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A stable class of modified Newton-like methods for multiple roots and their dynamics

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Abstract: There have appeared in the literature a lot of optimal eighth-order iterative methods for approximating simple zeros of nonlinear functions. Although, the similar ideas can be extended for the case of multiple zeros but the main drawback is that the order of convergence and computational efficiency reduce dramatically. Therefore, in order to retain the accuracy and convergence order, several optimal and non-optimal modifications have been proposed in the literature. But, as far as we know, there are limited number of optimal eighth-order methods that can handle the case of multiple zeros. With this aim, a wide general class of optimal eighth-order methods for multiple zeros with known multiplicity is brought forward, which is based on weight function technique involving function-to-function ratio. An extensive convergence analysis is demonstrated to establish the eighth-order of the developed methods. The numerical experiments considered the superiority of the new methods for solving concrete variety of real life problems coming from different disciplines such as trajectory of an electron in the air gap between two parallel plates, the fractional conversion in a chemical reactor, continuous stirred tank reactor problem, Planck's radiation law problem, which calculates the energy density within an isothermal blackbody and the problem arising from global carbon dioxide model in ocean chemistry, in comparison with methods of similar characteristics appeared in the literature.

Keywords: Kung-Traub conjecture; multiple roots; nonlinear equations; optimal iterative methods; stability.

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1 Introduction

Importance of solving nonlinear problems is justified by numerous physical and technical applications and intense growth of the field over the past decades. These problems arise in many areas of natural and physical sciences, which include initial and boundary value problems, heat and fluid flow problems, electrostatics problems, as well as problems associated with global positioning systems (GPS). In absence of analytical solutions, one of the possible ways to tackle the problem is to use appropriate numerical methods for finding approximate solutions. An algorithm, which is a cornerstone to the modern study of root-finding algorithms was made by Newton through his 'method of fluxions'. Later on this method was polished by Raphson to produce what we now know as the Newton-Raphson method [1–3]. It converges quadratically for simple roots but if the root is non-simple, the convergence becomes linear. Since then, a tremendous amount of effort has been made in the direction of improving the convergence resulting in modified Newton method (also known as Rall's method) for finding multiple roots of nonlinear equations of the form $f(x) = 0$, where $f(x)$ is real function defined in a domain $D \subseteq \mathbb{R}$. It is given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}. \tag{1}$$

Given the multiplicity $m \geq 1$ in advance, it converges quadratically for multipe roots. Although, there are many one-point iterative methods available in the literature but when seen from real context, they are not of practical interest because of their theoretical limitations regarding convergence order and efficiency index. Moreover, most of the one-point methods are computationally expensive and inefficient when they are tested on academic problems originating from real life. Therefore, multipoint iterative methods are better candidates to qualify as efficient solvers. The good thing with multipoint iterative methods without memory for scalar nonlinear equations is that we have a conjecture about their convergence order. According to the Kung-Traub conjecture [2], any multipoint method without memory can reach its convergence order of at most 2^{n-1} for n functional evaluations. A large community of researchers from the world wide turn towards the most important class of multipoint iterative methods and proposed various optimal fourth-order methods (requiring three functional evaluations, per iteration) [4–9] and non-optimal methods [10, 11] for approximating multiple zeros of nonlinear functions. In the literature, there are limited number of multipoint point iterative methods having sixth-order of convergence. For instance, Thukral [12] proposed the following sixth-order multipoint iteration scheme:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - m\frac{f(x_n)}{f'(x_n)}\sum_{i=1}^{3}\left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{i}{m}}, \\ x_{n+1} &= z_n - m\frac{f(x_n)}{f'(x_n)}\left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}\left[\sum_{i=1}^{3}\left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{i}{m}}\right]^2. \end{aligned} \tag{2}$$

In 2015, Geum et al. [13], presented a non-optimal class of two-point sixth-order methods as follows:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - Q(u_n, s_n)\frac{f(y_n)}{f'(y_n)}, \end{aligned} \tag{3}$$

where $u_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$, $s_n = \sqrt[m-1]{\frac{f'(y_n)}{f'(x_n)}}$ $(m > 1)$, and $Q : \mathbb{C} \to \mathbb{C}$ is a holomorphic function in the neighborhood of origin (0, 0). But, the main drawback of this scheme is that it does not work for simple zeros (i. e., for $m = 1$).

In 2016, Geum et al. [14] developed another non-optimal family of three-point sixth-order methods for multiple zeros and it is defined by

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ w_n &= y_n - mG(u_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= w_n - mK(u_n, v_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{4}$$

where $u_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$ and $v_n = \sqrt[m]{\frac{f(w_n)}{f(x_n)}}$. The weight functions $G : \mathbb{C} \to \mathbb{C}$ and $K : \mathbb{C}^2 \to \mathbb{C}$ are analytic in the neighborhood of 0 and (0, 0), respectively. It can be seen that the methods (3) and (4) require four function evaluations to achieve sixth-order of convergence. Therefore, these methods are not optimal in accordance with Kung and Traub conjecture. It is needless to mention that in the last decades much effort has been done to develop and analyze optimal eighth-order methods for multiple zeros but with no success. Motivated and inspired by this fact, Behl et al. [15] introduced an optimal family of eighth-order iterative methods in case of multiple roots for the first time. Its iterative expression is given by

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - u_nQ(h_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - u_nv_nG(h_n, v_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{5}$$

where the weight functions $Q : \mathbb{C} \to \mathbb{C}$ is analytic in the neighborhood of (0) and $G : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic in the neighborhoods of (0, 0), with $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $h_n = \frac{u_n}{a_1 + a_2u_n}$ and $v_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$, where a_1 and a_2 are free disposable real parameters.

Furthermore, Zafar et al. [16] presented another optimal eighth-order family using the weight function approach as follows:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - mu_nH(u_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - u_nv_n(A_2 + A_3u_n)P(v_n)G(w_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{6}$$

where A_2, A_3 are real parameters and weight functions $H, P, G : \mathbb{C} \to \mathbb{C}$ are analytic in the neighborhood of 0 with $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $v_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$ and $w_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$.

It is clear from the above discussions that there are a very small number of optimal eighth-order methods which can handle the case of multiple zeros. Moreover, these type of methods have not been discussed in deep till date. Therefore, the main motivation of the current research work is to present a new optimal class of iterative methods having eighth-order convergence which exploit weight function technique for computing multiple zeros. Our proposed scheme requires only four function evaluations ($f(x_n)$, $f'(x_n)$, $f(y_n)$ and $f(z_n)$) per full iteration which is in accordance with the classical Kung-Traub conjecture. Furthermore, we manifest that the proposed methods have good stability characteristics, reasonable errors in the estimation of multiple zeros.

Our presentation is unfolded in what follows. The new eighth-order scheme and its convergence analysis is presented in **Section 2**. In **Section 3**, some special cases are included based on the different choices of weight functions employed at second and third substeps of the designed family. In **Section 4**, numerical experiments and dynamical analysis are included which illustrate the efficiency, accuracy and stability of the scheme in comparison to other methods proposed in the scientific literature. Finally, **Section 5** is devoted to some conclusions.

2 Construction of the family

In this section, we intend to develop a new optimal eighth-order scheme for multiple roots with known multiplicity $m \geq 1$. We here establish a main theorem describing the convergence analysis of the proposed family. So, we present the three-step scheme as follows:

$$\begin{aligned} y_n &= x_n - m\lambda_n, \\ z_n &= y_n - m\left(\frac{u_n}{1-u_n}\right)\lambda_n H(t_n), \\ x_{n+1} &= z_n - u_n\lambda_n\left(\frac{v_n}{1+\alpha_1 v_n + \alpha_2 v_n^2}\right)(R(u_n) + P(w_n)), \end{aligned} \tag{7}$$

where $\lambda_n = \frac{f(x_n)}{f'(x_n)}$ and the weight functions $H, R, P: \mathbb{C} \to \mathbb{C}$ are analytic in the neighborhood of origin with $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $t_n = \frac{u_n}{b_1 + b_2 u_n}$, $v_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$ and $w_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$. Here, b_1, b_2, α_1 and α_2 are free disposable real parameters.

In Theorem 2.1, we demonstrate that how to construct weight functions H, R and P so that the proposed scheme (7) arrives at eighth order without consuming any additional functional evaluations.

Theorem 2.1. Assume that $f: \mathbb{C} \to \mathbb{C}$ is an analytic function in the region enclosing the multiple zero $x = \alpha$ with multiplicity m ≥ 1. Then, the iterative expression defined by (7) has eighth-order convergence when it satisfies the following conditions:

$$\begin{cases} H(0) = m,\ H'(0) = mb_1,\ R(0) = m - P(0),\ R'(0) = 2m, \\ R''(0) = \dfrac{6b_1^2 m - 2b_1 b_2 m + H''(0)}{b_1^2},\ \alpha_1 = -1, \\ R'''(0) = \dfrac{12b_1^3 m - 18b_1^2 b_2 m + 6b_1 b_2^2 m + (9b_1 - 6b_2)H''(0) + H'''(0)}{b_1^3}, \\ P'(0) = 2m. \end{cases} \tag{8}$$

Proof. Let $x = \alpha$ be a multiple zero of $f(x)$. Using Taylor's series expansion of $f(x_n)$ and $f'(x_n)$ about α, we obtain

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!}e_n^m(1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9)) \tag{9}$$

and

$$\begin{aligned} f'(x_n) = \frac{f^m(\alpha)}{m!}e_n^{m-1}(m &+ c_1(m+1)e_n \\ &+ c_2(m+2)e_n^2 + c_3(m+3)e_n^3 + c_4(m+4)e_n^4 \\ &+ c_5(m+5)e_n^5 + c_6(m+6)e_n^6 + c_7(m+7)e_n^7 \\ &+ c_8(m+8)e_n^8 + O(e_n^9)), \end{aligned} \tag{10}$$

respectively. Here, $e_n = x_n - \alpha$ and $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, k = 1, 2, 3, ….

Using the above expressions (9) and (10) in the first substep of (7), we get

$$y_n - \alpha = \frac{c_1 e_n^2}{m} + \frac{(-(1+m)c_1^2 + 2mc_2)e_n^3}{m^2} + \sum_{j=1}^{5}\Gamma_j e_n^{j+3} + O(e_n^9), \tag{11}$$

where $\Gamma_j = \Gamma_j(m, c_1, c_2, \ldots, c_8)$ are given in terms of $m, c_1, c_2, c_3, \ldots, c_8$ for $1 \leq j \leq 5$. The explicit expressions for the first two terms Γ_1 and Γ_2 are given by $\Gamma_1 = \frac{1}{m^3}\{3m^2c_3 + (m+1)^2c_1^3 - m(3m+4)c_2c_1\}$ and $\Gamma_2 = \frac{1}{m^4}\{(m+1)^3 c_1^4 - 2m(2m^2+5m+3)c_2c_1^2 + 2m^2(2m+3)c_3c_1 + 2m^2(c_2^2(m+2) - 2c_4 m)\}$.

Using again Taylor's series expansion, we obtain

$$\begin{aligned} f(y_n) = f^{(m)}(\alpha)e_n^{2m}\Bigg[&\frac{\left(\frac{c_1}{m}\right)^m}{m!} + \frac{(2c_2m - c_1^2(m+1))\left(\frac{c_1}{m}\right)^m e_n}{c_1 m!} \\ &+ \left(\frac{c_1}{m}\right)^{1+m}\frac{1}{2m!\,c_1^3}\{(3+3m+3m^2+m^3)c_1^4 - 2m(2+3m \\ &+ 2m^2)c_1^2c_2 + 4(-1+m)m^2c_2^2 + 6m^2c_1c_3\}e_n^2 \\ &+ \sum_{j=1}^{5}\bar{\Gamma}_j e_n^{j+3} + O(e_n^9)\Bigg] \end{aligned} \tag{12}$$

and

$$\begin{aligned} u_n = \frac{c_1 e_n}{m} &+ \frac{(2c_2m - c_1^2(m+2))e_n^2}{m^2} + \gamma_1 e_n^3 + \gamma_2 e_n^4 \\ &+ \gamma_3 e_n^5 + O(e_n^6), \end{aligned} \tag{13}$$

where

$$\begin{aligned} \gamma_1 &= \frac{1}{2m^3}[c_1^3(2m^2+7m+7) + 6c_3m^2 - 2c_2c_1m(3m+7)], \\ \gamma_2 &= -\frac{1}{6m^4}[c_1^4(6m^3+29m^2+51m+34) - 6c_2c_1^2m(4m^2+16m \\ &\quad +17) + 12c_3c_1m^2(2m+5) + 12m^2(c_2^2(m+3) - 2c_4m)], \\ \gamma_3 &= \frac{1}{24m^5}[-24m^3(c_2c_3(5m+17) - 5c_5m) + 12c_3c_1^2m^2(10m^2 \\ &\quad +43m+49) + 12c_1m^2\{c_2^2(10m^2+47m+53) \\ &\quad -2c_4m(5m+13)\} - 4c_2c_1^3m(30m^3+163m^2+306m \\ &\quad +209) + c_1^5(24m^4+146m^3+355m^2+418m+209)]. \end{aligned} \tag{14}$$

Now, using the above expression (14), we get

$$t_n = \frac{c_1}{mb_1}e_n + \sum_{i=1}^{4} \Theta_j e_n^{j+1} + O(e_n^6). \tag{15}$$

where $\Theta_j = \Theta_j(b_1, b_2, m, c_1, c_2, \ldots, c_8)$ are given in terms of b_1, b_2, m, c_1, c_2, …, c_8, where the two coefficients Θ_1 and Θ_2 are written explicitly as $\Theta_1 = -\frac{b_2c_1^2+b_1((2+m)c_1^2-2mc_2)}{m^2b_1^2}$, and $\Theta_2 = \frac{1}{2m^3b_1^3}[2b_2^2c_1^3 + 4b_1b_2c_1((2+m)c_1^2 - 2mc_2) + b_1^2\{(7+7m+2m^2)c_1^3 - 2m(7+3m)c_1c_2 + 6m^2c_3\}]$.

Due to the fact that $t_n = \frac{u_n}{b_1+b_2u_n} = \mathcal{O}(e_n)$, therefore, it suffices to expand weight function $H(t_n)$ in the neighborhood of origin by Taylor's expansion up to fourth-order term as follows:

$$H(t_n) \approx H(0) + H'(0)t_n + \frac{1}{2!}H''(0)t_n^2 + \frac{1}{3!}H^{(3)}(0)t_n^3 + \frac{1}{4!}H^{(4)}(0)t_n^4, \tag{16}$$

where $H^{(k)}$ represents the k-th derivative. By inserting the expressions (9)–(16) in the second substep of scheme (7), we have

$$z_n - \alpha = \frac{(m-H(0))c_1}{m^2}e_n^2 + \frac{2m(m-H(0))b_1c_2 - \left(H'(0) + \left(m+m^2-2H(0)-mH(0)\right)b_1\right)c_1^2}{(m^3b_1)}e_n^3 + \sum_{s=1}^{5}\Omega_s e_n^{s+3} + O(e_n^9). \tag{17}$$

where $\Omega_s = \Omega_s(H(0), H'(0), H''(0), H^{(3)}(0), H^{(4)}(0), m, b_1, b_2, c_1, c_2, \ldots, c_8)$, $s = 1, 2, 3, 4, 5$.

It is clear from error equation (17) that in order to obtain at least fourth-order convergence, the coefficients of e_n^2 and e_n^3 must vanish simultaneously. That is possible only for the following values of $H(0)$ and $H'(0)$, which can be calculated from the expression (17):

$$H(0) = m, \quad H'(0) = mb_1. \tag{18}$$

Substituting the above values of $H(0)$ and $H'(0)$ in (17), we obtain

$$z_n - \alpha = \frac{\left(m(5+m)b_1^2 - H''(0) + 2mb_1b_2\right)c_1^3 - 2m^2b_1^2c_1c_2}{2m^4b_1^2}e_n^4 + \sum_{r=1}^{4} L_r e_n^{s+4} + O(e_n^9), \tag{19}$$

where $L_r = L_r(H''(0), H^{(3)}(0), H^{(4)}(0), m, b_1, b_2, c_1, c_2, \ldots, c_8)$, $r = 1, 2, 3, 4$.

Now, again by using the Taylor series expansion, we have

$$f(z_n) = f^{(m)}(\alpha)e_n^{4m}\left[\frac{2^{-m}}{m!} \times \left(\frac{\left(m(5+m)b_1^2 - H''(0) + 2mb_1b_2\right)c_1^3 - 2m^2b_1^2c_1c_2}{m^4b_1^2}\right)^m + \sum_{s=1}^{5}\bar{P}_s e_n^s + O(e_n^6)\right], \tag{20}$$

$$v_n = \frac{\left(m(5+m)b_1^2 - H''(0) + 2mb_1b_2\right)c_1^2 - 2m^2b_1^2c_2}{2m^3b_1^2}e_n^2 + \gamma_0 e_n^3 + O(e_n^4) \tag{21}$$

and

$$w_n = \frac{\left(m(5+m)b_1^2 - H''(0) + 2mb_1b_2\right)c_1^2 - 2m^2b_1^2c_1c_2}{2m^4b_1^2}e_n^3 + \gamma_1 e_n^4 + O(e_n^5), \tag{22}$$

where

$$\gamma_0 = \frac{1}{6b_1^3m^4}\Big[\Big(6b_1^2b_2m(5+2m) + 2b_1^3m(19+15m+2m^2) - 6b_2H''(0) + 3b_1\left(2b_2^2m - 5H''(0) - 2mH''(0)\right) + H'''(0)\Big)c_1^3 - 12b_1m\left(2b_1b_2m + b_1^2m(5+m) - H''(0)\right)c_1c_2 + 12b_1^3m^3c_3\Big],$$

$$\gamma_1 = \frac{1}{6b_1^3m^5}\Big[\Big(6b_1^2b_2m(7+2m) + b_1^3m(68+51m+7m^2) - 6b_2H''(0) + 3b_1\left(2b_2^2m - 7H''(0) - 3mH''(0)\right)H'''(0)\Big)c_1^4 - 6b_1m\left(6b_1b_2m + b_1^2m(17+m) - 3H''(0)\right)c_1^2c_2 + 12b_1^3m^3c_2^2 + 12b_1^3m^3c_1c_3\Big].$$

It is clear from equations (13) and (22) that u_n and w_n are of order e_n and e_n^3, respectively. Therefore, we can expand weight function $R(u_n)$ and $P(w_n)$ in the neighborhood of origin by Taylor's series expansion up to fourth-order and third-order terms, respectively as follow:

$$R(u_n) \approx R(0) + R'(0)u_n + \frac{1}{2!}R''(0)u_n^2 + \frac{1}{3!}R^{(3)}(0)u_n^3 + \frac{1}{4!}R^{(4)}(0)u_n^4 \tag{23}$$

and

$$P(w_n) \approx P(0) + P'(0)w_n + \frac{1}{2!}P''(0)w_n^2 + \frac{1}{3!}P^{(3)}(0)w_n^3. \quad (24)$$

By using the expressions (9)–(24) in the last substep of the proposed scheme (7), we have

$$e_{n+1} = \frac{(m-P(0)-R(0))c_1\left(\left(H''(0)-(m+5)b_1^2-2b_1b_2\right)c_1^2-2mb_1^2c_2\right)}{2b_1^2m^5}e_n^4 + \sum_{i=1}^{4}\psi_i e_n^{i+4} + O(e_n^9), \quad (25)$$

where $\psi_i = \psi_i(m, b_1, b_2, \alpha_1, \alpha_2, H(0), H'(0), H''(0), H^{(3)}(0), P(0), P'(0), P''(0), P^{(3)}(0), R(0), R'(0), R''(0), R^{(3)}(0), c_1, c_2, \ldots, c_8)$, $i = 1, 2, 3, 4$.

For obtaining at least fifth-order convergence, we need to choose $R(0) = m - P(0)$. Further, substituting $R(0) = m - P(0)$ in $\psi_1 = 0$, one can obtain

$$R'(0) = 2m. \quad (26)$$

Now inserting $R(0) = m - P(0)$ and $R'(0) = 2m$ in $\psi_2 = 0$, we obtain the following relations

$$2b_1b_2\alpha_1 m - \alpha_1 H''(0) + b_1^2\left((11+5\alpha_1)m + (1+\alpha_1)m^2 - R''(0)\right) = 0$$
$$\text{and } 2b_1^2m^2(\alpha_1+1) = 0, \quad (27)$$

which further yields

$$\alpha_1 = -1, \qquad R''(0) = \frac{6b_1^2m - 2b_1b_2m + H''(0)}{b_1^2}. \quad (28)$$

By inserting the values of $R(0)$, $R'(0)$, $R''(0)$ and α_1 obtained from the above equations in $\psi_3 = 0$, we obtain the following independent expressions

$$-6b_1^2b_2m(m+P'(0)) + 3b_1\left(2b_2^2m^2 + (m+P'(0))H''(0)\right) + m(-6b_2H''(0) + H'''(0)) + b_1^3m(6m^2 - 3m(-14+P'(0)) - 15P'(0) - R^{(3)}(0)) = 0,$$
$$6b_1^3m^2(P'(0) - 2m) = 0, \quad (29)$$

which further gives

$$R'''(0) = \frac{12b_1^3m - 18b_1^2b_2m + 6b_1b_2^2m + 9b_1H''(0) - 6b_2H''(0) + H^{(3)}(0)}{b_1^3},$$
$$P'(0) = 2m. \quad (30)$$

Finally, using equations (26)–(30) in (7), one can get the following error equation:

$$e_{n+1} = \frac{1}{48m^{10}b_1^6}\left[c_1\left(\left(2b_1b_2m + b_1^2m(m+5) - H''(0)\right)c_1^2 - b_1^2m^2c_2\right)\right]\left(\left(24b_1^3b_2\alpha_2m^2(5+m) + 36b_2^2mH''(0) + 6\alpha_2H''(0)^2 + 12b_1^2m\left(2b_2^2(3+\alpha_2)m - \alpha_2(5+m)H''(0)\right) - 12b_1m\left(2b_2^3m + 2b_2(3+\alpha_2)H''(0) - H'''(0)\right) - 12b_2mH'''(0) + mH^{(4)}(0) + b_1^4m(2(134+75\alpha_2)m + 12(7+5\alpha_2)m^2 + (8+6\alpha_2)m^3 - R^{(4)}(0))\right)c_1^4 - 24b_1^2m^2\left(2b_1b_2\alpha_2m + b_1^2m(7+m+\alpha_2(5+m)) - \alpha_2H''(0)\right)c_1^2c_2 + 24b_1^4\alpha_2m^4c_2^2 + 24b_1^4m^4c_1c_3\right)e_n^8 + O(e_n^9). \quad (31)$$

The consequence of the above error analysis is that the proposed scheme (7) acquires eighth-order convergence using only four functional evaluations (viz. $f(x_n)$, $f'(x_n)$, $f(y_n)$ and $f(z_n)$) per full iteration. This completes the proof.

2.1 Some special cases of the proposed class

In this section, we will discuss some interesting special cases of our proposed class (7) by assigning different forms of weight functions $H(t_n)$, $R(u_n)$ and $P(w_n)$ employed at second and third step, respectively.

(1) Let us consider the following optimal class of eighth-order methods for multiple roots where polynomial weight functions are chosen satisfying the conditions appearing in Theorem 2.1:

$$y_n = x_n - m\lambda_n,$$
$$z_n = y_n - m\left(\frac{u_n}{1-u_n}\right)\lambda_n\left[1 + t_nb_1 + \frac{1}{2}t_n^2H''(0) + \frac{1}{3!}t_n^3H^{(3)}(0) + \frac{1}{4!}t_n^4H^{(4)}(0)\right],$$
$$x_{n+1} = z_n - u_n\left(\frac{v_n}{1-v_n+\alpha_2v_n^2}\right)\lambda_n\left[m + 2mu_n + \frac{6b_1^2m - 2b_1b_2m + H''(0)}{2b_1^2}u_n^2 + \frac{12b_1^3m - 18b_1^2b_2m + 6b_1b_2^2m + (9b_1 - 6b_2)H''(0) + H^{(3)}(0)}{6b_1^3}u_n^3 + \frac{R^{(4)}(0)}{24}u_n^4 + 2mw_n + \frac{P''(0)}{2}w_n^2 + \frac{P^{(3)}(0)}{6}w_n^3\right], \quad (32)$$

where $\lambda_n = \frac{f(x_n)}{f'(x_n)}$, $b_1, b_2, \alpha_2, H''(0), H^{(3)}(0), H^{(4)}(0)$, $R^{(4)}(0)$, $P''(0)$ and $P^{(3)}(0)$ are free parameters.

Sub cases of the given scheme (32):

(1) Let us consider $H''(0) = H^{(3)}(0) = H^{(4)}(0) = R^{(4)}(0) = P''(0) = P^{(3)}(0) = 0$ in expression (32), we obtain

$$
\begin{aligned}
y_n &= x_n - m\lambda_n,\\
z_n &= y_n - m\left(\frac{u_n}{1-u_n}\right)\lambda_n[1+t_nb_1],\\
x_{n+1} &= z_n - mu_n\left(\frac{v_n}{1-v_n+\alpha_2 v_n^2}\right)\lambda_n\\
&\left[1+2u_n+\frac{3b_1-b_2}{b_1}u_n^2+\frac{2b_1^2-3b_1b_2+b_2^2}{b_1^2}u_n^3+2w_n\right]
\end{aligned}
\tag{33}
$$

(2) Moreover, a combination of rational functions produce another optimal eighth-order scheme as follows:

$$
\begin{aligned}
y_n &= x_n - m\lambda_n,\\
z_n &= y_n - m\left(\frac{u_n}{1-u_n}\right)\lambda_n[1+t_nb_1],\\
x_{n+1} &= z_n - u_n\left(\frac{v_n}{1-v_n+\alpha_2 v_n^2}\right)\lambda_n\\
&\left[\frac{k_1+k_2u_n}{1+k_3u_n+k_4u_n^2}+\frac{\tau_1+w_n+w_n^2}{1+\tau_2w_n}\right],
\end{aligned}
\tag{34}
$$

where

$$
\begin{cases}
k_1 = m - P(0),\\
k_2 = \dfrac{\left(-2b_1^2+b_1b_2+b_2^2\right)m^2+\left(8b_1^2+2b_1b_2-2b_2^2\right)mP(0)+\left(2b_1^2-3b_1b_2+b_2^2\right)P(0)^2}{b_1\left((b_1+b_2)m+(3b_1-b_2)P(0)\right)},\\
k_3 = \dfrac{-4b_1^2m-b_1b_2m+b_2^2m-\left(2b_1^2-3b_1b_2+b_2^2\right)P(0)}{b_1\left(b_1m+b_2m+(3b_1-b_2)P(0)\right)},\\
k_4 = \dfrac{\left(5b_1^2-b_2^2\right)m}{b_1\left(b_1m+b_2m+(3b_1-b_2)P(0)\right)},\\
\tau_1 = P(0),\\
\tau_2 = \dfrac{1-2m}{\tau_1}.
\end{cases}
\tag{35}
$$

(3) Now, we suggest another rational function forms of weight functions satisfying the conditions as follow:

$$
\begin{aligned}
y_n &= x_n - m\lambda_n,\\
z_n &= y_n - m\left(\frac{u_n}{1-u_n}\right)\lambda_n[1+t_nb_1],\\
x_{n+1} &= z_n - u_n\left(\frac{v_n}{1-v_n+\alpha_2 v_n^2}\right)\lambda_n\left[\frac{1+\rho_1u_n+\rho_2u_n^2}{\rho_3+\rho_4u_n}\right.\\
&\left.+\frac{l_1+(l_1+2m)w_n}{1+w_n}\right],
\end{aligned}
\tag{36}
$$

where

$$
\begin{cases}
\rho_1 = \dfrac{4b_1^2m+b_1b_2m-b_2^2m+\left(2b_1^2-3b_1b_2+b_2^2\right)P(0)}{b_1(3b_1-b_2)(m-P(0))},\\
\rho_2 = \dfrac{\left(5b_1^2-b_2^2\right)m}{b_1(3b_1-b_2)(m-P(0))},\\
\rho_3 = \dfrac{1}{m-P(0)},\\
\rho_4 = \dfrac{-2b_1^2+3b_1b_2-b_2^2}{b_1(3b_1-b_2)(m-P(0))},\\
l_1 = P(0).
\end{cases}
\tag{37}
$$

Remark 2.1. Furthermore, it is important to note that weight functions $H(t_n)$, $R(u_n)$ and $P(w_n)$ play a significant role in the construction of eighth-order schemes. Therefore, it is customary to display different choices of weight functions, provided they must satisfy all the conditions in Theorem 2.1. Hence, we have mentioned above some special cases of new eighth-order schemes (33), (34) and (36) having simple body structures so that they can be easily implemented in the numerical experiments.

3 Dynamical analysis

In order to arrange this analysis, we apply our proposed families on the nonlinear function $p(z) = (z-a)^m(z-b)^m$, with two multiple roots of multiplicity m. This is the most simple nonlinear function containing two m-multiple roots and, although the results cannot be directly extrapolated to any nonlinear function, several analysis on different nonlinear problems confirm, in the numerical section, these results.

Firstly, we recall some dynamical concepts that we use in this paper (see, for example, [17]). Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function, where $\hat{\mathbb{C}}$ is the Riemann sphere, the orbit of a point $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}, \tag{38}$$

where R^k denotes the k-th composition of the map R with itself.

We analyze the phase plane of the map R by classifying the starting points from the asymptotic behavior of the orbits. In these terms, a point z_0 is called fixed point of R if $R(z_0) = z_0$; it is a periodic point of period $p > 1$ if $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, for $k < p$.

Moreover, a fixed point of R, z_0, is called attracting if $|R'(z_0)| < 1$, or superattracting if $|R'(z_0)| = 0$; it is repulsive if $|R'(z_0)| > 1$ and parabolic if $|R'(z_0)| = 1$.

Indeed, when R depends also on one or several parameters, then $\left|R'(z_0, \alpha_1, \ldots, \alpha_k)\right|$ is not an scalar, but a function of α_i, for $i = 1, 2, \ldots, k$. Then, $\left|R'(z_0, \alpha_1, \ldots, \alpha_k)\right|$ is called stability function of the fixed point and it gives us the character of the fixed point in terms of the value of α_i, for $i = 1, 2, \ldots, k$.

On the other hand, let us also remark that when fixed points are found such that they are not equivalent to the roots of the polynomial $p(z)$, then they are called strange fixed points.

The basin of attraction of an attractor α is defined as:

$$\mathcal{A}(\alpha) = \left\{z_0 \in \widehat{\mathbb{C}} : R^n(z_0) \to \alpha,\ n \to \infty\right\}. \tag{39}$$

The Fatou set of the rational function R, is the set of points $z \in \widehat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic point or infinity). Its complement in $\widehat{\mathbb{C}}$ is the Julia set, $\mathcal{J}(R)$. So the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basin of attraction belong to the Julia set.

Now, we are going to study the qualitative behavior of the rational functions related to the different special cases applied on low-degree polynomials. When class (33) with $b_1 = 1$ and $b_2 = -2$ is applied on polynomial $p(z)$, a rational function $O_p(z)$ is obtained, depending on the parameter of the class and also depending on the multiple roots a and b. To get a simpler operator, we use the conjugacy map (see the work of P. Blanchard in [17]) given by the Möbius transformation

$$M(z) = \frac{z-a}{z-b},\quad M^{-1}(z) = \frac{zb-a}{z-1},$$

with properties:

$$M(\infty) = 1,\quad M(a) = 0,\quad M(b) = \infty,$$

that yields a rational function that, being conjugated to $O_p(z)$ (and therefore, with equivalent dynamical behavior), does not longer depend on a and b:

$$M_1(z, \alpha_2) = z^8 \frac{N_1(z, \alpha_2)}{D_1(z, \alpha_2)},$$

being

$$\begin{aligned} N_1(z, \alpha_2) = {} & 35 + 48z + 167z^2 + 216z^3 + 327z^4 \\ & + 344z^5 + 322z^6 + 248z^7 + 157z^8 \\ & + 80z^9 + 31z^{10} + 8z^{11} + z^{12} \\ & + (1+z)^8 \alpha_2 \end{aligned}$$

and

$$\begin{aligned} D_1(z, \alpha_2) = {} & 1 + 8z + 31z^2 + 80z^3 + 8z^{11} \\ & (6 + \alpha_2) + 8z^5(31 + \alpha_2) \\ & + z^{12}(35 + \alpha_2) + z^4(157 + \alpha_2) \\ & + 14z^6(23 + 2\alpha_2) + 8z^9(27 + 7\alpha_2) \\ & + 8z^7(43 + 7\alpha_2) + z^{10}(167 + 28\alpha_2) \\ & + z^8(327 + 70\alpha_2). \end{aligned}$$

Let us remark that this rational operator does not even depend on the multiplicity m.

On the other hand, when the class described as Case 2, (34) (for $m = 2$), is applied on $p(z)$ and the resulting rational function is conjugated by a Möbius transformation, the following operator is obtained:

$$M_2(z, \alpha_2) = z^8 \frac{N_2(z, \alpha_2)}{D_2(z, \alpha_2)},$$

where two polynomials of degree 14 appear in both numerator and denominator,

$$\begin{aligned} N_2(z, \alpha_2) = {} & 14 + 52z + 107z^2 + 176z^3 + 235z^4 \\ & + 280z^5 + 306z^6 + 304z^7 + 278z^8 \\ & + 224z^9 + 152z^{10} + 88z^{11} + 40z^{12} \\ & + 12z^{13} + 2z^{14} + 2(1+z)^4(1 + 2z \\ & + 3z^2 - 2z^3 + 3z^4 + 2z^5 + z^6)\, \alpha_2 \end{aligned}$$

and

$$\begin{aligned} D_2(z, \alpha_2) = {} & 2 + 12z + 40z^2 + 88z^3 + 2z^{14}(7 + \alpha_2) \\ & + 2z^4(76 + \alpha_2) + 4z^{13}(13 + 3\alpha_2) \\ & + 4z^5(56 + 3\alpha_2) + 8z^9(35 + 4\alpha_2) \\ & + 4z^{11}(44 + 13\alpha_2) + 4z^7(76 + 13\alpha_2) \\ & + z^{12}(107 + 34\alpha_2) + z^6(278 + 34\alpha_2) \\ & + z^{10}(235 + 44\alpha_2) + z^8(306 + 44\alpha_2). \end{aligned}$$

Finally, is the family defined as Case 3 (36) is applied on polynomial $p(z)$, after a conjugation by the Möbius map, the rational function

$$M_3(z, \alpha_2) = z^8 \frac{N_3(z, \alpha_2)}{D_3(z, \alpha_2)},$$

is obtained, being

$$\begin{aligned} N_3(z, \alpha_2) = {} & (1 + z^2)^2(31 + 108z + 245z^2 + 435z^3 \\ & + 601Z^4 + 682Z^5 + 648Z^6 + 516Z^7 \\ & + 347Z^8 + 189Z^9 + 83Z^{10} + 28z^{11} + 5z^{12} \\ & + (1+z)^4(5 + 8z + 16z^2 + 29z^3 + 20z^4 \\ & + 29z^5 + 16z^6 + 8z^7 + 5z^8)\, \alpha_2 \end{aligned}$$

and

$$\begin{aligned} D_3(z, \alpha_2) = {} & 5 + 28z + 93z^2 + 245z^3 + z^{16}(31 + 5\alpha_2) \\ & + z^4(518 + 5\alpha_2) + 4z^{15}(27 + 7\alpha_2) \\ & + 7z^{13}(93 + 23\alpha_2) + 3z^6(475 + 26\alpha_2) \\ & + z^5(922 + 28\alpha_2) + 5z^{11}(332 + 71\alpha_2) \\ & + 5z^9(463 + 71\alpha_2) + z^{14}(307 + 78\alpha_2) \\ & + z^7(1903 + 161\alpha_2) + z^{12}(1122 + 269\alpha_2) \\ & + z^8(2244 + 269\alpha_2) + z^{10}(2095 + 384\alpha_2). \end{aligned}$$

Let us remark that it does not depend on the multiplicity m. In what follows, the dynamical properties of rational functions $M_i(z, \alpha_2)$, $i = 1, 2, 3$, will be analyzed.

By solving equation $M_i(z, \alpha_2) = z$, $i = 1, 2, 3$, the fixed points of the respective rational function are obtained. Among them, $z = 0$ and $z = \infty$, coming from the roots of the polynomial previous to the Möbius map. The asymptotic behavior of all the fixed points plays a key role in the stability of the iterative methods involved, as the convergence to fixed points different from the roots means an important drawback for an iterative method; so, we proceed below with this analysis.

Remark 3.1. A direct result of the Möbius transformation applied on these rational functions (and also those obtained by applying any other iterative method on $p(z)$) is the conjugacy by the inverse,

$$\frac{1}{M_i(z,\alpha_2)} = M_i\left(\frac{1}{z},\alpha_2\right),\quad i = 1,2,3.$$

Immediate consequences of this property are:

(a) If z^* is a fixed point of $M_i(z,\alpha_2)$, i = 1, 2, 3, that is $M_i(z^\star,\alpha_2) = z^\star$, i = 1, 2, 3, then also its conjugate $1/z^*$ is, $M_i(1/z^\star,\alpha_2) = 1/z^\star$.
(b) $z = 1$ is always an strange fixed point of $M_i(z,\alpha_2)$, $i = 1, 2, 3$, except maybe for some specific values of the parameter that simplify the operator.
(c) Given two conjugate fixed points, both have the same character, as their stability function coincide,

$$M'_i(1/z^\star,\alpha_2) = M'_i(z^\star,\alpha_2),\quad i = 1,2,3.$$

This is the case of the conjugate fixed points $z = 0$ and $z = \infty$ coming from the roots of the polynomial $p(z)$ previous to the Möbius map; they have the same asymptotic behavior. Indeed, as they are roots of $M_i(z,\alpha_2)$, $i = 1, 2, 3$, of multiplicity four, it can be concluded that $M'_i(0,\alpha_2) = M'_i(\infty,\alpha_2) = 0$, $i = 1, 2, 3$ and, therefore, they are superattracting fixed points. This is in concordance with the proven eighth-order of convergence of the proposed class of iterative methods.

3.1 Stability of strange fixed points

Firstly, in order to study the stability of the strange fixed points of $M_1(z,\alpha_2)$, we calculate its first derivative and evaluate it at every fixed point. Its stability function gives us information about the asymptotic behavior of the point. In our case, the stability of other fixed points than $z = 0$ or $z = \infty$ depends on the value of parameter α_2. We start this analysis with the strange fixed point coming from the divergence of the original method, $z = 1$. From this rational function, the following result can be stated.

Theorem 3.1. Rational function $M_1(z,\alpha_2)$ has $z = 1$ as an strange fixed point if $\alpha_2 \neq -\frac{31}{4}$. So, $z = 1$ is attracting if $\left|\alpha_2 - \frac{737}{60}\right| < \frac{17}{15}$, superattracting if $\alpha_2 = -12$, parabolic if $\left|\alpha_2 - \frac{737}{60}\right| = \frac{17}{15}$ and repulsive in other cases.

Proof. From the nature of Möbius transformation, $z = 1$ is a fixed point of the resulting rational function (except for specific values of the parameters that can make the rational function simpler). By definition, its stability is directly deduced from the derivative of the rational operator: The character of $z = 1$ is given by the stability function

$$\left|M'_1(1,\alpha_2)\right| = \left|-16\frac{12+\alpha_2}{31+4\alpha_2}\right|.$$

and then the thesis of the theorem is straightforward.

The relevance of this result is related with the role of $z = 1$ in Möbius map: as $M(\infty) = 1$, $z = 1$ in the conjugate operator corresponds to the divergence of the original one. So, an attracting or parabolic behavior derives in divergent behavior of the iterative process, and it must be avoided.

According to Theorem 3.1, there exists an infinite set of iterative methods in the original family (33) whose behavior on $p(z)$ does not include the divergence.

It can be checked that rational function $M_2(z,\alpha_2)$ is simplified for the specific value of parameter $\alpha_2 = -\frac{227}{32}$, that is, $z = 1$ (coming from the divergence of the original method) is not an strange fixed point of $M_2\left(z,-\frac{227}{32}\right)$ as is stated in the following result.

Theorem 3.2. Rational function $M_2(z,\alpha_2)$ has $z = 1$ as an strange fixed point if $\alpha_2 \neq -\frac{227}{32}$. In this case, $z = 1$ is attracting if $\left|\alpha_2 - \frac{5917}{480}\right| < \frac{157}{120}$, superattracting if $\alpha_2 = -12$, parabolic if $\left|\alpha_2 - \frac{5917}{480}\right| = \frac{157}{120}$ and repulsive in other cases.

Regarding Case 3, the following result sets the asymptotic behavior of the divergence ($z = 1$) depending on the value of the parameter.

Theorem 3.3. Rational function $M_3(z,\alpha_2)$ has $z = 1$ as an strange fixed point if $\alpha_2 \neq -\frac{1959}{272}$. In this case, z = 1 is attracting if $\left|\alpha_2 + \frac{3351}{272}\right| < \frac{87}{68}$, superattracting if $\alpha_2 = -12$, parabolic if $\left|\alpha_2 + \frac{3351}{272}\right| = \frac{87}{68}$ and repulsive in other cases.

The rest of strange fixed points of these rational functions and their stability (depending on their respective parameters) play also an important role in the general analysis of our class. As the stability of strange fixed point $z = 1$ has already been studied (see Theorems 3.1, 3.2 and 3.3), and $z = 0$ and $z = \infty$ are superattracting due to the eighth-order of convergence of the family, we focus ourselves in the calculation and analysis of the stability of the rest of strange fixed points.

Proposition 3.1. The strange fixed points of $M_1(z,\alpha_2)$ different from $z=1$ are

$$s1_i^+(\alpha_2) = \frac{s1_i(\alpha_2) + \sqrt{s1_i(\alpha_2)^2 - 4}}{2} = \frac{1}{s1_i^-(\alpha_2)}, i = 1, 2, \ldots, 9,$$

being $s1_i(\alpha_2)$, i = 1,2,…,9 the roots of the nineth-degree polynomial $r1(t) = t^9 + t^8 + 31t^7 + 112t^6 + (297 + \alpha_2)t^5 + (539 + 9\alpha_2)t^4 + (826 + 37\alpha_2)t^3 + (1145 + 92\alpha_2)t^2 + (1439 + 154\alpha_2)\,t + 1500 + 182\alpha_2$. From these 19 strange fixed points, only four can be attracting, two in a big area included in [–500, –200] × [–100, 100] and other four in region [–20, 20] × [–20, 20], as can be seen in Figure 1(a). Moreover, if $z = -\frac{3257}{95}$, the strange fixed points different from $z = 1$ are 16, being complex two-by-two conjugated.

The proof of this result, being numerical, is similar to those made in the previous subsection, by analyzing the value of the stability function of each one of the conjugate strange fixed points. Now, we present the corresponding result for rational function $M_2(z,\alpha_2)$.

Proposition 3.2. Rational function $M_2(z,\alpha_2)$ has as strange fixed points (different from $z = 1$) to

$$s2_i^+(\alpha_2) = \frac{s2_i(\alpha_2) + \sqrt{s2_i(\alpha_2)^2 - 4}}{2} = \frac{1}{s2_i^-(\alpha_2)}, i = 1, 2, \ldots, 10,$$

being $s2_i(\alpha_2)$, i = 1,2,…,10 the roots of the 10th-degree polynomial $r2(t) = 2t^{10} + 14t^9 + 34t^8 + 16t^7 + (-68 + 2\alpha_2)t^6 + (-98 + 14\alpha_2)t^5 + (12 + 36\alpha_2)t^4 + (64 + 28\alpha_2)t^3 + (-12 - 44\alpha_2)t^2 + (-23 - 96\alpha_2)t - 48\alpha_2$. From these 20 strange fixed points, four of them can be attracting in a small area included in [–10, 10] × [–10, 10], as can be seen in Figure 1(b). Moreover, if $z = -\frac{1309}{96}$, the strange fixed points different from $z = 1$ are 18, being complex two-by-two conjugated.

Regarding the rational function associated to Case 3, the stability of its strange fixed points different from $z = 1$ is stated in the following result.

Proposition 3.3. Rational function $M_3(z,\alpha_2)$ has as strange fixed points (different from $z = 1$) to

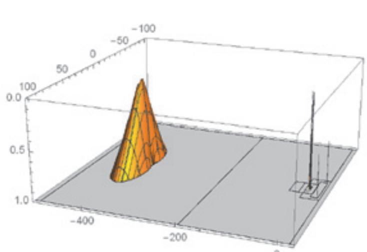
$$s3_i^+(\alpha_2) = \frac{s3_i(\alpha_2) + \sqrt{s3_i(\alpha_2)^2 - 4}}{2} = \frac{1}{s3_i^-(\alpha_2)}, i = 1, 2, \ldots, 11,$$

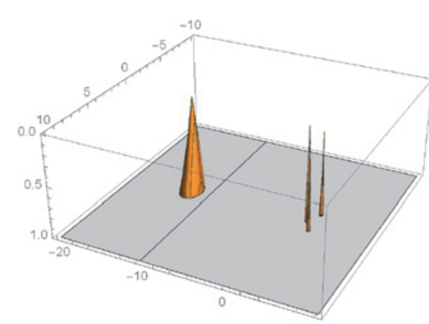
being $s3_i(\alpha_2)$, i = 1, 2, …, 11 the roots of the 11th-degree polynomial $r3(t) = 5t^{11} + 33t^{10} + 71t^9 + 41t^8 + (-25 + 5\alpha_2)t^7 + (-2 + 33\alpha_2)t^6 + (30 + 76\alpha_2)t^5 + (12 + 69\alpha_2)\,t^4 + (5 + 23\alpha_2)t^3 + (8 + 14\alpha_2)t^2 + (4620 + 4\alpha_2)t - 8\alpha_2$. From these strange fixed points, three can be attracting in small regions of [–20, 20] × [–20, 20] and another one can be attracting in a small area inside [350, 360] × [–10, 10]. The union of the stability functions of these strange fixed points appear in Figure 1(c).

The union of the respective stability functions of all the strange fixed point for each rational function $M_1(z,\alpha_2)$, $M_2(z,\alpha_2)$ and $M_3(z,\alpha_2)$, can be observed in Figure 1

3.2 Dynamical planes

The dynamical plane associated to a value of the parameter, that is, obtained by iterating an element of the family under study, is generated by using each point of the complex plane as initial estimation (we have used a mesh of 800 × 800 points). We paint in blue the points whose orbit converges to infinity, in orange the points converging to zero (with a tolerance of 10^{-3}), in other colors (green, red, etc.) those points whose orbit converges to one of the strange fixed points (all fixed points appear marked as a white star in the figures if they are attracting or by a white circle if they are repulsive). Moreover, a point appears in black if it reaches the maximum number of 200 iterations


(a) $M_1(z,\alpha_2)$


(b) $M_2(z,\alpha_2)$

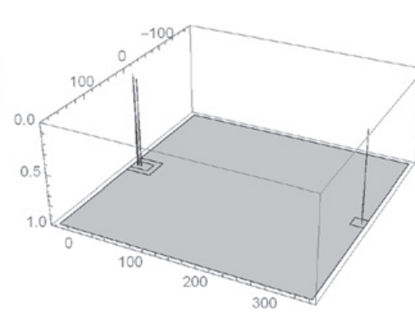

(c) $M_3(z,\alpha_2)$

Figure 1: Union of stability functions of all the strange fixed points of the rational functions under study.

without converging to any of the fixed points. The routines used appear in [18].

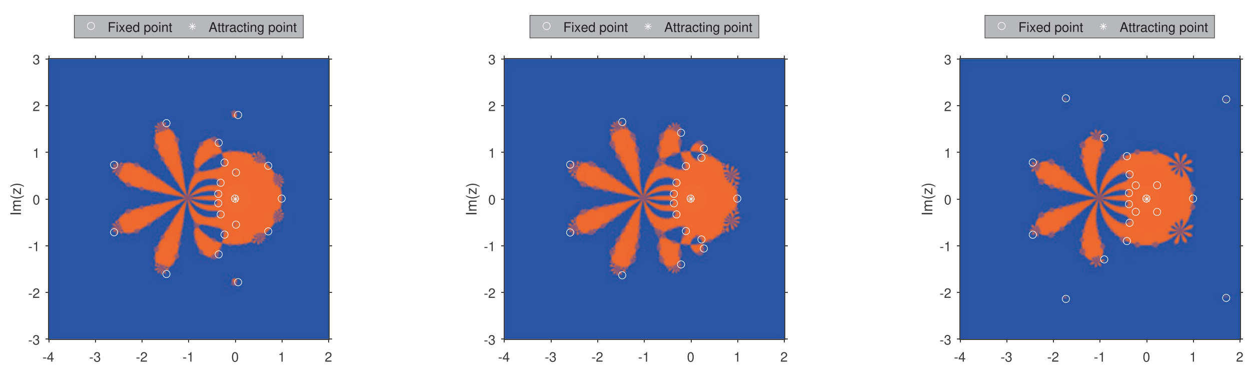
From these results, several stable elements can be selected from the rational class of iterative methods: those corresponding to values of parameter α_2 where there no exist attracting strange fixed nor periodic points. The dynamical planes corresponding to some of them can be seen at Figure 1. In Figure 2(a), the case of Q_3 null is presented, whose main characteristic is the existence of several poles of the rational function, giving as a result ”flowers” of slower convergence.

In a similar way, by means of the previous analysis, some elements can be selected as unstable members of the rational class of iterative methods. In Figure 3 some of them are presented, as in Figure 3(a), where the basin of attraction of $z = 1$ appears in green, being superattracting for this value of α_2. The case of Figure 3(b) corresponds to two simultaneously attracting strange fixed points $s1_3^{\pm}(-400) = -0.6036 \pm 0.7973i$, whose basins of attraction appear in green and yellow color. Finally, Figure 3(c) is related to the value $\alpha_2 = 13.35 - 15.5i$, lying in the complex area where $s1_7^{+}(13.35 - 15.5i) = -1.0421 + 1.8229i$ and $s1_7^{-}(13.35 - 15.5i) = -0.2364 - 0.4135i$ are simultaneously attracting.

Now, the corresponding analysis of the basins of attraction different from those of $z = 0$ and $z = \infty$ corresponding to $M_2(z, \alpha_2)$ for different values of α_2 leading to stable or unstable behavior is given below.

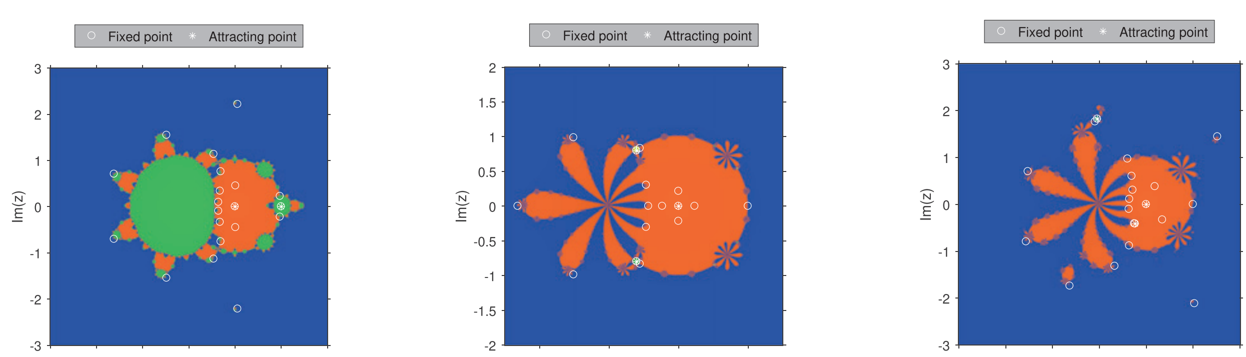
In Figure 4, the dynamical planes associated to the same values of parameter α_2 are used, as they also correspond to the wide are of the complex plane where all the strange fixed points are repulsive. In all of them, only two basins of attraction appear, corresponding to $z = 0$ and $z = \infty$, that is, to the original roots of polynomial $p(z)$.

Figure 5 corresponds to the dynamical planes obtained for several values of α_2 where the strange fixed points are superattracting, as it is the case of $z = 1$ for $\alpha_2 = -12$ (Figure 5(a)), or attracting $s3^{\pm}(2.4) = -0.8554 \pm 0.5180i$, that appear with their own basins of attraction in Figure 5(b) or $s2_3^{+}(3.25 + 1.25i) = -0.7669 + 1.0356i$ and $s2_3^{-}(3.25 + 1.25i) = -0.4618 - 0.6236i$ that are simultaneously attracting for $\alpha_2 = 3.25 + 1.25i$, see Figure 5(c).



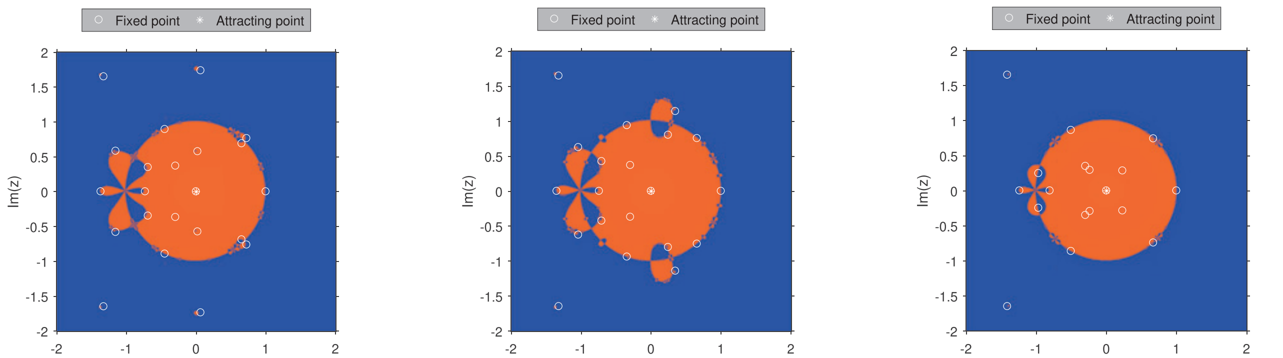
(a) α_2 =-3 (b) $\alpha_2 = 0$ (c) $\alpha_2 = 50$

Figure 2: Dynamical planes corresponding to stable performance of $M_1(z, \alpha_2)$.



(a) $\alpha_2 = -12$ (b) $\alpha_2 = -400$ (c) $\alpha_2 = 13.35 - 15.5i$

Figure 3: Dynamical planes corresponding to unstable performance of $M_1(z, \alpha_2)$.

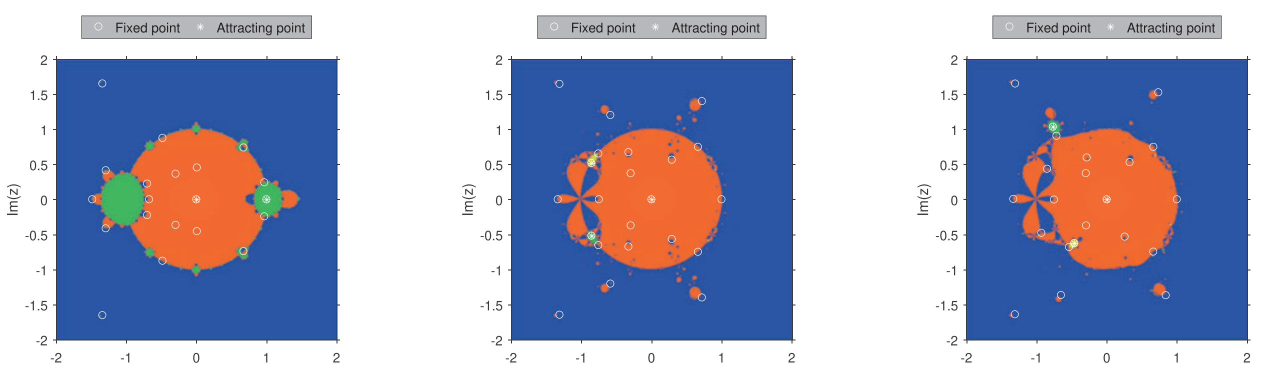


(a) $\alpha_2 = -3$ **(b)** $\alpha_2 = 0$ **(c)** $\alpha_2 = 50$

Figure 4: Dynamical planes corresponding to stable performance of $M_2(z, \alpha_2)$.

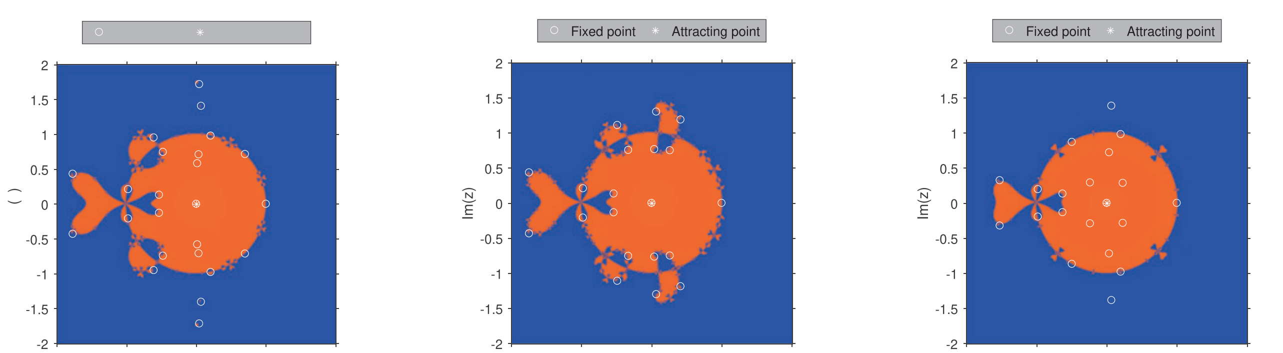
Finally, we present the dynamical planes associated to $M_3(z, \alpha_2)$, confirming the previous analysis of the stability of its strange fixed points.

The same values of α_2 can be used to visualize the stable performance of $M_3(z, \alpha_2)$ in wide areas of the complex plane (see Figure 6). Regarding the unstable cases, we can see in Figure 7 the dynamical plane corresponding to the value of α_2 assuring the behavior of $z = 1$ (original divergence of the method) as superattracting fixed point (Figure 7(a)), or those which result in strange attracting fixed points, as $s2_2^+(354) = -1.4169$ and $s2_2^-(354) = -0.7058$ that are simultaneously attracting for $\alpha_2 = 354$ (see Figure 5(b)) or $s2^{\pm}(-6.5) = -0.6203 \pm 0.7844i$, that appear with their own basins of attraction in Figure 5(c).



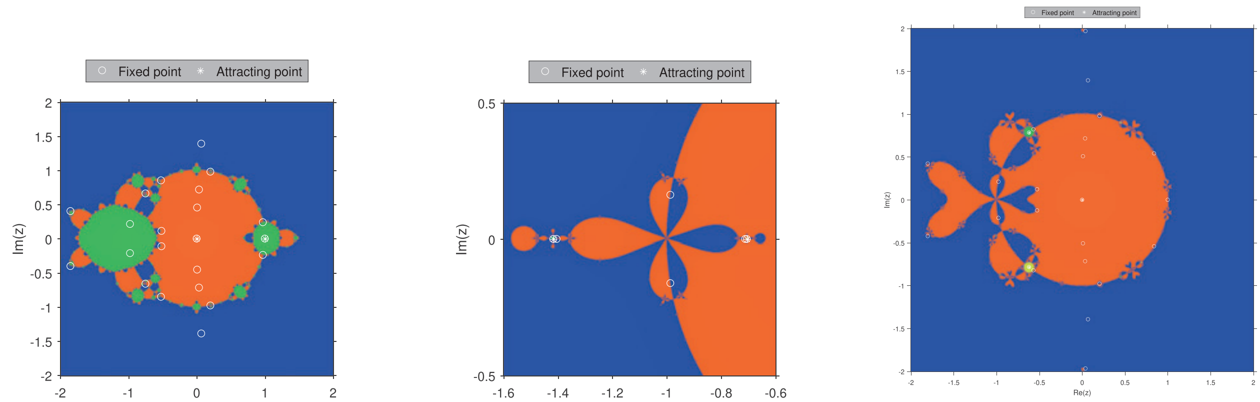
(a) $\alpha_2 = -12$ **(b)** $\alpha_2 = 2.4$ **(c)** $\alpha_2 = 3.25 + 1.25i$

Figure 5: Dynamical planes corresponding to unstable performance of $M_2(z, \alpha_2)$.



(a) $\alpha_2 = -3$ **(b)** $\alpha_2 = 0$ **(c)** $\alpha_2 = 50$

Figure 6: Dynamical planes corresponding to stable performance of $M_3(z, \alpha_2)$.



(a) $\alpha_2 = -12$ (b) $\alpha_2 = 354$ (c) $\alpha_2 = -6.5$

Figure 7: Dynamical planes corresponding to unstable performance of $M_3(z, \alpha_2)$.

4 Numerical experiments

In this section, we will check the computational aspects of the following proposed methods: expression (33) for $(b_1 = 1, b_2 = -2, \alpha_2 = -3)$, family (34) for $(b_1 = 1, b_2 = -2, \alpha_2 = -3, P(0) = \tau_1 = \frac{1}{2})$ and expression (36) for $(b_1 = 1, b_2 = -2, \alpha_2 = -3, P(0) = l_1 = \frac{1}{2})$ denoted by *MM*1, *MM*2 and *MM*3, respectively, comparing them with some already existing techniques.

In this regard, we consider several test functions coming from real life problems which are mentioned in examples 4.1 to 4.6. We compare our proposed methods with optimal eighth-order method (5) given by Behl et al. [15] for $Q(h_n) = m(1 + 2h_n + 3h_n^2)$ and $G(h_n, t_n) = m\left(\frac{1+2t_n+3h_n^2+h_n(2+6t_n+h_n)}{1+t_n}\right)$ and the method (6) given by Zafar et al. [16] taking $H(u_n) = 6u_n^3 - u_n^2 + 2u_n + 1$, $P(v_n) = 1 + v_n$ and $G(w_n) = \frac{2w_n+1}{A_2P_0}$ for $(A_2 = P_0 = 1)$. We denote these methods by *OM* and *ZM*, respectively. We also compare our proposed methods with family of two-point sixth-order method given by Geum et al. in [13], out of them we choose the case 2A, which is given by:

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, \; m > 1,$$
$$x_{n+1} = y_n - \left[\frac{m + b_1u_n}{1 + a_1u_n + a_2s_n + a_3s_nu_n}\right]\frac{f(y_n)}{f'(y_n)}, \qquad (40)$$

where $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $s_n = \left(\frac{f'(y_n)}{f(x_n)}\right)^{\frac{1}{m-1}}$, $b_1 = \frac{2m}{m-1}$, $a_1 = -\frac{2m(m-2)}{m-1}$, $a_2 = 2(m-1)$ and $a_3 = 3$.

Finally, we compare them with the non-optimal family of sixth-order methods based on weight function approach presented by the same authors Geum et al. [14], out of them we consider the case 5YD, which is defined as follows:

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, \; m \geq 1,$$
$$w_n = x_n - m\left[\frac{(u_n - 2)(2u_n - 1)}{(u_n - 1)(5u_n - 2)}\right]\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - m\left[\frac{(u_n - 2)(2u_n - 1)}{(5u_n - 2)(u_n + v_n - 1)}\right]\frac{f(x_n)}{f'(x_n)}. \qquad (41)$$

We denote methods (40) and (41) by *GK*1 and *GK*2, respectively.

In the numerical tests presented in Tables 1–6, we have compared our methods with the known ones on the basis of approximated zeros, residual error of the involved functions, difference between the two consecutive iterations, asymptotic error constants. In Tables 1–6, we display the number of iteration indices (n), approximated zeros (x_n), absolute residual error of the corresponding function $(|f(x_n)|)$, error in the consecutive iterations $|x_{n+1} - x_n|$, computational order of convergence $\rho \approx \frac{\log|f(x_{n+1})/f(x_n)|}{\log|f(x_n)/f(x_{n-1})|}$, $n \geq 2$, (the details of this formula can be seen in [19]), $\left|\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^p}\right|$ (where p is either 6 or 8 corresponding to the considered iteration function), the estimation of asymptotic error constant $\eta \approx \lim_{n\to\infty}\left|\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^p}\right|$ at the last iteration. We have maintained 4096 significant digits of minimum precision to minimize the round off error.

As mentioned in the above paragraph, we calculate the values of all the constants and functional residuals up to several number of significant digits but we display the value of approximated zero x_n up to 25 significant digits although minimum 4096 significant digits are

Table 1: Convergence behavior of different iterative methods on the test function $f_1(x)$.

Methods	n	x_n	$\|f(x_n)\|$	$\|x_{n+1}-x_n\|$	ρ	$\left\|\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^p}\right\|$	η
$GK1$	0	1.8	2.0(−4)	4.9(−2)			
	1	1.750895258580091535641280	2.5(−8)	9.0(−4)		6.385691220(+4)	3.536522620(+7)
	2	1.750000000014299761415271	6.1(−24)	1.4(−11)		2.777396484(+7)	
	3	1.750000000000000000000000	2.7(−117)	3.0(−58)	5.9816	3.536522620(+7)	
$GK2$	0	1.8	2.0(−4)	5.0(−2)			
	1	1.750388172793891559741273	4.6(−9)	3.9(−4)		2.603237303(+4)	3.215020576(+6)
	2	1.750000000000010343224637	3.2(−30)	1.0(−14)		3.023468138(+6)	
	3	1.750000000000000000000000	4.6(−157)	3.9(−78)	5.9959	3.215020576(+6)	
OM	0	1.8	2.0(−4)	4.9(−2)			
	1	1.750388172319823575363680	9.9(−9)	5.7(−4)		1.599594295(+7)	1.462834362(+11)
	2	1.750000000000001356336629	5.5(−32)	1.4(−15)		3.750857339(+11)	
	3	1.750000000000000000000000	8.4(−218)	1.7(−108)	7.9903	1.462834362(+11)	
ZM	0	1.8	2.0(−4)	5.0(−2)			
	1	1.750388172319823575363680	4.6(−9)	3.9(−4)		1.057651892(+7)	1.178394347(+11)
	2	1.750000000000000051608567	8.0(−35)	5.2(−17)		1.001210273(+11)	
	3	1.750000000000000000000000	1.1(−240)	5.9(−120)	7.9928	1.178394347(+11)	
$MM1$	0	1.8	2.0(−4)	5.0(−2)			
	1	1.750078744729477065897963	1.9(−10)	7.9(−5)		2.041444221(+6)	1.754865398(+10)
	2	1.750000000000000000000000	1.9(−47)	2.5(−23)		1.705919057(+10)	
	3	1.750000000000000000000000	2.5(−343)	2.9(−171)	7.9991	1.754865398(+10)	
$MM2$	0	1.8	2.0(−4)	5.0(−2)			
	1	1.750023647624207742848767	1.7(−11)	2.4(−5)		6.076745870(+5)	2.545224623(+9)
	2	1.750000000000000000000000	1.8(−57)	2.5(−28)		2.526328798(+9)	
	3	1.750000000000000000000000	3.7(−425)	3.5(−212)	7.9998	2.545224623(+9)	
$MM3$	0	1.8	2.0(−4)	5.0(−2)			
	1	1.750031099258857162422275	2.9(−11)	3.1(−5)		8.001136411(+5)	2.11655213(+9)
	2	1.750000000000000000000000	1.0(−55)	1.8(−27)		2.095705097(+9)	
	3	1.750000000000000000000000	2.2(−411)	2.7(−205)	7.9997	2.11655213(+9)	

available with us. The absolute residual error in the function $|f(x_n)|$ and error in the consecutive iterations $|x_{n+1} - x_n|$ are displayed up to 2 significant digits with exponent power which are mentioned in Tables 1–6. Moreover, computational order of convergence is up to 5 significant digits. In addition, we also display $\left|\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^p}\right|$ and η up to 10 significant digits. From Tables 1–6, it can be observed that the smaller asymptotic error constant implies that the corresponding method converge faster than the other ones. Although, it may happen in some cases that the method have smaller residual errors and smaller errors difference between two consecutive iterations but have also larger asymptotic error constant. All computations in numerical experiments have been carried out with *Mathematica* 10 programming package using multiple precision arithmetic. Further, the meaning of $a(\pm b)$ is $a \times 10^{(\pm b)}$ in Tables 1–6.

Example 4.1. (van der Waals equation of state):

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT,$$

where a and b are explains the behavior of a real gas by introducing in the ideal gas equations two parameters, a and b (known as van der Wall's constants), specific for each gas. The determination of the volume V of the gas in terms of the remaining parameters requires the solution of a nonlinear equation in V

$$PV^3 - (nbP + nRT)V^2 + an^2V - abn^2 = 0.$$

Given the constants a and b of a particular gas, one can find values for n, P and T, such that this equation has a three roots. By using the particular values, we obtain the following nonlinear function

$$f_1(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675,$$

having three zeros and one of them is a multiple zero $\alpha = 1.75$ of multiplicity of order two and other one is a simple zero $\alpha = 1.72$. However, our desired root is $\alpha = 1.75$.

Example 4.2. (Fractional conversion in a chemical reactor):

Let us consider the following expression (please, see [20] for more details of this problem)

Table 2: Convergence behavior of different iterative methods on the test function $f_2(x)$.

Methods	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	ρ	$\left\lvert\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^p}\right\rvert$	η
*GK*1	0	0.76	*	*			
	1		*	*		*	*
	2		*	*		*	
	3		*	*	*	*	
*GK*2	0	0.76	2.2(−1)	2.6(−3)			
	1	0.7573962460753336221899798	1.4(−8)	1.8(−10)		5.725910242(+5)	5.257130496(+5)
	2	0.7573962462537538794596413	1.4(−51)	1.7(−53)		5.257130467(+5)	
	3	0.7573962462537538794596413	1.0(−309)	1.3(−311)	6.0000	5.257130496(+5)	
OM	0	0.76	2.2(−1)	2.6(−3)			
	1	0.7573962463137703385994168	4.8(−9)	6.0(−11)		2.840999693(+10)	3.013467463(+10)
	2	0.7573962462537538794596413	4.0(−70)	5.1(−72)		3.013467461(+10)	
	3	0.7573962462537538794596413	1.1(−558)	1.3(−560)	8.0000	3.013467463(+10)	
ZM	0	0.76	2.2(−1)	2.6(−3)			
	1	0.7573962463048948508621891	4.1(−9)	5.1(−11)		2.420860580(+10)	3.421344786(+10)
	2	0.7573962462537538794596413	1.3(−70)	1.6(−72)		3.421344762(+10)	
	3	0.7573962462537538794596413	1.2(−562)	1.5(−564)	8.0000	3.421344786(+10)	
*MM*1	0	0.76	2.2(−1)	2.6(−3)			
	1	0.7573962462529556670756109	6.4(−11)	8.0(−13)		3.778498010(+8)	3.801147141(+8)
	2	0.7573962462537538794596413	5.0(−87)	6.3(−89)		3.801147141(+8)	
	3	0.7573962462537538794596413	7.2(−696)	9.0(−698)	8.0000	3.801147141(+8)	
*MM*2	0	0.76	2.2(−1)	2.6(−3)			
	1	0.7573962462537861829618272	2.6(−12)	3.2(−14)		1.529150906(+7)	1.263385529(+7)
	2	0.7573962462537538794596413	1.2(−99)	1.5(−101)		1.263385529(+7)	
	3	0.7573962462537538794596413	2.6(−798)	3.2(−800)	8.0000	1.263385529(+7)	
*MM*3	0	0.76	2.2(−1)	2.6(−3)			
	1	0.7573962462537905009805658	2.9(−12)	3.7(−14)		1.733552963(+7)	1.446059282(+7)
	2	0.7573962462537538794596413	3.7(−99)	4.7(−101)		1.446059282(+7)	
	3	0.7573962462537538794596413	2.1(−794)	3.3(−796)	8.0000	1.446059282(+7)	

$$f_2(x) = \frac{x}{1-x} - 5\log\left[\frac{0.4(1-x)}{0.4-0.5x}\right] + 4.45977. \tag{42}$$

In the above expression, x represents the fractional conversion of species A in a chemical reactor. Since, there is no physical meaning of above fractional conversion if x is less than zero or greater than one. In this sense, x is bounded in the region $0 \le x \le 1$. In addition, our required zero (that is simple) to this problem is $\alpha \approx 0.757396246253753879459641297929$. Moreover, it is interesting to note that the above expression is undefined in the region $0.8 \le x \le 1$ which is very close to our desired zero. Furthermore, there are some other properties to this function which make the solution more difficult. The derivative of the above expression is very close to zero in the region $0 \le x \le 0.5$ and there is an infeasible solution for $x = 1.098$.

We can see that the new methods possess smaller residual error and difference between the consecutive approximations in comparison to the existing ones. Moreover, the numerical estimation of the order of convergence coincide with the theoretical one in all cases. In Table 2, * means that the corresponding method does not converge to the desired root.

Example 4.3. (Continuous stirred tank reactor (CSTR)):

In our third example, we consider the isothermal continuous stirred tank reactor (CSTR) problem. The following reaction scheme develops in the reactor (see [21] for more details):

$$\begin{aligned} A + R &\to B \\ B + R &\to C \\ C + R &\to D \\ D + R &\to E, \end{aligned} \tag{43}$$

where, components A and R are fed to the reactor at rates of Q and q-Q respectively. The problem was analyzed in detail by Douglas [22] in order to design simple feedback control systems. In the analysis, he gave the following equation for the transfer function of the reactor:

$$K_C \frac{2.98(x+2.25)}{(s+1.45)(s+2.85)^2(s+4.35)} = -1, \tag{44}$$

where K_C is the gain of the proportional controller. The control system is stable for values of K_C that yields roots of the transfer function having negative real part. If we choose $K_C = 0$, we get the poles of the open-loop transfer function as roots of the nonlinear equation:

Table 3: Convergence behavior of different iterative methods on the test function $f_3(x)$.

Methods	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	ρ	$\left\lvert\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^p}\right\rvert$	η
*GK*1	0	−3.0	4.7(−2)	1.5(−1)			
	1	−2.850032149435759899649078	2.2(−9)	3.2(−5)		2.826079363(+0)	4.198827967(−5)
	2	−2.850000000000000000000000	4.5(−63)	4.6(−32)		4.191188565(−5)	
	3	−2.850000000000000000000000	3.6(−385)	4.1(−193)	6.0000	4.198827967(−5)	
*GK*2	0	−3.0	4.7(−2)	1.5(−1)			
	1	−2.845530536829933778640841	4.2(−5)	4.5(−3)		3.291554609(+2)	1.360955722(−3)
	2	−2.850002074441970615144759	9.0(−12)	2.1(−6)		2.595135041(+8)	
	3	−2.850000000000000000000000	2.5(−74)	1.1(−37)	9.3846	1.360955722(−3)	
OM	0	−3.0	4.7(−2)	1.6(−1)			
	1	−2.844042602118935658056506	7.5(−5)	6.0(−3)		1.703635813(+4)	6.783289282(−4)
	2	−2.850005050121091781574571	5.4(−11)	5.1(−6)		3.161585672(+12)	
	3	−2.850000000000000000000000	1.7(−91)	2.9(−46)	13.102	6.783289282(−4)	
ZM	0	−3.0	4.7(−2)	1.6(−1)			
	1	−2.840827596075196247341513	1.8(−4)	9.2(−3)		2.230697732(+4)	3.402776481(−4)
	2	−2.850019022777759525868734	7.6(−10)	1.9(−5)		3.734311208(+11)	
	3	−2.850000000000000000000000	7.1(−83)	5.8(−42)	13.609	3.402776481(−4)	
*MM*1	0	−3.0	4.7(−2)	1.5(−1)			
	1	−2.847981610389184901653897	8.6(−6)	2.0(−3)		7.077438026(+3)	9.674841648(−4)
	2	−2.850000186344162045752608	7.3(−14)	1.9(−7)		6.760129660(+14)	
	3	−2.850000000000000000000000	4.2(−114)	1.4(−57)	12.423	9.674841648(−4)	
*MM*2	0	−3.0	4.7(−2)	1.5(−1)			
	1	−2.847982098238578815439951	8.6(−6)	2.0(−3)		7.075908888(+3)	9.675353587(−4)
	2	−2.850000186252333907669066	7.3(−14)	1.9(−7)		6.769878897(+14)	
	3	−2.850000000000000000000000	4.1(−114)	1.4(−57)	12.423	9.675353587(−4)	
*MM*3	0	−3.0	4.7(−2)	1.5(−1)			
	1	−2.847981540231008673038257	8.6(−6)	2.0(−3)		7.077657926(+3)	9.675368192(−4)
	2	−2.850000186357369916831709	7.3(−14)	1.9(−7)		6.758728963(+14)	
	3	−2.850000000000000000000000	4.2(−114)	1.4(−57)	12.423	9.675368192(−4)	

$$f_3(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875 = 0 \tag{45}$$

given as: $x = -1.45, -2.85, -2.85, -4.35$. So, we see that there is one multiple root $x = -2.85$ with multiplicity 2.

Example 4.4. Let us consider another nonlinear test function from [3], which is given as follows:

$$f_4(x) = \left((x-1)^3 - 1\right)^{50}.$$

The above function has a multiple zero at $x = 2$ of multiplicity 50.

Table 4 shows the numerical results for this example. It can be observed from the numerical tests showed in this table that results are very good for all the methods, being lower the residuals at the newly proposed methods. Moreover, the asymptotic error constant (η) displayed in the last column of Table 4 is large for the methods *OM* and *ZM* in comparison to the other schemes.

Example 4.5. (Planck's radiation law problem):

Now, we consider the following Planck's radiation law problem which calculates the energy density within an isothermal blackbody and is given by [23]:

$$\Psi(\lambda) = \frac{8\pi ch\lambda^{-5}}{e^{\frac{ch}{\lambda BT}} - 1}, \tag{46}$$

where λ is the wavelength of the radiation, T is the absolute temperature of the blackbody, B is the Boltzmann constant, h is the Planck's constant and c is the speed of light. We are interested in determining wavelength λ which corresponds to maximum energy density $\Psi(\lambda)$.

Further, $\Psi'(\lambda) = 0$ implies that the maximum value of Ψ occurs when

Table 4: Convergence behavior of different iterative methods on the test function $f_4(x)$.

Methods	n	x_n	$\|f(x_n)\|$	$\|x_{n+1}-x_n\|$	ρ	$\left\|\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^p}\right\|$	η
*GK*1	0	2.1	9.8(−25)	1.0(−1)			
	1	2.000002777374011867781357	1.1(−254)	2.8(−6)		2.777836885(+00)	5.504789671(+00)
	2	2.000000000000000000000000	9.6(−1607)	2.5(−33)		5.504677538(+00)	
	3	2.000000000000000000000000	4.5(−9719)	1.4(−195)	6.0000	5.504789671(+00)	
*GK*2	0	2.1	9.8(−25)	1.0(−1)			
	1	2.000000200989638086020762	1.0(−311)	2.0(−7)		2.009920619(−1)	2.777777778(−1)
	2	2.000000000000000000000000	9.8(−2014)	1.8(−41)		2.777775861(−1)	
	3	2.000000000000000000000000	7.3(−12226)	1.0(−245)	6.0000	2.777777778(−1)	
OM	0	2.1	9.8(−25)	1.0(−1)			
	1	2.000000785189010712446522	4.0(−282)	7.9(−7)		7.852383342(+1)	2.269259259(+2)
	2	2.000000000000000000000000	4.4(−2301)	3.3(−47)		2.269242109(+2)	
	3	2.000000000000000000000000	8.3(−18453)	3.0(−370)	8.0000	2.269259259(+2)	
ZM	0	2.1	9.8(−25)	1.0(−1)			
	1	2.000000477890417235498042	6.6(−293)	4.8(−7)		4.779086880(+1)	2.084074074(+2)
	2	2.000000000000000000000000	3.4(−2389)	5.7(−49)		2.084057463(+2)	
	3	2.000000000000000000000000	1.6(−19159)	2.2(−384)	8.0000	2.084074074(+2)	
*MM*1	0	2.1	9.8(−25)	1.0(−1)			
	1	2.000000073305887479606243	1.3(−333)	7.3(−8)		7.330631738(+00)	2.066666667(+01)
	2	2.000000000000000000000000	4.7(−2765)	1.7(−56)		2.066664998(+01)	
	3	2.000000000000000000000000	1.5(−22216)	1.6(−445)	8.0000	2.066666667(+01)	
*MM*2	0	2.1	9.8(−25)	1.0(−1)			
	1	2.000000001927516381664629	1.3(−412)	1.9(−9)		1.927516679(−1)	1.326315789(+00)
	2	2.000000000000000000000000	9.7(−3457)	2.5(−70)		1.326315770(+00)	
	3	2.000000000000000000000000	1.1(−27809)	2.2(−557)	8.0000	1.326315789(+00)	
*MM*3	0	2.1	9.8(−25)	1.0(−1)			
	1	2.000000006966462333292930	1.0(−384)	7.0(−9)		6.966466216(−1)	1.466666667(+00)
	2	2.000000000000000000000000	2.4(−3231)	8.1(−66)		1.466666588(+00)	
	3	2.000000000000000000000000	2.2(−26004)	2.8(−521)	8.0000	1.466666667(+00)	

$$\frac{\frac{ch}{\lambda BT} e^{\frac{ch}{\lambda BT}}}{e^{\frac{ch}{\lambda BT}} - 1} = 5. \tag{47}$$

If $x = \frac{ch}{\lambda BT}$, then (47) is satisfied when

$$f_5(x) = e^{-x} + \frac{x}{5} - 1 = 0. \tag{48}$$

Therefore, the solutions of $f_5(x) = 0$ give the maximum wavelength of radiation λ by means of the following formula:

$$\lambda \approx \frac{ch}{\alpha BT}, \tag{49}$$

where α is a solution of (48). Our desired root is $x = 4.9651142317442$ with multiplicity $m = 1$.

The numerical results for the test equation $f_5(x) = 0$ are displayed in Table 5. It can be observed that the new methods *MM*1 and *MM*2 have small values of residual errors and asymptotic error constants (η) in comparison to the other methods when the accuracy is tested in multi-precision arithmetic.

Example 4.6. (Global CO_2 model by Bresnahan et al. [24] in ocean chemistry):

In this example, we will discuss the global CO_2 model by Bresnahan et al. [24] in ocean chemistry (please, see [25] for more details) which finally leads to the numerical solution of a nonlinear fourth order polynomial in the calculation of *pH* of the ocean. The effect of atmospheric CO_2 is very complex and varies with location of the ocean. Therefore, Babajee [25] considered a simplified approach based on the following assumptions:

(i) Only the ocean upper layer is considered (not the deep layer).
(ii) Approximation of the ocean upper layer carbon distribution by perfect mixing so that spatial variations are neglected.

As, CO_2 dissolves in ocean water and undergoes a series of chemical changes that ultimately leads to increased hydrogen ion concentration, denoted as $[H^+]$ and thus acidification. The problem was analyzed by Babajee [25] in order to find the solution of the following nonlinear function:

$$p([H^+]) = \sum_{n=0}^{4} r_n [H^+]^n, \tag{50}$$

Table 5: Convergence behavior of different iterative methods on the test function $f_5(x)$.

Methods	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	ρ	$\left\lvert \frac{x_{n+1}-x_n}{(x_n-x_{n-1})^p} \right\rvert$	η
$GK1$	0	5.0	*	*			
	1		*	*		*	*
	2		*	*		*	
	3		*	*	*	*	
$GK2$	0	5.0	6.7(−3)	3.5(−2)			
	1	4.965114231744277568317118	2.4(−16)	1.3(−15)		7.015679382(−7)	7.468020979(−7)
	2	4.965114231744276303698759	5.9(−97)	3.1(−96)		7.468020979(−7)	
	3	4.965114231744276303698759	1.2(−580)	6.1(−580)	6.0000	7.468020979(−7)	
OM	0	5.0	6.7(−3)	3.5(−2)			
	1	4.965114231744276303744811	8.9(−21)	4.6(−20)		2.099233812(−8)	2.312146664(−8)
	2	4.965114231744276303698759	9.0(−164)	4.7(−163)		2.312146664(−8)	
	3	4.965114231744276303698759	1.0(−1307)	5.3(−1307)	8.0000	2.312146664(−8)	
ZM	0	5.0	6.7(−3)	3.5(−2)			
	1	4.965114231744276303727319	5.5(−21)	2.9(−20)		1.301869270(−8)	1.435568470(−8)
	2	4.965114231744276303698759	1.2(−165)	6.4(−165)		1.435568470(−8)	
	3	4.965114231744276303698759	7.4(−1323)	3.8(−1322)	8.0000	1.435568470(−8)	
$MM1$	0	5.0	6.7(−3)	3.5(−2)			
	1	4.965114231744276303681372	3.4(−21)	1.7(−20)		7.925586248(−9)	8.529952965(−9)
	2	4.965114231744276303698759	1.4(−167)	7.1(−167)		8.529952965(−9)	
	3	4.965114231744276303698759	1.1(−1338)	5.7(−1338)	8.0000	8.529952965(−9)	
$MM2$	0	5.0	6.7(−3)	3.5(−2)			
	1	4.965114231744276303680705	3.5(−21)	1.8(−20)		8.229748515(−9)	8.881048023(−9)
	2	4.965114231744276303698759	1.9(−167)	1.0(−166)		8.881048023(−9)	
	3	4.965114231744276303698759	1.7(−1337)	9.1(−1337)	8.0000	8.881048023(−9)	
$MM3$	0	5.0	6.7(−3)	3.5(−2)			
	1	4.965114231744276303680702	3.5(−21)	1.8(−20)		8.231154237(−9)	8.882681023(−9)
	2	4.965114231744276303698759	1.9(−167)	1.0(−166)		8.882681023(−9)	
	3	4.965114231744276303698759	1.8(−1337)	9.2(−1337)	8.0000	8.882681023(−9)	

where

$$\begin{cases} r_0 = 2K_0K_1K_2P_tK_B, \\ r_1 = K_0K_1P_tK_B + 2K_0K_1K_2P_t + K_WK_B, \\ r_2 = K_0K_1P_t + BK_B + K_W - AK_B, \\ r_3 = -K_B - A, \\ r_4 = -1. \end{cases} \tag{51}$$

Here, K_0, K_1, K_2, K_W and K_B are equilibrium constants. The parameter A represents alkalinity which expresses the neutrality of ocean water and P_t is the gas phase CO_2 partial pressure. We assume the values of $A = 2.050$ and $B = 0.409$ taken by L & Gruber [26] and Bacastow and Keeling [27], respectively. Furthermore, choosing the values of K_0, K_1, K_2, K_W, K_B and P_t given by Babajee [25], we obtain the following nonlinear equation:

$$f_7\,(x) = x^4 - \frac{2309x^3}{250} - \frac{65226608163x^2}{500000} + \frac{425064009069x}{25000} - \frac{10954808368405209}{62500000} = 0. \tag{52}$$

The roots of $f_7\,(x) = 0$ are given by $x = -411.452$, 11.286, 140.771, 268.332. Our desired root is −411.452 having multiplicity $m = 1$. Finally, we are interested to find the solution $x = [H^+]$ of the above equation (52) to calculate the pH of the ocean.

The numerical experiments of this example are given in Table 6. The methods $MM1$, $MM2$ and $MM3$ have small residual errors and asymptotic error constants as compared to the other existing methods. The computational order of convergence for all methods coincides with the theoretical ones in all cases.

5 Conclusions

In this paper, we have developed a wide general three-step class of methods for approximating multiple zeros of nonlinear functions numerically. Optimal iteration schemes having eighth-order for multiple zeros have been considered very seldom in the literature, so the presented methods may be regarded as an advancement in the topic. Weight functions based on function-to-function ratios and free parameters are employed at second and third steps of the family which enable us to achieve desired convergence order eight.

Table 6: Convergence behavior of different iterative methods on the test function $f_6(x)$.

Cases	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	ρ	$\left\lvert\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^p}\right\rvert$	η
$GK1$	0	−412	*	*			
	1		*	*		*	*
	2		*	*		*	
	3		*	*	*	*	
$GK2$	0	−412	1.3(+8)	8.5(−1)			
	1	−411.1521869660545671537300	9.6(−5)	6.1(−13)		1.636932403(−12)	1.668249119(−12)
	2	−411.1521869660539592549395	1.3(−77)	8.4(−86)		1.668249119(−12)	
	3	−411.1521869660539592549395	9.4(−515)	5.9(−523)	6.0000	1.668249119(−12)	
OM	0	−412	1.3(+8)	8.5(−1)			
	1	−411.1521869660539699457729	1.7(−6)	1.1(−14)		4.005076808(−14)	4.198566741(−14)
	2	−411.1521869660539592549395	1.1(−117)	7.2(−126)		4.198566741(−14)	
	3	−411.1521869660539592549395	4.6(−1007)	2.9(−1015)	8.0000	4.198566741(−14)	
ZM	0	−412	1.3(+8)	8.5(−1)			
	1	−411.1521869660539687486432	1.5(−6)	9.5(−15)		3.556599499(−14)	3.852323642(−14)
	2	−411.1521869660539592549395	4.0(−118)	2.5(−126)		3.852323642(−14)	
	3	−411.1521869660539592549395	1.1(−1010)	6.7(−1019)	8.0000	3.852323642(−14)	
$MM1$	0	−412	1.3(+8)	8.5(−1)			
	1	−411.1521869660539602280746	1.5(−7)	9.7(−16)		3.645628543(−15)	3.846055662(−15)
	2	−411.1521869660539592549395	4.9(−127)	3.1(−135)		3.846055662(−15)	
	3	−411.1521869660539592549395	5.1(−1083)	3.2(−1091)	8.0000	3.846055662(−15)	
$MM2$	0	−412	1.3(+8)	8.5(−1)			
	1	−411.1521869660539593310835	1.2(−8)	7.6(−17)		2.852561685(−16)	2.964107964(−16)
	2	−411.1521869660539592549395	5.3(−137)	3.3(−145)		2.964107964(−16)	
	3	−411.1521869660539592549395	7.4(−1164)	4.7(−1172)	8.0000	2.964107964(−16)	
$MM3$	0	−412	1.3(+8)	8.5(−1)			
	1	−411.1521869660539593268602	1.1(−8)	7.2(−17)		2.694344101(−16)	2.799008203(−16)
	2	−411.1521869660539592549395	3.2(−137)	2.0(−145)		2.799008203(−16)	
	3	−411.1521869660539592549395	1.2(−1165)	7.3(−1174)	8.0000	2.799008203(−16)	

In the numerical section, we have incorporated variety of real life problems to confirm the efficiency of the proposed technique in comparison to the existing robust methods. From the computational results, we find that the new methods show better performance in terms of precision, residual errors for the considered test functions $f_{1-6}(x)$. Finally, we point out that the easy structure and high convergence order of the proposed class, makes it not only interesting from theoretical point of view but also in practice.

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