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Article

An ETD Method for American Options under the Heston Model

Rafael Company[1], Vera N. Egorova[2], Lucas Jódar[1,*] and Ferran Fuster Valls[3]

[1]Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, Valencia, 46022, Spain
[2]Depto. de Matemática Aplicada y Ciencias de la Computación, Universidad de Cantabria, Santander, 39005, Spain
[3]Nfoque Advisory Services, Madrid, 28001, Spain
*Corresponding Author: Lucas Jódar. Email: ljodar@mat.upv.es
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Abstract: A numerical method for American options pricing on assets under the Heston stochastic volatility model is developed. A preliminary transformation is applied to remove the mixed derivative term avoiding known numerical drawbacks and reducing computational costs. Free boundary is treated by the penalty method. Transformed nonlinear partial differential equation is solved numerically by using the method of lines. For full discretization the exponential time differencing method is used. Numerical analysis establishes the stability and positivity of the proposed method. The numerical convergence behaviour and effectiveness are investigated in extensive numerical experiments.

Keywords: Heston model; American option pricing; exponential time differencing; semi-discretization

1 Introduction

The price of options is one of the key topics in quantitative finance. The classic Black-Scholes model is based on unrealistic assumptions, such as the log-normality of the asset price. One possible solution is to presume that the volatility of the asset follows a stochastic process. In the literature there are various stochastic volatility (SV), such as Heston model [1], constant elasticity of variance (CEV) model proposed by Cox [2], widely used the generalized autoregressive conditional heteroskedasticity (GARCH) model and others. SV models assume realistic dynamics for the underlying asset and explain in a self-consistent way why options with different strikes and expirations have different Black-Scholes implied volatilities. Thus, study of such models is of great interest to the financial markets.

In this paper, the Heston model [1] is considered, which assumes that the variance follows the Cox–Ingersoll–Ross (CIR) process. Previous studies show that the non-negative and mean-reverting process is more consistent with the real markets [3]. Apart from the independent variables appearing in the Black-Scholes equation, i.e., the time t and the asset price S, the Heston pricing partial differential equation (PDE) has an additional variable, the variance v, due to the included stochastic volatility. Due to the possibility of early exercise, from the mathematical point of view, we deal with two-dimensional free boundary PDE that is challenging task especially in presence of mixed derivative term.

Various studies attempted to solve this PDE problem associated with the valuation of American options under the Heston model. One of the common approaches for treating the optimal exercise boundary is to add some penalty term [4]. Clarke et al. [5] formulate the American put pricing problem as a linear complementarity problem (LCP) and proposed a multigrid finite difference method (FDM) for numerical solution taking an advantage that the number of iterations required to solve a LCP is inherently independent of the grid size, but a complicated smoother procedure is necessary. An improvement of this method suggested in [6] is based on Fourier analysis to optimize the smoothing procedure. Alternative approach is an operator splitting method is proposed in [7], that allows to improve the accuracy by introducing an auxiliary variable. In [8] a space-time adaptive FDM based on operator splitting is proposed for the generalized Black-Scholes model. Zhu et al. [9] develop a predictor-corrector scheme based on the alternating direction implicit (ADI) method paying special attention to the boundary conditions. Alternative approaches based on contour integral method are proposed in [10].

However, much of the research up to now has been focused only on the case with low correlation, that has slight influence to the qualitative characteristics of a numerical method. This fact motivates us to develop an algorithm that includes a canonical form transformation that removes the source of computational problems–the cross derivative term. After that, numerical techniques are required. We apply method of lines (MOL), resulting in a system of ordinary differential equations (ODE) that has to be solved numerically by, for instance, widely used Runge-Kutta method [11] or the full FDM discretization [12–14].

In this paper, the exponential time differencing method (ETD) [15] is used. This approach based on exact integration of the system of ODEs has been applied to various problems [16,17], including the American option under the Heston model valuation [18]. An additional computational challenge arises due to the necessity of inverse matrices, not always well conditioned [19]. To overcome this difficulty the accurate Simpson's rule is used. Furthermore, taking advantage of logarithmic matrix norm and exponential matrix properties a stability and positivity analysis is performed to guarantee conditionally the boundedness of the solution independently of the semi-discrete system step-size.

This work is organized as follows. The Heston model for American put option is presented in Section 2. Section 3 addresses the problem transformation aimed to remove cross derivatives and explain the new rhomboid numerical domain. The semi-discretization and the ETD scheme are proposed in Section 4. Section 5 studies the positivity, stability and boundedness of the numerical solution. Section 6 presents numerical results comparing the proposed method with other authors and compute the numerical order of convergence. Concluding remarks are given in Section 7.

2 American Options with Stochastic Volatility

One of most widely used SV models is the Heston model [1], when the asset price process S and squared volatility process v are governed by the following risk-neutral dynamics

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{v(t)}S(t)dW_1, \\ dv(t) &= \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dW_2, \\ dW_1 dW_2 &= \rho dt \end{aligned} \tag{1}$$

where W_1, W_2 are standard Brownian motions, μ represents the deterministic drift, κ is the mean reversion rate, θ is the long-run variance, σ is the volatility of the variance and ρ is the instantaneous correlation $\rho \in (-1, 1)$. The variance process follows a square root, also known as a CIR process [3,20]. This kind of processes is bounded below by zero and, if the Feller condition is satisfied: $2\kappa\theta \geq \sigma^2$, the boundary cannot be achieved.

Applying Itô lemma and standard arbitrage arguments to (1) we achieve a PDE for the price $U = U(S, v, t)$ of a contingent claim

$$\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} + (\kappa(\theta - v) - \phi)\frac{\partial U}{\partial v} - rU = 0, \tag{2}$$

where r is the interest rate, ϕ represents the market price of volatility risk, $\phi = \alpha v$ for some constant α [1]; $\kappa(\theta - v) - \alpha v$ is the risk-neutral drift rate.

When the contingent claim is an European vanilla put option with strike price E and maturity at T, the function $U(S, v, t)$ satisfies the PDE (2) subject to the final and boundary conditions

$$U(S, v, T) = \max(E - S, 0), \tag{3}$$

$$U(0, v, t) = E, \tag{4}$$

$$\lim_{S\to\infty} \frac{\partial U}{\partial S}(S, v, t) = -1, \tag{5}$$

$$\frac{\partial U}{\partial t}(S, 0, t) + rS\frac{\partial U}{\partial S}(S, 0, t) + \kappa\theta\frac{\partial U}{\partial v}(S, 0, t) - rU = 0, \tag{6}$$

$$\lim_{v\to\infty} U(S, v, t) = 0. \tag{7}$$

American put option is described by a free boundary PDE problem, which is treated by adding the penalty function $f(E, S, U) = \lambda \max\{E - S - U, 0\}$, where λ is a positive penalty parameter, $\lambda \to \infty$, for more details see [21]. Thus, free boundary PDE is replaced by the following nonlinear problem posed on fixed domain:

$$\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} + \bar{\kappa}(\bar{\theta} - v)\frac{\partial U}{\partial v} - rU + f(E, S, U) = 0, \tag{8}$$

$$0 < S < \infty, \quad 0 < v < \infty, \quad 0 \le t < T. \tag{9}$$

Problem (8) is a time dependent two-dimensional Diffusion-Advection-Reaction (DAR) problem. Since the closed-form solution is not available, several numerical methods have been proposed, for instance, a tree-based method [22], PSOR method [23], sparse wavelet [24] or FDM [25]. In the case of FDM, presence of the mixed derivative term involves the appearance of negative coefficients in the numerical scheme, deteriorates the quality of the numerical solution augment the computational cost and possible rounding accumulation error [26]. Thus, the next section addresses to the classical technique for the reduction of second-order PDE to canonical form.

3 Canonical Form Equation

Following the classical techniques for canonical form transformation, see for instance [27], chapter 3, we proceed to classify the spatial part of (8) by the discriminant

$$D = B^2 - 4AC = \sigma^2 v^2 S^2(\rho^2 - 1), \quad B = \rho\sigma v S, \quad A = \frac{1}{2} v S^2, \quad C = \frac{1}{2} v\sigma^2. \tag{10}$$

Assuming $-1 < \rho < 1$, the spatial differential operator of (8) becomes of elliptic type. Then, the suitable change of variables is given by solving the following ODE

$$\frac{dv}{dS} = \frac{B + i\sqrt{4AC - B^2}}{2A} = \frac{\sigma(\rho + i\tilde{\rho})}{S}, \quad \tilde{\rho} = \sqrt{1 - \rho^2}. \tag{11}$$

Solving (11) one gets

$$x = \tilde{\rho}\sigma \ln S, \quad y = \rho\sigma \ln S - v. \tag{12}$$

Now, applying the time inverse transformation $\tau = T - t$ and denoting $U(S, v, \tau) = P(x, y, \tau)$, Eq. (8) takes the form

$$\frac{\partial P}{\partial \tau} = \frac{1}{2}\tilde{\rho}^2\sigma^2 v\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) + \tilde{\rho}\sigma\left(r - \frac{1}{2}v\right)\frac{\partial P}{\partial x} + \left(\rho\sigma\left(r - \frac{1}{2}v\right) - \bar{\kappa}(\bar{\theta} - v)\right)\frac{\partial P}{\partial y} - rP + f(E, S, P), \tag{13}$$

where $f(E, S, P)$ is the penalty term defined by

$$f(E, S, P) = \lambda \max(E - e^{\frac{x}{\tilde{\rho}\sigma}} - P, 0). \tag{14}$$

Inside a bounded domain, the PDE numerical solution will not be crucially affected by artificial boundary conditions, then some simplified strategies can be taken into consideration, see proposition 4.1 in [28]. In this paper, the artificial boundary conditions are chosen to be equal to the payoff function, as it is proposed in [29]:

$$\lim_{S \to 0} U(S, v, \tau) = E, \tag{15}$$

$$\lim_{S \to \infty} U(S, v, \tau) = 0, \tag{16}$$

$$\lim_{v \to 0} U(S, v, \tau) = \max(E - S, 0), \tag{17}$$

$$\lim_{v \to \infty} U(S, v, \tau) = \max(E - S, 0). \tag{18}$$

Note that there are other types of boundary conditions in literature, see [9,18,30].

From (12), v takes the expression in terms of x and y as follows

$$v = mx - y, \quad m = \frac{\rho}{\tilde{\rho}}. \tag{19}$$

The variance v has to be positive, thus, the variable y must be upper bounded by mx. Finally, the domain of the transformed problem is as follows

$$D = \{(x, y, \tau);\ x \in \mathbb{R},\ y < mx,\ 0 < \tau \leq T\}. \tag{20}$$

Since the spatial domain (20) is not bounded, a truncated computational domain has to be defined for numerical solution. First, following the ideas of [31], we chose S_1 and v_1 close to zero and S_2 and v_2 large enough. In order to perform practical simulations the choice is specified in Section 6. Then, the computational rectangle domain $[S_1, S_2] \times [v_1, v_2]$ is transformed into rhomboid $\overline{ABCD}$, see Fig. 1, with the sides

$$\begin{aligned}
&\overline{AD} = \left\{(x,y) \in \mathbb{R}^2 \mid x = a = \tilde{\rho}\sigma lnS_1,\ y = ma - v,\ v_1 \le v \le v_2\right\},\\
&\overline{AB} = \left\{(x,y) \in \mathbb{R}^2 \mid a \le x \le b = \tilde{\rho}\sigma lnS_2,\ y = mx - v_2,\ \right\},\\
&\overline{BC} = \left\{(x,y) \in \mathbb{R}^2 \mid x = b,\ y = mb - v,\ v_1 \le v \le v_2\right\},\\
&\overline{CD} = \left\{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ y = mx - v_1\right\}.
\end{aligned} \tag{21}$$

Note that if S and v are fully correlated: $|\rho| = 1$, from (10), the discriminant $\Delta = 0$ and the right-hand side of (8) becomes a parabolic PDE. Following the techniques for reduction to canonical form, an appropriate substitution is $x = S$; $y = v - \rho\sigma$ that leads to the following equation

$$\frac{\partial P}{\partial \tau} = \frac{1}{2} v x^2 \frac{\partial^2 P}{\partial x^2} + rx\frac{\partial P}{\partial x} + \left(\sigma\rho\left(\frac{1}{2}v - r\right) - \bar{\kappa}(\bar{\theta} - v)\right)\frac{\partial P}{\partial y} + f(E,S,P). \tag{22}$$

4 Numerical Algorithm

For the transformed DAR problem (13), the closed-form solution is not available and some numerical technique is required. In this paper, we proposed MOL-ETD combined method. MOL lies in discretization of the spatial derivatives, which leads to a system of ODEs. This system is solved numerically by ETD method [15,17]. This approach has to afford the computation challenge of the inverse matrices, not always well conditioned when eigenvalues are close to zero [19].

4.1 Semi-Discretization

Firstly, the spatial mesh is introduced:

$$x_i \in [a,b], \quad h = \Delta x, \quad x_i = a + ih, \quad 0 \le i \le N_x \tag{23}$$

$$y_{ij} \in [y_1 = mx - v_1,\ y_2 = mx - v_2], \quad mh = \Delta y, \quad y_{ij} = y_0 + (i+j)mh, \quad 0 \le j \le N_y, \tag{24}$$

where

$$N_x = \frac{b-a}{h},\ N_y = \frac{y_1 - y_2}{mh} = \frac{v_2 - v_1}{mh},\ y_0 = ma - v_2. \tag{25}$$

Thus, the numerical rhomboid domain includes all the mesh-points of the discretization. Let us denote the set of all mesh points by Γ, the subset located at the numerical domain boundary by $\partial\Gamma$ and the interior nodes by $\dot{\Gamma} = \Gamma - \partial\Gamma$. Now, the rhomboid boundary sides are partitioned in the following way:

$$\begin{aligned}
&P(\overline{AB}) = \{(x_i, y_{i0}) \mid 0 \le i \le N_x,\ \ j = 0\},\\
&P(\overline{BC}) = \left\{(x_{N_x}, y_{N_x j}) \mid i = N_x,\ \ 0 \le j \le N_y\right\},\\
&P(\overline{CD}) = \left\{(x_i, y_{iN_y}) \mid 0 \le i \le N_x,\ \ j = N_y\right\},\\
&P(\overline{AD}) = \left\{(x_0, y_{0j}) \mid i = 0,\ \ 0 \le j \le N_y\right\}.
\end{aligned} \tag{26}$$

The total number $N+1$ of points ξ in the mesh-grid is $N+1 = (N_y+1)(N_x+1)$. Further, let us reorder the mesh-points denoting

$$\xi_D = (x_i, y_j), \quad D = (N_y+1)i + j, \quad 0 \le i \le N_x, \quad 0 \le j \le N_y. \tag{27}$$

Taking into account that $j = D - (N_y + 1)i$, for given point ξ_D, we define the recovered coordinates as x_D and y_D:

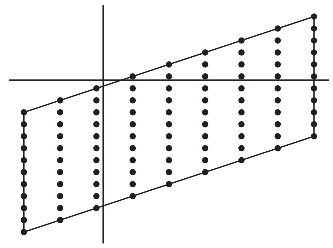

Figure 1: Rhomboid domain

$$x_D = a + \left[\frac{D}{N_y+1}\right]h, \quad y_D = ma - v_2 + \left(D + \left[\frac{D}{N_y+1}\right] - (N_y+1)\left[\frac{D}{N_y+1}\right]\right)mh. \tag{28}$$

where $[\cdot]$ stands for the integer part. Further, an expression for v_D from (19) is found to be

$$v_D = v_2 - jmh. \tag{29}$$

Taking into account (28) and omitting τ argument, the spatial derivatives are approximated by the centred finite difference as follows

$$\frac{\partial P}{\partial x}(\xi_D) \approx \frac{P(\xi_{D+N_y}) - P(\xi_{D-N_y})}{2h}, \qquad \frac{\partial^2 P}{\partial x^2}(\xi_D) \approx \frac{P(\xi_{D+N_y}) - 2P(\xi_D) + P(\xi_{D-N_y})}{h^2}, \tag{30}$$

$$\frac{\partial P}{\partial y}(\xi_D) \approx \frac{P(\xi_{D+1}) - P(\xi_{D-1})}{2mh}, \qquad \frac{\partial^2 P}{\partial y^2}(\xi_D) \approx \frac{P(\xi_{D+1}) - 2P(\xi_D) + P(\xi_{D-1})}{(mh)^2}. \tag{31}$$

Applying (30)–(31) to (13) and denoting $P(\xi_{D\pm z})$ as $P_{D\pm z}$ we achieve the following system of ODEs

$$\begin{aligned}\frac{\partial P}{\partial \tau}(\xi_D) = &\left(-\frac{2\alpha_D}{h^2} - \frac{2\alpha_D}{(mh)^2} - r\right)P_D + \left(\frac{\alpha_D}{(mh)^2} + \frac{\gamma_D}{mh}\right)P_{D+1} + \left(\frac{\alpha_D}{(mh)^2} - \frac{\gamma_D}{mh}\right)P_{D-1} \\ &+ \left(\frac{\alpha_D}{h^2} + \frac{\beta_D}{h}\right)P_{D+N_y} + \left(\frac{\alpha_D}{h^2} - \frac{\beta_D}{h}\right)P_{D-N_y} + f(P_D),\end{aligned} \tag{32}$$

where

$$\alpha_D = \frac{1}{2}\tilde{\rho}^2\sigma^2 v_D, \quad \beta_D = \frac{1}{2}\tilde{\rho}\sigma\left(r - \frac{1}{2}v_D\right), \quad \gamma_D = \frac{1}{2}\left(\rho\sigma\left(r - \frac{1}{2}v_D\right) - \bar{\kappa}(\bar{\theta} - v_D)\right). \tag{33}$$

The previous system (32) is presented in the vectorial form

$$\frac{dP}{d\tau} = A(\xi)P(\tau) + f(\xi, P), \tag{34}$$

where $P = P(\tau) \in R^{N+1}$ denotes the vector of all values $P_0, \ldots, P_N$, such that $P = [P_0, \ldots, P_N]^T$ and $f(\xi, P) = [f_0, \ldots, f_N]^T$ is the penalty term vector for every P_D. Matrix $A(\xi) = (a_{DL})_{D,L=0}^{N} \in \mathbb{R}^{(N+1)\times(N+1)}$ is a singular matrix whose non-zero entries are

$$\left.\begin{array}{l} a_{D,D} = -\left(\dfrac{2\alpha_D}{h^2} + \dfrac{2\alpha_D}{(mh)^2} + r\right) \\ a_{D,D\pm1} = \dfrac{\alpha_D}{(mh)^2} \pm \dfrac{\gamma_D}{mh} \\ a_{D,D\pm N_y} = \dfrac{\alpha_D}{h^2} \pm \dfrac{\beta_D}{h} \end{array}\right\} \text{if } \xi_D \in \dot{\Gamma}. \tag{35}$$

Note that the rows of $A(\xi)$ corresponding to boundary points $\xi_D \in \partial\Gamma$ are zero rows. Matrix $A(\xi) \in \mathbb{R}^{(N+1)\times(N+1)}$ dimension depends on h

$$(N+1) = (N_x+1)(N_y+1) = \left(\frac{b-a}{h}+1\right)\left(\frac{v_2-v_1}{mh}+1\right) = O\left(\frac{1}{h^2}\right) \to \infty, \quad h \to 0. \tag{36}$$

Denoting $f(\xi, P)_D$ as the Dth element of vector $f(\xi, P)$, one gets

$$f(\xi,P)_D = \begin{cases} 0 & \text{if } \ \xi_D \in \partial\Gamma \\ f(P_D) & \text{if } \ \xi_D \in \dot{\Gamma} \end{cases}. \tag{37}$$

4.2 ETD Method

System (34) is equivalent to a non-linear integral equation (semigroups approach, see [32]):

$$P(\tau) = e^{A(\tau-\tau_0)}P(\tau_0) + \int_{\tau_0}^{\tau} e^{A(\tau-v)} f(\xi, P(v))\, dv, \quad \tau > \tau_0. \tag{38}$$

The ETD methods [15] deal with the numerical approximation of the integral part of (38). First, let us introduce the temporal discretization

$$\tau^n = kn, \quad 0 \le n \le N_\tau, \quad \Delta\tau = k = \frac{T}{N_\tau}. \tag{39}$$

Then, for each sub-interval $[\tau^n, \tau^{n+1}]$, the approximation of (38) takes the form

$$P(\tau^{n+1}) = e^{Ak}P(\tau^n) + \int_{\tau^n}^{\tau^{n+1}} e^{A(\tau^{n+1}-v)} f(\xi, P(v))\, dv, \tag{40}$$

that is under the change of variable $s = t^{n+1} - v$ is equivalent to the following

$$P(\tau^{n+1}) = e^{Ak}P(\tau^n) + \int_0^k e^{As} f(\xi, P(\tau^{n+1} - s))\, ds. \tag{41}$$

As it is proposed in [17], a first explicit integral approximation $P(\tau^{n+1} - s)$ is replaced by the known value $P(\tau^n)$ corresponding to $s = k$:

$$P(\tau^{n+1}) = e^{Ak}P(\tau^n) + \left(\int_0^k e^{As} ds\right) f(\xi, P(\tau^n)) + O(k^2). \tag{42}$$

It is well known that if A is a regular matrix, then $\int_0^k e^{As} ds = A^{-1}\left(e^{Ak} - I\right)$. Indeed, this formula can fail due to matrix inverse computation in the case of singular or ill-conditioned matrix. In our case, matrix A, see (35), is not regular due to the existence of zero rows, meanwhile $\int_0^k e^{As} ds$ exists. It motivates us to approximate the integral term by the accurate Simpson's quadrature rule see [33],

$$\int_0^k e^{As}ds = k\varphi(A,k) + O(k^5), \quad \varphi(A,k) = \frac{1}{6}\left(I + 4e^{A\frac{k}{2}} + e^{Ak}\right). \tag{43}$$

Finally, denoting $P(\tau^n)$ as P^n, the numerical solution to (13) is given by

$$P^{n+1} = e^{Ak}P^n + k\,\varphi(A,k)\,f(\xi, P^n). \tag{44}$$

5 Numerical Analysis

In this section, the qualitative properties of the proposed method, such as positivity and stability, are studied.

The positivity can be assured if all the matrix elements of e^{Ak} and $\varphi(A,k)$ are positive. For the sake of clarity, we recall some definitions and results from [34].

A matrix $A \in \mathbb{R}^{n\times n}$ is called *Metzler* if its off-diagonal elements are non-negative, i.e., $a_{i,j} \geq 0$, $1 \leq i \neq j \leq n$. If A is Metzler, then $e^{At} \geq 0$ for $t \geq 0$, that can be shown taking $a_0 = \min_D a_{D,D}$.

Lemma 5.1 Matrix A defined by (35) is Metzler, if spatial step-size h satisfies

$$h \leq \frac{\tilde{\rho}^2\sigma^2 v_1}{2\max\{\beta,\ m\gamma\}}, \tag{45}$$

where $\beta = \max_D |\beta_D|$, $\gamma = \max_D |\gamma_D|$.

Proof. Let us consider the non-zero off-diagonal elements of matrix A. From (35), $a_{D,D\pm1}$ are non-negative, if

$$h \leq \frac{\alpha_D}{m|\gamma_D|}, \quad 0 \leq D \leq N+1. \tag{46}$$

Analogously, from (35), $a_{D,D\pm N_y} \geq 0$, if $h \leq \frac{\alpha_D}{|\beta_D|}$. Combining both conditions one gets that the off-diagonal elements of A are non-negative, if

$$h \leq \frac{\alpha_D}{\max\{m|\gamma_D|, |\beta_D|\}}, \quad \forall D = 0, 1, \ldots, N+1. \tag{47}$$

Under (45) all the off-diagonal elements of A are non-negative, thus, matrix A is Metzler.

Corollary 5.1. If (45) is fulfilled, then matrix A is Metzler, that assures $e^{At} \geq 0$ and $\varphi(A,k) \geq 0$ and, consequently, the positivity of the numerical solution by the proposed scheme.

Now, we define the infinite vector norm as

$$\|v\|_\infty = \max_n |v_n|, \quad \forall v \in \mathbb{R}^n. \tag{48}$$

The scheme (44) is $\|\cdot\|_\infty$-stable on the domain $\Gamma \times [0,T]$ if for every domain partition and for some positive constant K independent of h, k, and n, it is verified

$$\|P^n\|_\infty \leq K, \quad 0 \leq n \leq N_\tau. \tag{49}$$

Exponential matrix norm is bounded by the exponential of logarithmic norm $\mu[A]$, see [35,36]:

$$\|e^{Ak}\| \leq e^{k\mu[A]}, \quad \mu_\infty[A] = \max_i\left(\Re(a_{ii}) + \sum_{j\neq i}^{n}|a_{ij}|\right), \tag{50}$$

where $\Re(x)$ is the real part of a complex number x.

From (35), the row's sum of matrix A could take only two values

$$\sum_{j=1}^{N+1} a_{ij} = \begin{cases} 0 & \text{if } \xi_D \in \partial\Gamma \\ -r & \text{if } \xi_D \in \dot{\Gamma} \end{cases}, \; i = 1, 2 \; \ldots \; N+1. \tag{51}$$

From (50) one gets $\mu_\infty[A] = 0$, and consequently, $\|e^{Ak}\|_\infty \leq e^0 = 1$ and from (43) $\|\varphi(A,k)\|_\infty \leq 1$.

From other hand, matrix A has some zero rows, and their corresponding rows in e^{Ak} have only one entry equal to 1 and 0 at the others, consequently, $\|e^{Ak}\|_\infty = \|\varphi(A,k)\|_\infty \geq 1$. Then,

$$\left\|e^{Ak}\right\|_\infty = \|\varphi(A,k)\|_\infty = 1. \tag{52}$$

Let us represent each $i-th$ row of P^{n+1} in (44) as a function $g_i(P_0^n, \ldots, P_N^n)$, i.e.,

$$P_i^{n+1} = g_i(P_0^n, \cdots, P_N^n) = (e^{Ak})_i P^n + k\varphi(A,k)_i f(\xi, P^n), \quad 0 \leq i,j \leq N, \quad 0 \leq n \leq N_\tau. \tag{53}$$

Assuming the boundedness of the derivative $\left|\dfrac{\partial f(\xi_i, P)}{\partial P}\right| \leq \lambda$, $\xi_i \in \dot{\Gamma}$, $0 \leq P \leq E$, from non-negativity of e^{Ak} and $\varphi(A, k)$ one gets

$$\frac{\partial g_i}{\partial P_j^n} \geq (e^{Ak})_{ij} - \lambda \, k \, \varphi(A,k)_{ij}, \quad 0 \leq i,j \leq N. \tag{54}$$

Let us denote the lower bound of derivative $\psi(A, k) = e^{Ak} - \lambda \; k \; \varphi(A, k)$ and the vector function $g(P_0^n, \cdots, P_N^n)$ as $[g_0, \ldots, g_N]^T$. Then, from (54), the Jacobian matrix $\dfrac{\partial g}{\partial P^n}$ satisfies

$$\frac{\partial g}{\partial P^n} \geq \psi(A,k). \tag{55}$$

Under the assumption (45), a matrix defined by $B = A - a_0 I$, where $a_0 = \min\limits_D \; a_{D,D}$, verifies $B \geq 0$. Taking into account $e^{Ak} = e^{a_0 k} e^{Bk}$ and $e^{Bk} = I + \sum_{s=1}^{\infty} \dfrac{B^s k^s}{s!}$, the lower bound derivative $\psi(A, k)$ can be written in terms of B powers as follows

$$\psi(A,k) = \phi_0(k) + \sum_{s=1}^{\infty} \phi_s(k) \frac{B^s k^s}{s!}, \tag{56}$$

where

$$\phi_0(k) = e^{a_0 k} - \frac{\lambda\, k}{6}\left(1 + 4e^{a_0 \frac{k}{2}} + e^{a_0 k}\right), \quad \phi_s(k) = e^{a_0 k} - \frac{\lambda\, k}{6}\left(\frac{4}{2^s} e^{a_0 \frac{k}{2}} + e^{a_0 k}\right), \quad a_0 = \min_D a_{D,D}. \tag{57}$$

From (57), $\phi_s(k) > \phi_0(k)$ for $s \geq 1$. A Taylor expansion of $\phi_0(k)$, with $0 < \xi < k$, gives

$$\phi_0(k) = \phi_0(0) + \phi_0'(0)k + \frac{\phi''{}_0(\xi)}{2} k^2, \tag{58}$$

where

$$\phi_0(0) = 1, \tag{59}$$

$$\phi_0'(0) = a_0 - \lambda, \tag{60}$$

$$\phi_0''(\xi) = a_0^2 + \frac{\lambda}{3}|a_0|e^{a_0\frac{\xi}{2}} + \frac{\lambda|a_0|}{6}(2 - |a_0|\xi)(e^{a_0\xi} + e^{a_0\frac{\xi}{2}}). \tag{61}$$

Note that the two first terms the Taylor expansion (58) are positive if $k < \dfrac{1}{\lambda + |a_0|}$, moreover, $|a_0|$ satisfies

$$|a_0| = 2\alpha_m\left(\frac{1}{h^2} + \frac{1}{(mh)^2}\right) + r, \quad \alpha_m = \max_D \ \alpha_D \tag{62}$$

So, if temporal step-size k satisfies

$$k \leq \frac{h^2}{(\lambda + r)h^2 + 2\alpha_m\left(\dfrac{1 + m^2}{m^2}\right)}, \tag{63}$$

then $(2 - |a_0|\xi) > 0$, so $\dfrac{\phi''_0(\xi)}{2}k^2 > 0$. Consequently, $\phi_0(k) \geq 0$, that implies non-negativity of $\psi(A, k)$, that guarantees also the non-negativity of the Jacobian matrix $\dfrac{\partial g}{\partial P^n}$. Moreover, initial vector P^0 is bounded

$$0 \leq P^0 = \left[P_0^0, \cdots, P_N^0\right]^T \leq [E, E, \cdots, E]^T. \tag{64}$$

The boundedness of the solution $P_D^n \leq E$ is proven using the induction principle. The base case is given by (64). Induction hypothesis is established as follows: $0 \leq P_i^n \leq E, \ 0 \leq i \leq N$ and conditions (63) and (45) are fulfilled. In addition, as $\dfrac{\partial g}{\partial P^n} \geq 0$ every g_i is increasing in each direction P_j^n: $g_i(P_0^n, \cdots, P_N^n) \leq g_i(E, \cdots, E)$. Hence,

$$\begin{aligned} P_i^{n+1} &= g_i(P_0^n, \cdots, P_N^n) = (e^{Ak})_i P^n + k\varphi(A, k)_i f(\xi, P^n) \\ &\leq g_i(E, \cdots, E) = (e^{Ak})_i \mathbf{E} + k\varphi(A, k)_i f(\xi, \mathbf{E}). \end{aligned} \tag{65}$$

where $\mathbf{E} = [E, \cdots, E]^T$.

Since

$$\mu_\infty[A] = \max_i \left\{ a_{ii} + \sum_{j \neq i} |a_{ij}| \right\} = \max\{0, \ -r\} = 0, \tag{66}$$

from (65) one gets

$$(e^{Ak})_i \mathbf{E} = E\sum_{j=1}^{N+1}(e^{Ak})_{ij} \overset{Metz.}{=} E\sum_{j=1}^{N+1}\left|(e^{Ak})_{ij}\right| \leq \ E\|e^{Ak}\|_\infty \leq \ E\ e^{k\mu_\infty[A]} = E. \tag{67}$$

Then, by taking into account (67) and the fact that $f(\xi, \mathbf{E}) = 0$, the boundedness is established

$$P_i^{n+1} \leq E, \quad 0 \leq i \leq N, \quad 0 \leq n \leq N_\tau - 1, \tag{68}$$

that implies

$$\|P^n\|_\infty \le E. \tag{69}$$

Summarizing all above, the main result of the paper is established as follows.

Theorem 5.1. With the previous notation under conditions (45) and (63) the numerical solution P^n of the scheme (44) is non-negative and $\|\cdot\|_\infty$-stable, with $\|P^n\|_\infty \le E$ for $0 \le n \le N_\tau$.

Note that apart from the positivity and stability of the numerical scheme, it has been proven that the solution remains between zero and the strike price as it is expected dealing with put options.

6 Numerical Results and Discussions

Numerical convergence is studied by providing a sequence of simulations with time stepping starting with k = 0.125 and keep on halving. The results are presented in Tab. 1. "Difference" is calculated by $\|P_{2k} - P_k\|_\infty$, where P_k and P_{2k} are the consecutive solutions taking values $2k$ and k for the time step, R is defined as the ratio of consecutive differences:

$$R = \frac{\|P_{4k} - P_{2k}\|_\infty}{\|P_{2k} - P_k\|_\infty} = \frac{\varepsilon_2}{\varepsilon_1}. \tag{70}$$

Order of numerical convergence α is defined by

$$\alpha = \frac{\log \varepsilon_2 - \log \varepsilon_1}{\log 2}. \tag{71}$$

Table 1: Numerical convergence table for successive values of k

k	Difference	R	α
0.125	–	–	–
0.0625	2.87874	–	–
0.03125	0.613901	4.68926	2.22936
0.015625	0.108354	5.66572	2.50226
0.0078125	0.0196251	5.52117	2.46497

As expected, the scheme maintains the second order of convergence in time.

We check the verity of the established stability conditions by setting the temporal step-size k that does not satisfy (63). The results plotted in Fig. 2 show that if stability condition (63) is broken, some option values surpass the strike (E = 10) and the boundedness of the solution is lost.

Further, we compare the numerical results for American put options with other known methods in literature. The prices are presented for the asset values S = 8, 9, 10, 11, 12, for variance v = 0.0625, 0.25, and correlation ρ = 0.1, 0.7. Rest of parameters are given in Tab. 2. To obtain the solution at the point of interest a linear interpolation is used for the v values, and a cubic spline interpolation–for asset values.

The test case with low correlation ρ = 0.1 is widely considered in literature that gives us the possibility to compare the results with other known methods, such as predictor-corrector ADI scheme of Zhu et al. [9], multigrid method of Oosterlee [6], space-time adaptive FDM proposed in [8], etc. The results are plotted in Fig. 3 and compared in Tab. 3. Since the exact solution is not known, the error cannot be estimated, however, the proposed method is found to be competitive and efficient.

Note that this set of parameters which has been used for the past thirty years do not show the advantages of the proposed method. When $|\rho|$ is close to zero, the mixed derivative term has not a significant influence on

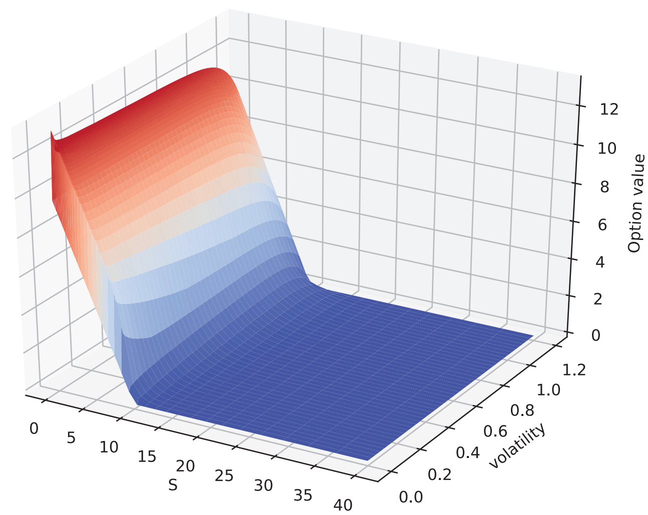

Figure 2: Unstable solution for large time-step

Table 2: Parameters of the American put option problem

Parameters	S_1	S_2	v_1	v_2	λ	E	T	r	κ	θ	σ
Values	0.25	40	0.002	1.2	200	10	0.25	0.1	5	0.16	0.9

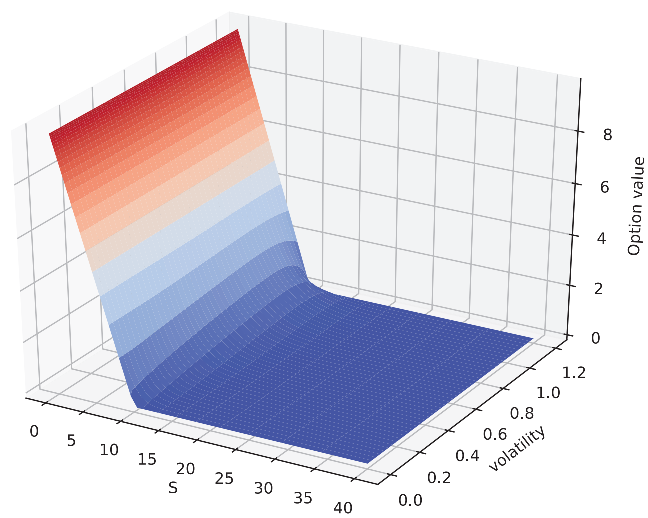

Figure 3: Numerical solutions with low correlation $\rho = 0.1$

the numerical method. The importance of the proposed canonical form transformation is shown by considering $\rho = 0.7$. The results are given in Tab. 4 and in Fig. 4.

Note that the proposed scheme (44) provides positive and stable solutions under conditions on the step sizes discretization (45) and (63) even if the correlation parameter has absolute value close to one.

Properties of the numerical solution also can be studied by considering the first order greeks $\Delta = \frac{\partial P}{\partial S}$ and $v = \frac{\partial P}{\partial v}$. For the case with low correlation, the greeks are plotted in Figs. 5 and 6, correspondingly.

Table 3: Comparison of the computed option prices with $\rho = 0.1$

v value	(N_x, N_y, N_τ)	Asset values				
		8	9	10	11	12
$v = 0.0625$	(65, 169, 5000)	1.9941	1.1020	0.5139	0.2122	0.0843
	[18] (400, 80, 20)	1.9958	1.1051	0.5167	0.2119	0.0815
	[37]	2.00	1.108	0.5316	0.2261	0.0907
	[7] (4096, 2048, 4098)	2.0000	1.10763	0.52004	0.21368	0.08205
	[6] (256, 256)	2.000	1.107	0.517	0.212	0.0815
	[8] (81, 21, 21)	1.9976	1.10768	0.51837	0.21424	0.08193
	[9] (100, 100, 50000)	2.0000	1.0987	0.5082	0.2106	0.0861
	[4] (177, 103)	2.000	1.1076	0.5202	0.2138	0.0821
$v = 0.25$	(65, 169, 5000)	2.0744	1.3291	0.7920	0.4467	0.2437
	[18] (400, 80, 20)	2.0760	1.3316	0.7945	0.4473	0.2423
	[37]	2.0733	1.3290	0.7992	0.4536	0.2502
	[7] (4096, 2048, 4098)	2.0784	1.3336	0.7960	0.4483	0.2428
	[6] (256, 256)	2.0790	1.3340	0.7960	0.4490	0.2430
	[8] (81, 21, 21)	2.0777	1.33219	0.79377	0.44621	0.2417
	[9] (100, 100, 50000)	2.0781	1.3337	0.7965	0.4496	0.2441
	[4] (177, 103)	2.0784	1.3337	0.7961	0.4483	0.2428

Table 4: Comparison of the computed option prices with $\rho = 0.7$

v value	(N_x, N_y, N_τ)	Asset values				
		8	9	10	11	12
$v = 0.0625$	(150, 57, 2500)	2.0022	1.1382	0.5163	0.1573	0.0317
$v = 0.25$		2.1160	1.3665	0.7937	0.4062	0.1803

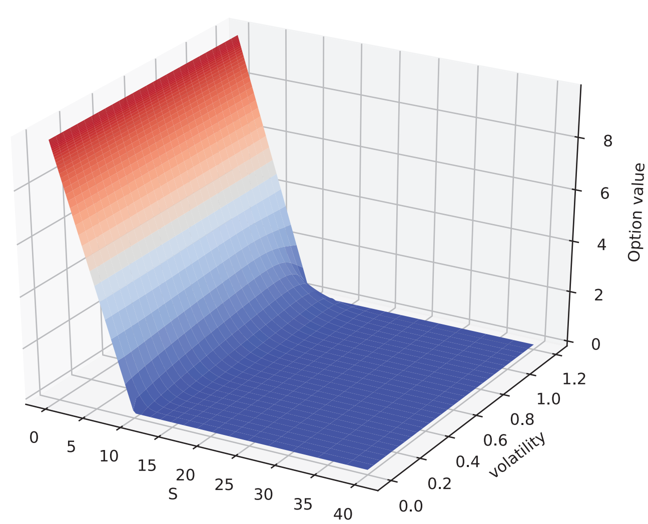

Figure 4: Numerical solution with high correlation $\rho = 0.7$

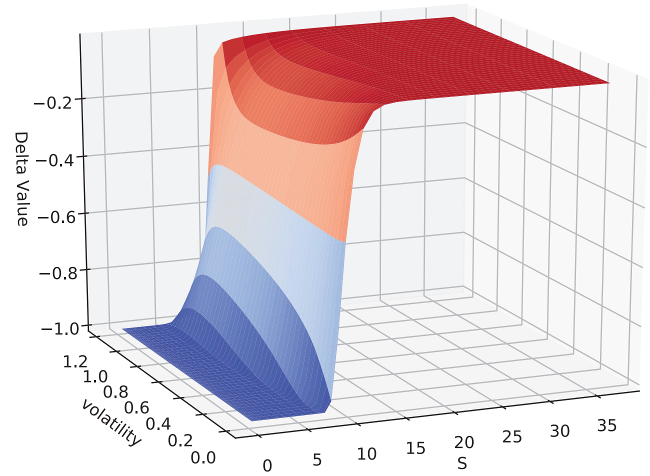

Figure 5: Delta of the option

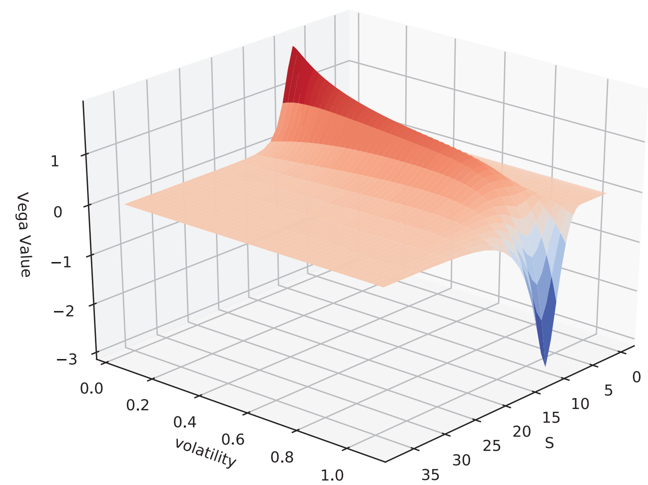

Figure 6: Vega of the option

For small values of S, Δ is approaching to -1 showing the linear decreasing behaviour of the option price with respect to the asset price. For $S \approx E$, Δ is increasing fast up to zero, and then remains there for larger values of S. As expected, the put option price tends to zero for large asset price.

7 Conclusions

In present paper, a numerical method for solving American options pricing problems under stochastic volatility has been developed. A suitable transformation is employed to remove the cross derivative term. The semi-discretization approach combined with the ETD method is proposed, whose positivity and stability is guaranteed under conditions (45) and (63). Comparing the results with other known numerical methods, the competitiveness of the proposed method has been shown. A case with high correlation is considered to show the advantages of the mixed derivative removing transformation.

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