_1)$. Thus, we have

$$\|F^{'}(\alpha)^{-1}(P_1 - F^{'}(\alpha))\| \leq K_0 \left(\|\alpha - \alpha\| + \|\beta - \alpha\|\right) \leq K_0 R_1 < 1,$$

therefore, by Banach Lemma, P_1^{-1} exists. Then,

$$0 = F(\alpha) - F(\beta) = P_1(\alpha - \beta),$$

we obtain $\alpha = \beta$. ■

3.1. Numerical examples

In this subsection, we consider numerical examples to demonstrate the applicability of our work. Moreover, we compare our results with the local convergence of a modified Halley-Like method (2) and Chebyshev-Halley-type methods (3) respectively.

Example 4. *Let $X = Y = \mathbb{R}$, $D = (-1, 1)$. Define F on D by*

$$F(x) = e^x - 1.$$

Clearly, $\alpha = 0$, $F'(\alpha) = F'(\alpha)^{-1} = 1$, $K_0 = \frac{e-1}{2}$, and $L = e$. Then, we have

$$r_3 = 0.100343 < r_2 = 0.101834 < r_1 = 0.109801.$$

Example 5. *Let $X = Y = \mathbb{R}$, $D = (-1, 1)$. Define F on D by*

$$F(x) = x^2 - 1.$$

Clearly, $\alpha = 1$, $K_0 = K = \frac{1}{2}$, and $L = 2$. Then, we have

$$r_3 = 0.265055 < r_2 = 0.267949 < r_1 = 0.285714.$$

Example 6. *Consider the nonlinear Hammerstein type integral equation given by*

$$F(x(s)) = x(s) - 5\int_0^1 s\,t\,x(t)^3 dt, \tag{33}$$

with $x(s)$ in $\mathcal{C}[0, 1]$.

Clearly $\alpha = 0$. Taking $K_0 = 3.75$, $K = 7.5$ and $L = 8.5$ we get

$$r_3 = 0.008636 < r_2 = 0.008708 < r_1 = 0.009039.$$

The value of parameters are listed in Table 1. The radius of a convergence ball of a derivative free fifth order method (19) is compared with the existing methods and showed in Table 3. We can observe that except in example 5, all other examples larger radius of convergence ball is obtained by our approach. In example 5, we observe that the larger radius of convergence is obtained as compared to the Method (3).

Table 3: Comparison of radius of a ball

Examples	Method (19)	Method (2)	Method (3)
4	0.100343	0.02726	0.00892
5	0.265055	0.55264	0.06989
6	0.008636	0.00709	0.00755

4. Conclusions

In this paper, we discussed the local convergence of two fifth order iterative methods for solving nonlinear equations in Banach spaces, including the corresponding study when we consider the derivative free method obtained by approximating the derivatives by divided difference, getting a complete analysis of this iterative method. This analysis established under the assumption that the first order Fréchet derivative satisfies the Lipschitz continuity condition for first case and similar condition for the derivative free method involving only the derivative at the exact solution. Finally, for each method, some numerical examples worked out and computed the radii of convergence. Also, we have compared these results with existing methods and observed that our results are more efficient.

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