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On nonlinear Fredholm integral equations with non-differentiable Nemystkii operator

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Summary

From decomposition method for operators, we consider a Newton-Steffensen iterative scheme for approximating a solution of nonlinear Fredholm integral equations with non-differentiable Nemystkii operator. By means of a convergence study of the iterative scheme applied to this type of nonlinear Fredholm integral equations, we obtain domains of existence and uniqueness of solution for these equations. In addition, we illustrate this study with a numerical experiment

KEYWORDS:
Fredholm integral equation, two-steps Newton iterative scheme, domain of existence of solution, domain of uniqueness of solution.

1 | INTRODUCTION

The various types of integral equations are important mathematical tools for describing knowledge models that appear in different areas of applied science. Because of extensive application of integral equations and not having the exact solutions in many cases, numerical solution of integral equations has attracted researcher's attention to develop numerical method for approximating solution of these equations. For example, we can consider different Fredholm-type integral equations (18, 19, 27) , Volterra-Fredholm integral equations (20, 25, 14), nonlinear Fredholm integro-differential equations (21), systems of Fredholm-Volterra integral equations (22), etc.

In this paper, we consider a special case of nonlinear Fredholm integral equation ((13, 24, 27))

$$x(s) = f(s) + \lambda \int_a^b K(s,t)\mathcal{N}(x)(t)\,dt, \quad s \in [a,b], \tag{1}$$

where $\lambda \in \mathbb{R}$, $-\infty < a < b < +\infty$, the function $f(s)$ is continuous on $[a,b]$ and given, the kernel $K(s,t)$ is a known continuous function in $[a,b]\times[a,b]$, the Nemytskii operator $\mathcal{N} : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$, where Ω is a nonempty open convex domain in $\mathscr{C}([a,b])$, given by $\mathcal{N}(x)(t) = N(x(t))$, where N is a known continuous but non-differentiable function in $\mathbb{R}$ and x is a solution to be determined in $\mathscr{C}([a,b])$, where $\mathscr{C}([a,b])$ denotes the space of continuous real functions in $[a,b]$.

These equations are related to boundary value problems for differential equations, since they can be reformulated as two-point boundary value problems or elliptic partial differential equations with nonlinear boundary conditions (6, 27). Moreover, these equations appear in several applications to real world: the theory of elasticity, engineering, mathematical physics, potential theory, electrostatics and radiative heat transfer problems (3). As the Fredholm equations of form (1) cannot be solved exactly, we can use different numerical techniques to solve them. Some of them will be discussed below.

First, we can approximate a solution of (1) by applying directly an iterative scheme. If we pay attention to the iterative schemes that can be applied, the method of successive approximations plays an important role, as we can see in (23, 29). This method consists of applying the Fixed Point

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Theorem to the equation

$$x(s) = [\Psi(x)](s) \quad \text{with} \quad [\Psi(x)](s) = f(s) + \lambda \int_a^b K(s,t)\mathcal{N}(x)(t)\,dt \tag{2}$$

and obtaining a sequence $\{x_n = \Psi(x_{n-1})\}_{n\in\mathbb{N}}$ that converges to a solution $x^*(s)$ of (1). This technique has two problems: the sequence $\{x_n\}$ converges slowly to x^* and the condition required to the operator involved Ψ is very restrictive, since Ψ must be a contraction from a domain to itself (26). As a consequence of both problems, other iterative schemes can be used, as for example, Whittaker-type methods (11), Newton's method (10, 15), direct modifications of Newton's method (24), Newton-type methods (16) or particular iterative schemes of high order of convergence (8, 12).

Second, we can usually find techniques based on processes of discretization that transform the continuous problem given in (1) into a finite dimensional problem. A procedure to achieve this consists of applying formulas of numerical quadrature that are used to approximate the integral appearing in equation (1), so that systems of equations are then obtained, and finally solved (28). Another procedure for the discretization of the problem consists on the application of the discrete collocation method (2, 6, 25), that consists of choosing a finite dimensional space of functions that are possible candidates to be solutions of (1). An easy choice is polynomials of a certain degree. After that, a number of points in the interval $[a,b]$ are chosen and the polynomial chosen is forced to satisfy (1) in such points, so that we obtain a finite dimensional problem that has to be solved. Notice that different choices of the finite dimensional space have notably proliferated. Another method that can be also included under this kind of techniques, although with some differences, is the Adomian decomposition method (1, 7), which is a semi-analytical method that consists of using the Adomian polynomials to approximate a solution of (1) by imposing that the Adomian polynomials satisfy equation (1).

Third, an interesting technique to approximate a solution of (1) is the homotopy analysis method (3, 14), that provides an analytical approximation to the solution of (1) from an homotopy, so that a continuous mapping of an initial guess approximation to the exact solution of (1) is constructed.

On the other hand, observe that the equation (1) can be defined as $\mathcal{H}(x) = 0$ for $\mathcal{H} : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$ where Ω is a nonempty open convex domain in $\mathscr{C}([a,b])$ and

$$[\mathcal{H}(x)](s) = x(s) - f(s) - \lambda \int_a^b K(s,t)\mathcal{N}(x)(t)\,dt, \quad s \in [a,b]. \tag{3}$$

It is well-known that Newton's method,

$$x_{n+1} = x_n - \left[\mathcal{H}'(x_n)\right]^{-1}\mathcal{H}(x_n), \quad n \geq 0; \quad x_0 \in \Omega \text{ is given}, \tag{4}$$

is one of the most used iterative schemes to approximate a solution x^* of $\mathcal{H}(x) = 0$. But this method has a serious shortcoming: the derivative $\mathcal{H}'(x)$ has to be evaluated at each iteration. This makes the method not applicable to equations with non-differentiable operators. It is common to approximate derivatives by divided differences (((4, 5, 9)) for obtaining derivative free iterative schemes. So, given an operator $D : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$, let us denote by $\mathcal{L}(\Omega, \mathscr{C}([a,b]))$ the space of bounded linear operators from Ω to $\mathscr{C}([a,b])$, an operator $[x,y;D] \in \mathcal{L}(\Omega, \mathscr{C}([a,b]))$ is called a first order divided difference for the operator D on the continuous real functions x and y $(x \neq y)$ if

$$[x,y;D](x-y) = D(x) - D(y). \tag{5}$$

Therefore, as $\mathcal{N}$ is a non-differentiable operator then $\mathcal{H}(x)$ is not either. So, if we want to apply Newton's method to approximate a solution of the equation $\mathcal{H}(x) = 0$, if each step of Newton's method $\mathcal{H}'$ is approximated by the divided difference of first order $[x, x+\mathcal{H}(x); \mathcal{H}]$, the Steffensen's method is obtained. Stetffensen's method has been widely studied ((4, 5, 9)) and its algorithm is

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n, x_n + \mathcal{H}(x_n); \mathcal{H}]^{-1}\mathcal{H}(x_n), \quad n \geq 0. \end{cases} \tag{6}$$

The method has quadratic convergence and the same computational efficiency as Newton's method. Although, the iterative schemes using divided differences in their algorithm have a drawback, the accessibility of these iterative schemes to the solution of the equation is poor (17), so that the domains of starting points are reduced. However, this is one of the favorable features of Newton's method (4). So, , this is the first main aim of this paper, we try to improve the accessibility of Steffensen's method. For this, we use the decomposition method. So, for the operator $\mathcal{H}$ that defines the equation $\mathcal{H}(x) = 0$, we consider

$$\mathcal{H}(x) = \mathcal{F}(x) + \mathcal{G}(x), \tag{7}$$

where $\mathcal{F}, \mathcal{G} : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$, being $\mathcal{F}$ a Fréchet differentiable operator and $\mathcal{G}$ a continuous but non-differentiable operator. Then, consider the Newton-Steffensen-type iterative scheme (17):

$$\begin{cases} x_0 \in \mathscr{C}([a,b]) \text{ is given}, \\ x_{n+1} = x_n - (\mathcal{F}'(x_n) + [x_n, x_n + \mathcal{H}(x_n); \mathcal{G}])^{-1}\mathcal{H}(x_n), \quad n \geq 0, \end{cases} \tag{8}$$

that is applied as

$$\begin{cases} x_0 \in \mathscr{C}([a,b]) \text{ is given}, \\ (\mathcal{F}'(x_n) + [x_n, x_n + \mathcal{H}(x_n); \mathcal{G}])\alpha_n = -\mathcal{H}(x_n) \\ x_{n+1} = x_n + \alpha_n, \quad n \geq 0. \end{cases}$$

The Newton-Steffensen-type method (8) improves greatly the accessibility of Steffensen's method by approximating its domain of starting points to that of Newton's method (17). Moreover, as we have already indicated above, a fundamental problem in the application of Newton's method is the fact that the operator $\mathcal{H}$ is non-differentiable, so that we cannot obtain $\mathcal{H}'$ to do the iterates. However, this technique of decomposition of the operator $\mathcal{H}$ plays a key role for the equation $\mathcal{H}(x) = 0$. On the other hand, the Newton-Steffensen-type method (8) improves the approximations of solutions of equations defined from non-differentiable operators. In this case, there are two advantages of (8): first, the differentiable part of the operator is considered in the optimal situation, namely $F'(x_n)$; and second, for the non-differentiable part, iteration (6), is considered with $[x_n, x_n + \mathcal{H}(x_n); \mathcal{G}]$, which has quadratic convergence and the same efficiency as Newton's method. In addition, as we see, we obtain an efficient iterative scheme to solve the equation $\mathcal{H}(x) = 0$.

On the other hand, we have other two aims for the paper: drawing conclusions about the existence and uniqueness of solution of some integral equations of type (1), from using the theoretical significance of the iterative scheme applied to solve (1), and approximating numerically a solution.

Section 2, is devoted to introduce Nemytskii operators and the use of them for applying the decomposition method. In Section 3 we study the existence of solution x^* of equation (1), obtaining recurrence relations for the sequence $\{x_n\}$. In Section 4, the uniqueness domain is established. Next, in Section 5 we apply the results to a nonlinear Fredholm integral equation, obtaining convergence radii and a numerical solution. Finally, we perform a comparative study of the iterative scheme (8) with an already existing one and we drove some conclusions.

2 | PRELIMINARIES

To establish a proper notation for the study indicated in the introduction and developed in this work, we use the known Nemytskii operators. So, from a function $x \in \Omega \subseteq \mathscr{C}([a,b])$, we define the operator $\mathcal{N} : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$ by $\mathcal{N}(x)(t) = N(x(t))$, which is a Nemytskii operator. Besides, from a function $y \in \mathscr{C}([a,b])$, we define the operator $\mathcal{K} : \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$ by $\mathcal{K}(y)(s) = \lambda \int_a^b K(s,t)y(t)\,dt$, which is a linear integral operator with kernel $K(s,t)$. Finally, as a consequence of both operators, equation (3) can be written as

$$[\mathcal{H}(x)](s) = [(I - \mathcal{K}\mathcal{N})(x) - f](s)$$

with $\mathcal{H} : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$.

Now, as the Nemystkii operator $\mathcal{N}$ is non-differentiable, since that N is continuous but non-differentiable in $\mathbb{R}$, we consider the Nemystkii operators $\mathcal{N}_i : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$, for $i = 1, 2$, such that $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$ with $\mathcal{N}_i(x)(t) = N_i(x(t))$, for $i = 1, 2$, where $N = N_1 + N_2$ with N_1 a differentiable operator in $\mathbb{R}$ and N_2 a continuous but non-differentiable operator in $\mathbb{R}$. So,

$$\begin{aligned} [\mathcal{H}(x)](s) &= [(I - \mathcal{K}\mathcal{N})(x) - f](s) = [(I - \mathcal{K}(\mathcal{N}_1 + \mathcal{N}_2))(x) - f](s) \\ &= [(I - \mathcal{K}\mathcal{N}_1)(x) - f](s) - \mathcal{K}\mathcal{N}_2(x)(s). \end{aligned}$$

Then, for applying the decomposition method of operator $\mathcal{H}$ (17), we can consider $\mathcal{H}(x) = \mathcal{F}(x) + \mathcal{G}(x)$ for $\mathcal{F}(x)(s) = [(I - \mathcal{K}\mathcal{N}_1)(x) - f](s)$ and $\mathcal{G}(x)(s) = -\mathcal{K}\mathcal{N}_2(x)(s)$, with $\mathcal{F}, \mathcal{G} : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$, where $\mathcal{F}$ is a Fréchet differentiable operator and $\mathcal{G}$ is a continuous but non-differentiable operator. So, we have:

$$\mathcal{F}(x)(s) = x(s) - f(s) - \lambda \int_a^b K(s,t)\mathcal{N}_1(x)(t)\,dt, \quad s \in [a,b], \tag{9}$$

and

$$\mathcal{G}(x)(s) = -\lambda \int_a^b K(s,t)\mathcal{N}_2(x)(t)\,dt, \quad s \in [a,b], \tag{10}$$

To continue, for applying the method (8) to approximate a solution of equation $\mathcal{H}(x) = 0$, we must calculate $\mathcal{F}'(x)$ and $[x, x + \mathcal{H}(x); \mathcal{G}]$.

In first place, notice that $\mathcal{K}'(x)z(s) = \mathcal{K}(z)(s)$ and, taking into account (9), the first derivative of $\mathcal{F}$ at x is then given by

$$\begin{aligned} [\mathcal{F}'(x)y](s) &= y(s) - [(\mathcal{K}\mathcal{N}_1)'(x)y](s) = y(s) - \mathcal{K}(\mathcal{N}_1'(x)y)(s) \\ &= y(s) - \lambda \int_a^b K(s,t)\mathcal{N}_1'(x)y(t)\,dt, \end{aligned}$$

where $\mathcal{F}'(x) : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b]$.

In second place, taking into account (10), given the continuous real functions x and y $(x \neq y)$, we define $[x, y; \mathcal{G}] : \Omega \subseteq \mathscr{C}([a, b]) \to \mathscr{C}([a, b]$ with

$$[x, y; \mathcal{G}](u)(s) = -\lambda \int_a^b K(s, t)[x, y; \mathcal{N}_2](t)u(t)\, dt,$$

we consider $[x, y; \mathcal{N}_2] : \mathbb{R} \to \mathbb{R}$ with

$$[x, y; \mathcal{N}_2](t) = \begin{cases} \dfrac{N_2(x(t)) - N_2(y(t))}{x(t) - y(t)} & \text{if } t \in [a, b] \text{ such that } x(t) \neq y(t), \\ \\ 0 & \text{if } t \in [a, b] \text{ such that } x(t) = y(t). \end{cases}$$

So, for the continuous real functions x and y $(x \neq y)$, obviously $[x, y; \mathcal{G}] \in \mathcal{L}(\Omega, \mathscr{C}([a, b]))$ and

$$[x, y; \mathcal{G}](x - y) = \mathcal{G}(x) - \mathcal{G}(y).$$

Then, $[x, y; \mathcal{G}]$ is a first order divided difference for the operator $\mathcal{G} : \Omega \subseteq \mathscr{C}([a, b]) \to \mathscr{C}([a, b])$.

3 | DOMAIN OF EXISTENCE OF SOLUTION

This section concerns with the study of the semilocal convergence of iterative scheme (8). The analysis of the semilocal convergence is based on demanding conditions to the initial approximations, from certain conditions on the operator $\mathcal{H}$, and provide the conditions required to the initial approximations that guarantee the convergence to a solution x^* of the equation $\mathcal{H}(x) = 0$. From the semilocal convergence of iterative scheme (8), we draw conclusions about the existence of a solution x^*.

We shall show the semilocal convergence of iterative scheme (8) based on the following conditions:

(I) $\mathcal{N}$ is K_0-Lipschitz continuous operator such that

$$\|\mathcal{N}(x) - \mathcal{N}(y)\| \leq K_0\|x - y\|, \text{ where } x, y \in \Omega \subseteq \mathscr{C}([a, b]) \text{ and } K_0 \geqslant 0. \tag{11}$$

(II) $\mathcal{N}_1'$ is a K_1-Lipschitz continuous operator such that

$$\|\mathcal{N}_1'(x) - \mathcal{N}_1'(y)\| \leq K_1\|x - y\|, \text{ where } x, y \in \Omega \subseteq \mathscr{C}([a, b])\ K_1 \geqslant 0. \tag{12}$$

(III) $[-, -; \mathcal{N}_2]$ is an operator such that

$$\|[x, y; \mathcal{N}_2] - [u, v; \mathcal{N}_2]\| \leq L + K_2\left(\|x - u\| + \|y - v\|\right), \text{ where } x, y, u, v \in \Omega \subseteq \mathscr{C}([a, b]), \tag{13}$$

$K_2 \geqslant 0$ and $L > 0$.

As first step, from the previous conditions we easily obtain the following result:

Lemma 1. *Under conditions (**I**)–(**III**) we obtain the following items:*

(i) *$\mathcal{H}$ is Lipschitz continuous operator such that*

$$\|\mathcal{H}(x) - \mathcal{H}(y)\| \leq (1 + \lambda M K_0)\|x - y\|, \text{ where } x, y \in \Omega \subseteq \mathscr{C}([a, b], \tag{14}$$

with $M = \max_{s\in[a,b]} \int_a^b |\mathcal{K}(s, t)|\, dt$.

(ii) *$\mathcal{F}'$ is a Lipschitz continuous operator such that*

$$\|\mathcal{F}'(x) - \mathcal{F}'(y)\| \leq \lambda M K_1\|x - y\|, \text{ where } x, y \in \Omega \subseteq \mathscr{C}([a, b], \tag{15}$$

(iii) *$[-, -; \mathcal{G}]$ is an operator such that*

$$\|[x, y; \mathcal{G}] - [u, v; \mathcal{G}]\| \leq \lambda M \left(L + K_2\left(\|x - u\| + \|y - v\|\right)\right), \tag{16}$$

for pairs of distinct functions $(x, y), (u, v) \in \Omega \times \Omega$.

3.1 | Main result

At this point we have established in Lemma 1 the bounding conditions for the operators we use in our iterative proccess. Now, our aim is to perform the semilocal convergence study for setting the domain of existence of the solution. First of all, we give the main result, but in order to prove it, we have to analyze the well definition of iterative process (8) by starting from any suitable guess and obtain the relations that verify the iterates.

Theorem 1. *Let $\mathcal{H}$ be a nonlinear operator, $\mathcal{H}:\Omega\subseteq\mathscr{C}([a,b])\longrightarrow\mathscr{C}([a,b])$, defined on a nonempty open convex domain Ω, with $\mathcal{H}(x)=\mathcal{F}(x)+\mathcal{G}(x)$, where $\mathcal{F}$ is a Fréchet differentiable operator and $\mathcal{G}$ is a continuous but non-differentiable operator. Suppose that conditions (**I**)–(**III**) are satisfied and consider $x_0\in\Omega$ verifying that $A_0=\mathcal{F}'(x_0)+[x_0,x_0+\mathcal{H}(x_0);\mathcal{G}]$ has inverse with $\|A_0^{-1}\|\leq\beta$. Let $||\mathcal{H}(x_0)||\leq\alpha,\eta=\beta\alpha$, and $h=\lambda M\beta$.*

If the equation

$$t=\left(1+\frac{f(a_0)b_0}{1-\frac{e(t)}{(1-d(t))(1-2d(t))}}\right)\eta, \tag{17}$$

has at least one positive real root and the smallest positive real root, denoted by R, satisfies

$$\overline{B(x_0,R)}\subset\Omega,\ \ \eta<R,\ \text{ and }\ a<\frac{1}{2}\ , \tag{18}$$

where

$$\begin{aligned}
&d(t)=h(L+(K_1+K_2(3+\lambda MK_0))t),\\
&e(t)=h(L+(\frac{1}{2}K_1+K_2)\eta+K_2(\alpha+(1+\lambda MK_0)t)),\\
&f(t)=\frac{1}{1-t},\\
&a_0=d(\eta),\ \ b_0=e(0),\\
&a=d(R),\ \ b=e(R),
\end{aligned} \tag{19}$$

then, the sequence $\{x_n\}$ generated by the iterative process given by (8), converges to a solution x^* of the equation $\mathcal{H}(x)=0$, verifying that $x_n,x^*\in\overline{B(x_0,R)}$, for all $n\in\mathbb{N}$.

In order to clarify the proof of this main theorem, we give the following lemmas where we analyze the iterative process step by step.

Lemma 2. *Under conditions of Theorem 1 the value of R obtained from* (17) *verifies*

$$\begin{aligned}
&(i)\ \ f(a)f(2a)b<1\\
&(ii)\ \ R=\left[1+\frac{f(a_0)b_0}{1-f(a)f(2a)b}\right]\eta,\\
&(iii)\ \ R>\left[1+f(a_0)b_0\left(1+f(a)f(2a)b+(f(a)f(2a)b)^2+\ldots+(f(a)f(2a)b)^{n-1}\right)\right]\eta,\ \ \forall n\geq 1.
\end{aligned}$$

***Proof*.** *This Lemma follows obviously, (i) and (ii) by definition of R, taking into account for (i) that $\eta<R$, and substituting in* (17) *the values of parameters a_0,b_0,a and b. To prove (iii), note that in second part of relation (ii) appears the sum of a geometric sequence of positive ratio $f(a)f(2a)b<1$, for (i).* ■

Lemma 3. *Under conditions of Theorem 1 and by denoting $a_{-1}=0$, $a_n=a$ and $b_n=b$ for all $n\geq1$, the following assertions hold for $n\geq1$,*

$$\begin{aligned}
&(I_1)\ \exists\, A_n^{-1},\text{ such as }\ ||A_n^{-1}||\leq\beta f(a_{n+1}),\ \text{ and }||A_n^{-1}A_0||\leq f(a_{n+1}).\\
&(I_2)\ ||x_{n+1}-x_n||\leq b_{n-1}f(a_{n+1})f(2a_{n-2})||x_n-x_{n-1}||.\\
&(I_3)\ ||x_{n+1}-x_n||\leq(bf(a)f(2a))^{n-1}||x_2-x_1||.\\
&(I_4)\ ||x_{n+1}-x_0||\leq\left[1+f(a_0)b_0(1+f(a)f(2a)b+(f(a)f(2a)b)^2+\ldots+(f(a)f(2a)b)^{n-1})\right]\eta.\\
&(I_5)\ x_{n+1}\in\overline{B(x_0,R)}.
\end{aligned}$$

***Proof*.** *We prove this lemma by an induction procedure.*

We start analyzing the first steps. For $n=1$ we have by hypothesis that $\exists A_0^{-1}$, so x_1 is well defined and is obtained:

$$||x_1-x_0||\leq||A_0^{-1}\mathcal{H}(x_0)||\leq||A_0^{-1}||||\mathcal{H}(x_0)||\leq\beta\alpha=\eta$$

now by (18)*, we have that $x_1\in\overline{B(x_0,R)}$.*

When $n=2$, first of all we need to obtain the existence of A_1^{-1}, so by using bounds obtained in Lemma 1 we have

$$
\begin{aligned}
||I - A_0^{-1}A_1|| &\leq ||A_0^{-1}||||A_0 - A_1|| \\
&\leq ||A_0^{-1}|| \left(||\mathcal{F}'(x_1) - \mathcal{F}'(x_0)|| + ||[x_1, x_1 + \mathcal{H}(x_1); \mathcal{G}] - [x_0, x_0 + \mathcal{H}(x_0); \mathcal{G}]||\right) \\
&\leq \beta\left(\lambda M K_1 ||x_1 - x_0|| + \lambda M (L + K_2(||x_1 - x_0|| + ||x_1 - x_0|| + ||\mathcal{H}(x_1) - \mathcal{H}(x_0)||))\right) \\
&\leq h\left(L + (K_1 + K_2(2 + (1 + \lambda M K_0)))||x_1 - x_0||\right) \\
&\leq h(L + (K_1 + K_2(3 + \lambda M K_0))\eta) = a_0
\end{aligned}
$$

By (18)*, we have* $a_0 < a < 1$*, then by applying Banach Lemma we have that* $\exists A_1^{-1}$*. It also fulfills these bounds:*

$$
\begin{aligned}
||A_1^{-1}|| &\leq \beta f(a_0) \\
||A_1^{-1}A_0|| &\leq f(a_0)
\end{aligned}
\tag{20}
$$

so x_2 *is well defined and it is derived:*

$$
||x_2 - x_1|| = ||A_1^{-1}\mathcal{H}(x_1)|| \leq ||A_1^{-1}A_0||||A_0^{-1}\mathcal{H}(x_1)|| \tag{21}
$$

We use the expression of the iterative scheme (8) *to deduce:*

$$
\begin{aligned}
A_0^{-1}\mathcal{H}(x_1) &= A_0^{-1} - (x_1 - x_0) - A_0^{-1}\mathcal{H}(x_0) \\
&= A_0^{-1}(\mathcal{F}(x_1) + \mathcal{G}(x_1)) - (x_1 - x_0) - A_0^{-1}(\mathcal{F}(x_0) + \mathcal{G}(x_0)) \\
&= A_0^{-1}(\mathcal{F}(x_1) - \mathcal{F}(x_0)) - (x_1 - x_0) + A_0^{-1}(\mathcal{G}(x_1) - \mathcal{G}(x_0)) \\
&= \int_{x_0}^{x_1} (A_0^{-1}\mathcal{F}'(z) - Id)dz + A_0^{-1}(\mathcal{G}(x_1) - \mathcal{G}(x_0)) \\
&= A_0^{-1}\int_{x_0}^{x_1} (\mathcal{F}'(z) - A_0)dz + A_0^{-1}(\mathcal{G}(x_1) - \mathcal{G}(x_0)) \\
&= A_0^{-1}\int_{x_0}^{x_1} (\mathcal{F}'(z) - \mathcal{F}'(x_0) - [x_0, x_0 + \mathcal{H}(x_0; \mathcal{G})])dz + A_0^{-1}([x_1, x_0; \mathcal{G}](x_1 - x_0)) \\
&= A_0^{-1}\int_{x_0}^{x_1} (\mathcal{F}'(z) - \mathcal{F}'(x_0))dz + A_0^{-1}([x_1, x_0; \mathcal{G}] - [x_0, x_0 + \mathcal{H}(x_0); \mathcal{G}])(x_1 - x_0)
\end{aligned}
$$

Then, by taking norms and using bounds obtained in Lemma 1 we get

$$
\begin{aligned}
||A_0^{-1}\mathcal{H}(x_1)|| &\leq ||A_0^{-1}|| \left[\frac{1}{2}\lambda M K_1 ||x_1 - x_0||^2 + \lambda M(L + K_2(||x_1 - x_0|| + ||\mathcal{H}(x_0)||))||x_1 - x_0||\right] \\
&\leq h\left(\frac{1}{2}K_1||x_1 - x_0|| + L + K_2(||x_1 - x_0|| + ||\mathcal{H}(x_0)||)\right)||x_1 - x_0|| \\
&\leq h\left(L + (\frac{1}{2}K_1 + K_2)\eta + K_2\alpha\right)||x_1 - x_0|| = b_0||x_1 - x_0||
\end{aligned}
\tag{22}
$$

Therefore, going back to (21) *and using* (20) *and* (22) *we get*

$$
||x_2 - x_1|| \leq f(a_0)b_0||x_1 - x_0||
$$

So, by using (18) *it is obvious that* $f(a_0)b_0 < 1$*, then we we get that*

$$
\begin{aligned}
&||x_2 - x_1|| < ||x_1 - x_0|| \leq \eta \\
&||x_2 - x_0|| \leq ||x_2 - x_1|| + ||x_1 - x_0|| \leq (f(a_0)b_0 + 1)\eta < R
\end{aligned}
$$

and so by using Lemma 3 it follows that $x_2 \in B(x_0, R)$*.*

We have to perform another step before establishing the recurrence, so for $n = 3$ *in order to apply Banach Lemma we need the following:*

$$\begin{aligned}||I - A_0^{-1}A_2|| &\leq ||A_0^{-1}||||A_2 - A_0||\\ &\leq ||A_0^{-1}|| \left(||\mathcal{F}'(x_2) - \mathcal{F}'(x_0)|| + ||[x_2, x_2 + \mathcal{H}(x_2); \mathcal{G}] - [x_0, x_0 + \mathcal{H}(x_0); \mathcal{G}]||\right)\\ &\leq \beta\lambda M \left(K_1||x_2 - x_0|| + (L + K_2(||x_2 - x_0|| + ||x_2 - x_0|| + ||\mathcal{H}(x_2) - \mathcal{H}(x_0)||))\right)\\ &\leq \lambda M\beta(L + (K_1 + K_2(2 + (1 + \lambda MK_0)))||x_2 - x_0||)\\ &< h(L + (K_1 + K_2(3 + \lambda MK_0))R) = a\end{aligned}$$

Again, by (18), as $a < 1$, then $\exists A_2^{-1}$ and it verifies that:

$$\begin{aligned}||A_2^{-1}|| &\leq \beta f(a)\\ ||A_2^{-1}A_0|| &\leq f(a).\end{aligned} \tag{23}$$

So x_3 is well defined and we can obtain the following inequality

$$||x_3 - x_2|| \leq ||A_2^{-1}A_0||||A_0^{-1}A_1||||A_1^{-1}\mathcal{H}(x_2)||. \tag{24}$$

Now, we need to bound $||A_0^{-1}A_1||$, just applying Lema 1 we have:

$$\begin{aligned}||I - A_1^{-1}A_0|| &\leq ||A_1^{-1}||||A_1 - A_0||\\ &\leq \beta f(a_0)\lambda M(L + (K_1 + K_2(3 + \lambda MK_0)))\eta = f(a_0)a_0\end{aligned}$$

but from (18) and $a_0 < \frac{1}{2}$ is easy to obtain that $f(a_0)a_0 < 1$, then by Banach Lemma it is established:

$$||A_0^{-1}A_1|| \leq \frac{1}{1 - f(a_0)a_0} = \frac{1 - a_0}{1 - 2a_0} = \frac{f(2a_0)}{f(a_0)} \tag{25}$$

But, also with a similar reasoning that the one used before for bounding $||A_1^{-1}\mathcal{H}(x_1)||$, now we have

$$\begin{aligned}||A_1^{-1}\mathcal{H}(x_2)|| &\leq ||A_1^{-1}||(\frac{1}{2}\lambda MK_1||x_2 - x_1||^2 + \lambda M(L + K_2(||x_2 - x_1|| + ||\mathcal{H}(x_1)||))||x_2 - x_1||)\\ &< f(a_0)h(L + (\frac{1}{2}K_1 + K_2)\eta + K_2(||\mathcal{H}(x_0)|| + (1 + \lambda MK_0)||x_1 - x_0||))||x_2 - x_1||\\ &< f(a_0)h(L + (\frac{1}{2}K_1 + K_2)\eta + K_2(\alpha + (1 + \lambda MK_0)R))||x_2 - x_1||\\ &= f(a_0)b||x_2 - x_1||,\end{aligned} \tag{26}$$

where in the last inequality we have used that $x_1 \in B(x_0, R)$ and $\|\mathcal{H}(x_1)\| \leq ||\mathcal{H}(x_1) - \mathcal{H}(x_0)|| + \|\mathcal{H}(x_0)||$.

Now, coming back to (24) and using (23), (25) and (26), we obtain

$$||x_3 - x_2|| \leq f(a)f(2a_0)b||x_2 - x_1||,$$

from (18) we have that $f(a)f(2a)b < f(a)f(2a_0)b < 1$, then,

$$||x_3 - x_2|| < ||x_2 - x_1|| < ||x_1 - x_0|| \leq \eta.$$

On the other hand, by using Lemma 3

$$\begin{aligned}||x_3 - x_0|| \leq ||x_3 - x_2|| + ||x_2 - x_0|| &\leq (f(a)f(2a_0)bf(a_0)b_0 + f(a_0)b_0 + 1)\eta\\ &\leq (1 + f(a_0)b_0(1 + f(a)f(2a)b)\eta < R\end{aligned}$$

so we get $x_3 \in B(x_0, R)$, having that $(I_1) - (I_5)$ hold for all $k = 1, 2$.

Now by an induction procedure we assume that $(I_1) - (I_5)$ hold for all $k = 3, \ldots, n - 1$. Then, for completing the proof we have to obtain these assertions for $k = n$.

First of all, we need to obtain the existence of A_n^{-1}, so we have

$$
\begin{aligned}
||I - A_0^{-1}A_n|| &\leq ||A_0^{-1}||||A_n - A_0|| \\
&\leq ||A_0^{-1}|| \left(||\mathcal{F}'(x_n) - \mathcal{F}'(x_0)|| + ||[x_n, x_n + \mathcal{H}(x_n); \mathcal{G}] - [x_0, x_0 + \mathcal{H}(x_0); \mathcal{G}]||\right) \\
&\leq \beta\left(\lambda M K_1||x_n - x_0|| + (L + K_2(||x_n - x_0|| + ||x_n - x_0|| + ||\mathcal{H}(x_n) - \mathcal{H}(x_0)||))\right) \\
&\leq h(L + (K_1 + K_2(2 + (1 + \lambda M K_0)))R) = a
\end{aligned}
$$

Where in the last inequality we have used the induction hypothesis. Now, by (18)*, it follows that* $a < 1$*, then, by applying Banach Lemma we have that* $\exists A_n^{-1}$ *and also fulfills these bounds:*

$$
||A_n^{-1}|| \leq \beta f(a)
$$
$$
||A_n^{-1}A_0|| \leq f(a)
$$

So x_{n+1} *is well defined and with the same reasoning than for previous steps we obtain*

$$
||x_{n+1} - x_n|| \leq ||A_n^{-1}A_0||||A_0^{-1}A_{n-1}||||A_{n-1}^{-1}\mathcal{H}(x_n)||. \tag{27}
$$

Now, the bound for $A_0^{-1}A_{n-1}$ *is having as in previous steps, so it follows*

$$
||I - A_{n-1}^{-1}A_0|| \leq ||A_{n-1}^{-1}||||A_{n-1} - A_0|| \leq \beta f(a)a
$$

but from (18) *and* $a < \frac{1}{2}$ *it is given vthat* $f(a)a < 1$*, then by Banach Lemma it is established:*

$$
||A_0^{-1}A_{n-1}|| \leq \frac{1}{1 - f(a)a} = \frac{1-a}{1-2a} = \frac{f(2a)}{f(a)}
$$

But, also with a similar reasoning that we have used before and by the induction procedure we obtain

$$
\begin{aligned}
||A_{n-1}^{-1}\mathcal{H}(x_n)|| &\leq ||A_{n-1}^{-1}||(\frac{1}{2}\lambda M K_1||x_n - x_{n-1}||^2 + \lambda M(L + K_2(||x_n - x_{n-1}|| \\
&+ ||\mathcal{H}(x_{n-1})||))||x_n - x_{n-1}||) \\
&< f(a)h(L + (\frac{1}{2}K_1 + K_2)\eta + K_2(||\mathcal{H}(x_0)|| + (1 + \lambda M K_0)||x_{n-1} - x_0||))||x_n - x_{n-1}|| \\
&< f(a)b||x_n - x_{n-1}||
\end{aligned}
$$

so coming back to (27) *we obtain*

$$
||x_{n+1} - x_n|| \leq f(a)f(2a)b||x_n - x_{n-1}||
$$

from (18) *we have that* $f(a)f(2a)b < 1$*, then,*
$||x_{n+1} - x_n|| < ||x_n - x_{n-1}|| < \cdots < ||x_1 - x_0|| \leq \eta$*.*

Moreover, by using the induction for (I_3) *we have:*

$$
||x_{n+1} - x_n|| \leq bf(a)f(2a)||x_n - x_{n-1}|| \leq (bf(a)f(2a))^{n-1}||x_2 - x_1|| \qquad \leq (bf(a)f(2a))^{n-1}f(a_0)b_0\eta
$$

As a consequence we get,

$$
\begin{aligned}
||x_{n+1} - x_0|| &\leq ||x_{n+1} - x_n|| + ||x_n - x_0|| \\
&\leq (bf(a)f(2a))^{n-1}f(a_0)b_0\eta + 1 + f(a_0)b_0[f(a)f(2a)b + (f(a)f(2a)b)^2 \\
&+ \ldots + (f(a)f(2a)b)^{n-2}]\eta \\
&\leq \left[1 + f(a_0)b_0(f(a)f(2a)b + (f(a)f(2a)b)^2 + \ldots + (f(a)f(2a)b)^{n-1})\right]\eta < R
\end{aligned}
$$

so we deduce that $x_{n+1} \in \overline{B(x_0, R)}$ *and the induction procedure is completed.* ■

After proving these Lemmas we are in conditions of stating the main result established in Theorem 1, that now we write in the same terms but just in an abbreviated form without specifying again the values of the parameters, (19).

Proof of Theorem 1:

The iterative process is well defined as we have proved in the previous Lemma. So, in order to prove that $\{x_n\}$ is a Cauchy sequence we get

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq \sum_{i=1}^{m}\|x_{n+i}-x_{n+i-1}\| \leq \sum_{i=1}^{m}(bf(a)f(2a))^{n+i-2}\|x_2-x_1\| \\ &\leq \left[bf(a)f(2a))^{n-1}+(bf(a)f(2a))^{n}+\cdots(bf(a)f(2a))^{n+m-2}\right]\|x_2-x_1\| \\ &\leq \frac{(bf(a)f(2a))^{n-1}-(bf(a)f(2a))^{n+m-1}}{1-bf(a)f(2a)}f(a_0)b_0\eta\end{aligned} \tag{28}$$

where in the last inequality we have sum the m terms of a geometric sequence of ratio $bf(a)f(2a)<1$. So, we can conclude that $\{x_n\}$ is a Cauchy sequence, and then it has a limit x^*. By taking $m\to\infty$ we obtain an a priori error estimation:

$$\|x_n-x^*\| \leq \frac{(bf(a)f(2a))^{n-1}}{1-bf(a)f(2a)}f(a_0)b_0\eta$$

and taking $n=0$ in (28) and $m\to\infty$ and the characterization of R in Lemma 3 we have that $\|x^*-x_0\|\leq R$ and then $x^*\in\overline{B(x_0,R)}$.

Moreover x^* is a solution of $\mathcal{H}(x)=0$ since $\|A_n\mathcal{H}(x_n)\|=\|x_{n+1}-x_n\|$, but this difference tends to zero and also A_n^{-1} is bounded, so by using that $\|\mathcal{H}(x_n)\|\leq\|A_n^{-1}\|\|A_n\mathcal{H}(x_n)\|$, by the continuity of $\mathcal{H}$ it follows that $\mathcal{H}(x^*)=0$.

4 | DOMAIN OF UNIQUENESS OF SOLUTION

Concerning to the uniqueness of the solution x^*, we have the following result.

Theorem 2. *Under conditions of Theorem 1, then, the solution x^* is the unique solution of the equation $\mathcal{H}(x)=0$ in $\overline{B(x_0,S)}\cap\Omega$, being $S=\dfrac{1-\lambda M\beta(L+K_2\alpha)}{2\lambda M\beta(K_1+K_2)}$.*

***Proof*.** *Let $y^*\in\overline{B(x^*,R)}\cap\Omega$ and $\mathcal{H}(y^*)=0$. We then define the following operator*

$$P=\int_0^1\mathcal{F}'\left(x^*+t(y^*-x^*)\right)\,dt+[y^*,x^*;\mathcal{G}]$$

*and, using **(ii)** and **(iii)** of Lemma 1, we obtain*

$$\begin{aligned}\|A_0^{-1}P-I\| &\leq \|A_0^{-1}\|\,\|P-A_0\| \\ &\leq \|A_0^{-1}\|\left[\int_0^1\|\mathcal{F}'\left(x^*+t(y^*-x^*)\right)-\mathcal{F}'(x_0)\|\,dt+\|[y^*,x^*;\mathcal{G}]-[x_0,x_0+\mathcal{H}(x_0);\mathcal{G}]\|\right] \\ &\leq \lambda M\beta\left(K_1(\|x^*-x_0\|+\int_0^1 t\|y^*-x^*\|dt)+L+K_2(\|y^*-x_0\|+\|x^*-x_0\|+\|\mathcal{H}(x_0)\|)\right) \\ &< \lambda M\beta\left(2(K_1+K_2)S+L+K_2\alpha\right).\end{aligned}$$

By using Banach Lemma and the value of S we have that it exists $P^{-1}\in\mathcal{L}(X,Y)$, and, by the identity $P(y^*-x^*)=0$, we deduce $x^*=y^*$. ■

5 | NUMERICAL EXPERIMENT

Now, we present a numerical example where we illustrate all the above results. The max-norm has been considered.

We consider a nonlinear integral equation of Fredholm, which can be used to describe applied problems in the fields of electro-magnetics, fluid dynamics, in the kinetic theory of gases and, in general, in the reformulation of boundary value problems. So, we consider an equation of the form given in (1). Then, to solve this equation, we apply the iterative scheme (8) to the operator equation (3).

Next, we consider $f(s)=(1-\frac{11\lambda}{80})s-\frac{1}{2}$, $K(s,t)=st$ and $\mathcal{N}(x)(t)=x(t)^3+|x(t)|$.

So, we approximate a solution of the nonlinear integral equation of Fredholm of type (1) given by:

$$x(s)=(1-\frac{11\lambda}{80})s-\frac{1}{2}+\lambda\int_0^1 st\left(x(t)^3+|x(t)|\right)dt,\quad a\leq s\leq b, \tag{29}$$

By direct substitution of $x^*(s)=s-\frac{1}{2}$ in the above equation we have that $x^*(s)$ is a solution of (29).

For applying the decomposition method (8), for the operator:

$$\mathcal{H}(x)(s) = x(s) - (1 - \frac{11\lambda}{80})s + \frac{1}{2} - \lambda s \int_0^1 t\left(x(t)^3 + |x(t)|\right) dt, \quad a \le s \le b, \tag{30}$$

we consider

$$\mathcal{F}(x)(s) = x(s) - (1 - \frac{11\lambda}{80})s + \frac{1}{2} - \lambda s \int_0^1 t\, x(t)^3\, dt, \quad s \in [a, b], \tag{31}$$

and

$$\mathcal{G}(x)(s) = -\lambda s \int_0^1 t\, |x(t)|\, dt, \quad s \in [a, b], \tag{32}$$

such that $\mathcal{H}(x) = \mathcal{F}(x) + \mathcal{G}(x)$.

5.1 | Existence and uniqueness of solutions

At this point by taking $\Omega = B(0, 1)$ from (11), (12) and (13) we have for all $x(t), y(t), u(t), v(t) \in \Omega$ that:

$$\begin{aligned} \|\mathcal{N}(x(t)) - \mathcal{N}(y(t)) &\le \|x(t)^3 - y(t)^3\| + \||x(t)| - |y(t)|\| \\ &\le \left(\|x(t)^2 + y(t)^2 - x(t)y(t)\| + 1\right) \|x(t) - y(t)\|, \end{aligned}$$

moreover,

$$\|\mathcal{N}_1'(x(t)) - \mathcal{N}_1'(y(t))\| \le \|3(x(t)^2 - y(t)^2)\| \le 3\|x(t) + y(t)\|\|x(t) - y(t)\|,$$

and finally,

$$\|[x(t), y(t); \mathcal{N}_2] - [u(t), v(t); \mathcal{N}_2]\| \le \|\frac{|x(t)| - |y(t)|}{\|x(t) - y(t)\|}\| + \|\frac{|u(t)| - |v(t)|}{\|u(t) - v(t)\|}\|$$

So, we deduce the value of the constans introduced in Section 3 in order to perform the study, these are, $K_0 = 4, K_1 = 6, L = 2$ and $K_2 = 0$. Then, we can apply Lemma 1 with $M = 1/2$ and by choosing a starting point $x_0(s)$, and $\lambda = 1/11$, we use, first, (30) for obtaining the value of α that can be seen in Table 1 . Secondly, for obtaining the existence and bound of A_0^{-1} we calculate:

$$\begin{aligned} [A_0 y](s) &= \left[(\mathcal{F}'(x_0) + [x_0, x_0 + \mathcal{H}(x_0); \mathcal{G}])y\right](s) \\ &= [(\mathcal{F}'(x_0)y](s) + [[x_0, x_0 + \mathcal{H}(x_0); \mathcal{G}])y](s) \\ &= y(s) - \lambda s \int_0^1 3t x_0(t)^2 y(t)\, dt + \lambda s \int_0^1 t \frac{|x_0(t)| - |x_0(t) + \mathcal{H}(x_0)(t)|}{\mathcal{H}(x_0)(t)} y(t)\, dt \\ &= y(s) - \lambda s \int_0^1 t \left[3x_0(t)^2\, dt - \frac{|x_0(t)| - |x_0 + \mathcal{H}(x_0)(t)|}{\mathcal{H}(x_0)(t)}\right] y(t)\, dt = \omega(s). \end{aligned} \tag{33}$$

From (33) we deduce that

$$\|I - A_0\| \le \|\lambda s \int_0^1 t \left[3x_0(t)^2\, dt - \frac{|x_0(t)| - |x_0 + \mathcal{H}(x_0)(t)|}{\mathcal{H}(x_0)(t)}\right] dt\|$$

and then for each starting guess $x_0(s)$ we obtain the bound m_{x_0} such as:

$$\|I - A_0\| \le m_{x_0} < 1,$$

so by applying Banach Lemma we obtain the bound of A_0^{-1} mentioned in Theorem 2:

$$\beta = \frac{1}{1 - m_{x_0}}.$$

These values can be seen in Table 1 for different starting points. Once these bounds have been obtained we applied Theorem 2 for concluding the radius of existence and uniqueness that can be also checked in Table 1 .

$x_0(s)$	α	β	R	S
$s-1/4$	1/4	1.080845	0.333531	1.68863
$(2s-1)/4$	1/4	1.028132	0.291515	1.77559
$(4s-1)/6$	1/3	1.061859	0.454784	1.71896

TABLE 1 Semilocal convergence radii for different starting guesses and $\lambda = 1/11$.

5.2 | A numerical solution

Now, we obtain a new algorithm to apply our iterative scheme as follows: from the development obtained in (33) we have: $y(s) = A_0^{-1}\omega(s)$, but, we can consider $y(s) = \omega(s) + \lambda s\mathcal{I}$ where

$$\mathcal{I} = \int_0^1 t\left(3x_0(t)^2 - \frac{|x_0(t)| - |x_0(t) + \mathcal{H}(x_0)(t)|}{\mathcal{H}(x_0)(t)}\right) y(t)\,dt$$

If the last equality of (33) is multiplied by $s\left(3x_0(s)^2 - \dfrac{|x_0(s)| - |x_0(s) + \mathcal{H}(x_0)(s)|}{\mathcal{H}(x_0)(s)}\right)$ and integrated between 0 and 1, we obtain

$$\mathcal{I} = \frac{\int_0^1 t\left(3x_0(t)^2 - \frac{|x_0(t)| - |x_0(t) + \mathcal{H}(x_0)(t)|}{\mathcal{H}(x_0)(t)}\right)\omega(t)\,dt}{1 - \lambda\int_0^1 t^2\left(3x_0(t)^2 - \frac{|x_0(t)| - |x_0(t) + \mathcal{H}(x_0)(t)|}{\mathcal{H}(x_0)(t)}\right)dt},$$

provided that $\lambda\displaystyle\int_0^1 t^2\left(3x_0(t)^2 - \frac{|x_0(t)| - |x_0(t) + \mathcal{H}(x_0)(t)|}{\mathcal{H}(x_0)(t)}\right)dt \neq 1$. Therefore,

$$y(s) = A_0^{-1}\omega(s) = \omega(s) + \lambda\, s\,\frac{\int_0^1 t\left(3x_0(t)^2 - \frac{|x_0(t)| - |x_0(t) + \mathcal{H}(x_0)(t)|}{\mathcal{H}(x_0)(t)}\right)\omega(t)\,dt}{1 - \lambda\int_0^1 t^2\left(3x_0(t)^2 - \frac{|x_0(t)| - |x_0(t) + \mathcal{H}(x_0)(t)|}{\mathcal{H}(x_0)(t)}\right)dt}.$$

So, the application of the decomposition method (8) is given by the following algorithm:

- **First step:** Calculate:

$$\mathcal{H}(x_n)(s) = x_n(s) - (1 - \frac{11\lambda}{80})s + \frac{1}{2} - \lambda s\int_0^1 t\left(x_n(t)^3 + |x_n(t)|\right)dt.$$

- **Second step:** Calculate:

$$A_n = \int_0^1 3t x_n(t)^2\mathcal{H}(x_n)(t)\,dt, \quad B_n = \int_0^1 t(|x_n(t)| - |x_n(t) + \mathcal{H}(x_n)(t)|)\,dt.$$

$$C_n = \int_0^1 t^2\left(3x_n(t)^2 - \frac{|x_n(t)| - |x_n(t) + \mathcal{H}(x_n)(t)|}{\mathcal{H}(x_n)(t)}\right)dt.$$

- **Third step:** Calculate:

$$W_n = \frac{A_n - B_n}{1 - \lambda C_n},$$
$$x_{n+1}(s) = x_n(s) - \mathcal{H}(x_n)(s) - \lambda s\, W_n.$$

In Tables 2 and 3 we can see the behavior of this algorithm that we apply by choosing different starting guesses and imposing as stopping criterion $\|x_{n+1}(s) - x_n(s)\| \leq 10^{-32}$. We work by using program Matlab 2016b working in variable precision arithmetic with 50 digits of mantissa. The integrals for obtaining coefficient W_n have been solved analytically.

Once the solutions for each iteration n have been obtained, we calculate different norms. The first column shows the difference between two consecutive iterates, and the second column shows the norm between the solution obtained and the exact solution, $x^*(s) = s - 1/2$. In the last two columns we obtain the value W_n indicated in third step of the algorithm and the computational convergence order. As one can check quadratic convergence is reached for all analyzed cases.

5.3 | A comparative study

Finally, we are interested in comparing the numerical application of iterative scheme (8) with Steffensen's method (6), which is the usually iterative scheme used to solve non-differentiable equations. For this purpose, we use the algorithm described in Section 5.2 for obtaining the solution with

n	$\|x_n(s) - x_{n-1}(s)\|$	$\|x_n(s) - x^*(s)\|$	W_n	p
1	8.3916e-01	1.1806e-02	2.0941e-01	
2	1.1803e-02	3.1641e-06	-2.0755e-03	
3	3.1641e-06	2.2566e-13	-5.6440e-07	1.92873
4	2.2566e-13	1.1478e-27	-4.0253e-14	2.00093
5	1.1478e-27	4.8383e-53	-2.0475e-28	2
6	4.8827e-53	4.4389e-55	2.4884e-54	1.77501

TABLE 2 Errors for starting point $x_0(s) = s - 1/4$ and tolerance 10^{-32}.

n	$\|x_n(s) - x_{n-1}(s)\|$	$\|x_n(s) - x^*(s)\|$	W_n	p
1	5.2375e-01	1.2354e-03	-6.6512e-02	
2	1.2354e-03	3.4426e-08	-2.2003e-04	
3	3.4426e-08	2.6714e-17	-6.1408e-09	1.73367
4	2.6714e-17	1.6086e-35	-4.7651e-18	2.00007
5	1.6086e-35	4.4342e-53	-2.8693e-36	2

TABLE 3 Errors for starting point $x_0(s) = \dfrac{2s-1}{4}$ and tolerance 10^{-32}.

(8), when we split equation by decomposing operator $\mathcal{H}$ in $\mathcal{F} + \mathcal{G}$ as it is expressed in (31) and (32). In order to apply Steffensen's method, we discretize the integral equation (30) by taking $n = 20$ and $n = 40$ subintervals in $[0, 1]$ and using the Simpson quadrature to transform equation (30) into a finite dimensional problem:

If we denote the nodes by $S = (s_i),\ i = 1, ..., n+1$ and the approximations of $x(s_i)$ by x_i , then equation (30) is equivalent to the following system of nonlinear equations:

$$x_i = (1 - \frac{11\lambda}{80})s_i - \frac{1}{2} + \lambda s_i \sum_{j=1}^{n+1} s_j p_j \left(x_j^3 + |x_j|\right), \quad i = 1, 2, \ldots, n+1. \tag{34}$$

where $P = (p_j),\ j = 1, ..., n+1$ are Simpson's weights, given by $P = \frac{1}{n+1}(1, 4, 2, \ldots, 2, 4, 1)$.

Then, the system of nonlinear equations given in (34) is of the form

$$\mathbb{H}(\mathbf{x}) = \mathbf{x} - (1 - \frac{11\lambda}{80})S - \frac{1}{2} + \lambda S(P\mathbf{v_x}) = 0, \qquad \mathbb{H} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}, \tag{35}$$

where

$$\mathbf{v_x} = (s_1(x_1^3 + |x_1|), s_2(x_2^3 + |x_2|), \ldots, s_{n+1}(x_{n+1}^3 + |x_{n+1}|))^T.$$

For applying Steffensen's method we consider divided difference of first order, (9), given by $[\mathbf{u}, \mathbf{v}; \mathbb{H}] = ([\mathbf{u}, \mathbf{v}; \mathbb{H}]_{ij})_{i,j=1}^{n+1} \in \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$, where

$$[\mathbf{u}, \mathbf{v}; \mathbb{H}]_{ij} = \frac{1}{u_j - v_j}\left(\mathbb{H}_i(u_1, \ldots, u_j, v_{j+1}, \ldots, v_{n+1}) - \mathbb{H}_i(u_1, \ldots, u_{j-1}, v_j, \ldots, v_{n+1})\right),$$

$\mathbf{u} = (u_1, u_2, \ldots, u_{n+1})^T$ and $\mathbf{v} = (v_1, v_2, \ldots, v_{n+1})^T$.

We solve the discretized nonlinear problem obtained by using program Matlab 2016b working in variable precision arithmetic with 50 digits of mantissa and iterating until the distance between consecutive iterates is less than the tolerance 10^{-32}. Table 4 shows the number of iterations, $iter$, the distance between the last iterates, $\|x_n(s) - x_{n-1}(s)\|$ and the max-norm of $\mathbb{H}(x)$ at the approximated solution, by taking different initial guesses. We have omitted in Table 4 the value of the computational order of convergence because in all cases the quadratically convergence is reached with same precision.

Finally, in order to compare the numerical results between both methods, (6) and (8), we obtain in Table 5 the distance between the numerical approximation to the solution and the exact solution $x^*(s) = s - 1/2$. That is, for both methods we start by $x_0 = s - 1/4$ and we stop when the distance between two consecutive iterates is less than 10^{-32}. Then, we call $x(s)$ the solution obtained with the Steffensen's method and $y(s)$ the solution with Newton-Steffensen's method, obtained as has been exposed in Section 5.2, so, we can compare both of them with the exact solution of the problem by evaluating these in some nodes. Results can be seen in Table 5 .

At this moment we notice that the solution $y(s)$ obtained with the new algorithm we design in Section 5.2 is enormously better, but this is because working as working as shown throughout this paper allows us not to discretizise the problem when it is not differentiable, while Steffensen's method does.

Method	**Starting guess:** x_0	**iter**	$\|x_n(s) - x_{n-1}(s)\|$	$\|\mathbb{H}(x_n(s))\|$
Steffensen	$(s-1)/4$	6	4.4534e-44	2.2641e-58
	$(2s-1)/4$	5	2.7805e-34	2.8787e-58
	$(4s-1)/6$	6	4.3374e-44	3.0641e-58
	$s+1/2$	7	4.9825e-44	2.6811e-58
	1	7	4.3497e-44	2.6659e-58
	2	8	1.8644e-45	2.2194e-58

TABLE 4 Numerical results for solving equation (30) with $\lambda = 1/11$.

	Steffensen, $n = 20$, (6)	**Steffensen,** $n = 40$, (6)	**Newton-Steffenson,** (8)
s	$\|x^*(s_i) - x(s_i)\|$	$\|x^*(s_i) - x(s_i)\|$	$\|x^*(s_i) - y(s_i)\|$
0.0	0	0	0
0.2	1.1746e-08	7.3386e-10	4.5244e-56
0.4	2.3493e-08	1.4677e-09	9.0468e-56
0.6	3.5239e-08	2.2016e-09	1.3573e-55
0.8	4.6985e-08	2.9354e-09	1.8098e-55
1.0	5.8731e-08	3.6693e-09	2.2622e-55

TABLE 5 Comparing different methods applied to nonlinear problem (30).

6 | CONCLUSIONS

In this paper, we consider the Newton-Steffensen iterative scheme by using the decomposition method. Specifically, we apply this iterative scheme for approximating a solution of nonlinear Fredholm integral equations with non-differentiable Nemystkii operator and to obtain domains of existence and uniqueness of solution for these equations.

Finally, we apply the theoretical results obtained to a numerical experiment in order to show the applicability of the study. Moreover, we design an algorithm in order to apply the Newton-Steffensen iterative scheme to approximate a solution of the nonlinear Fredholm integral equation with non-differentiable Nemystkii operator considered. We observe that the numerical results obtained indicate the competitiveness of the proposed development.

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8 | CONFLICT OF INTEREST

There are no conflicts of interest to this work.

9 | DATA TYPE

No data were used to support this study.

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