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Additional Information

Mean square convergent numerical solutions of random fractional differential equations: Approximations of moments and density

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Abstract

A fractional forward Euler-like method is developed to solve initial value problems with uncertainties formulated via the Caputo fractional derivative. The analysis is conducted by using the so-called random mean square calculus. Under mild conditions on the data, the mean square convergence of the numerical method is proved. This type of stochastic convergence guarantees the approximations of the mean and the variance of the solution stochastic process, computed via the aforementioned numerical scheme, will converge to their corresponding exact values. Furthermore, from this probability information, we calculate reliable approximations to the first probability density function of the solution by taking advantage of the Maximum Entropy Principle. The theoretical analysis is illustrated by two examples.

Keywords: Fractional differential equations with randomness, Random mean square calculus, Random mean square Caputo fractional derivative, Random numerics, Maximum Entropy Principle.

1. Introduction and motivation

Engineers and scientists have developed new models formulated via fractional differential equations. These models have been applied in a wide range of areas including viscoelasticity and viscoplasticity problems, to study polymers and proteins, to analyze the dynamics of transmission of ultrasound waves, to deal with human tissue under mechanical loads, etc., [1, 2, 3, 4]. When they are applied to describe the dynamics of physical phenomena on the basis of sampled data, the parameters of fractional differential equations (coefficients, forcing/control terms, initial/boundary conditions) need to be fixed. This is usually done by assigning a nominal or averaged value (estimate), thus deterministic, to each model parameter. Although this is often accepted, in the context of modelling it is more natural to interpret parameters of fractional differential equations as random variables or stochastic processes rather than constants and deterministic functions, respectively. These facts make modelling with random fractional differential

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13 equations more appropriate than considering deterministic fractional differential equations.
 14 Random ordinary and random fractional differential equations have been theoretically developed
 15 and applied in the last decades to deal with errors and uncertainties [5, 6, 7] and [8, 9, 10, 11, 12,
 16 13, 14, 15, 10, 13, 14, 16, 17], respectively.

17 In this paper, we deal with random fractional initial value problems (IVP) of the form

$$\begin{cases} ({}^C D_a^\alpha X)(t) &= f(X(t), t), \quad t \in [a, b], \quad 0 < \alpha \leq 1, \\ X(a) &= X_0, \end{cases} \quad (1.1)$$

18 where $({}^C D_a^\alpha X)(t) := \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-u)^{-\alpha} X'(u) du$ is the random mean square Caputo fractional
 19 derivative of order α of the stochastic process $X(t)$ (see [18] and references therein). Henceforth,
 20 we will work in an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and in the Hilbert space
 21 $(L^2(\Omega), \|\cdot\|_2)$ of second-order random variables defined by

$$L^2(\Omega) = \{X : \Omega \longrightarrow \mathbb{R} : \mathbb{E}[X^2] < +\infty\}, \quad \|X\|_2 = (\mathbb{E}[X^2])^{1/2}.$$

22 The norm $\|\cdot\|_2$ is inferred from the inner product $\langle X, Y \rangle = \mathbb{E}[XY]$, $X, Y \in L^2(\Omega)$, being $\mathbb{E}[\cdot]$ the
 23 expectation operator. Since the existence of $\mathbb{E}[X^2]$ entails that $\mathbb{E}[X]$ does, and the variance can
 24 be expressed in terms of these two first moments, $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, every random variable
 25 with finite variance belongs to $L^2(\Omega)$. Given $\mathcal{T} \subset \mathbb{R}$, if $X(t) \equiv \{X(t)(\omega) : t \in \mathcal{T}, \omega \in \Omega\}$ is
 26 a second-order random variable for every $t \in \mathcal{T}$, then $X(t)$ is termed a second-order stochastic
 27 process. These kind of random variables and stochastic processes are met in the most physical
 28 problems involving randomness. The convergence inferred by the $\|\cdot\|_2$ -norm is referred to as
 29 mean square convergence. Throughout this paper, the initial condition, X_0 is assumed to be
 30 second-order random variable, i.e., $X_0 \in L^2(\Omega)$. Additionally, we will assume that the function f
 31 defining the right-hand side of fractional differential equation (1.1), i.e. $f : S \times [a, b] \rightarrow L^2(\Omega)$,
 32 $S \subseteq L^2(\Omega)$, satisfies the following conditions:

H1: f is mean square Lipschitz, that is, there exists $\kappa > 0$ such that

$$\|f(X, t) - f(Y, t)\|_2 \leq \kappa \|X - Y\|_2, \quad X, Y \in L^2(\Omega),$$

H2: f satisfies the mean square modulus of continuity property, i.e.,

$$\lim_{h \rightarrow 0} W(S, h) = 0 \quad W(S, h) = \sup_{X \in S \subseteq L^2(\Omega)} \sup_{|t-t'| \leq |h|} \|f(X, t) - f(X, t')\|_2, \quad S \text{ bounded.}$$

33 The paper is organized as follows. Section 2 addresses the relation between the IVP (1.1) and
 34 a random Volterra integral equation. Sections 3 and 4 are devoted to formulate the random frac-
 35 tional forward Euler-like numerical scheme and to prove its mean square convergence, respec-
 36 tively. In Section 5, we give explicit approximations of the mean, the variance and the covariance
 37 obtained by the random numerical scheme. Section 6 presents an approach to approximate the
 38 first probability density function of the solution stochastic process to the random fractional IVP
 39 (1.1) based on the Maximum Entropy Principle. Section 7 presents several illustrative examples.
 40 Conclusions are drawn in last section.

41 **2. Relation between the random fractional differential and integral equations**

42 As it has been previously pointed out, the aim of the paper is to construct mean square
 43 approximations to the solution stochastic process to the random fractional IVP given by (1.1).
 44 To this end, we will first study the mean square solution of the following random integral equation

$$X(t) = X_0 + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(X(s), s) ds, \quad t \in [a, b], \quad 0 < \alpha \leq 1, \quad (2.1)$$

and its connection with the solution of the random fractional IVP given by (1.1). In the following,
 the operator $J_{a^+}^\alpha : L^2(\Omega) \rightarrow L^2(\Omega)$ is defined as

$$J_{a^+}^\alpha(X(t)) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} X(s) ds & \text{if } 0 < \alpha \leq 1 \\ X(t) & \text{if } \alpha = 0, \end{cases}$$

45 for each $t \in [a, b]$, see [1, p. 13]. From hypotheses H1 and H2, we know that $f(X(t), t)$ is mean
 46 square continuous for $t \in [a, b]$, hence $f(X(t), t)$ is also mean square uniformly continuous in
 47 $t \in [a, b]$, i.e, there exists $M > 0$ such that $\|f(X(t), t)\|_2 \leq M$, for $t \in [a, b]$. As a consequence,
 48 $\left\| \int_a^t (t-s)^{\alpha-1} f(X(s), s) ds \right\|_2 \leq M \int_a^t (t-s)^{\alpha-1} ds = M(t-a)^\alpha / \alpha$ and hence, since $X_0 \in L^2(\Omega)$,
 49 using the triangular inequality in (2.1), we derive that $X(t) \in L^2(\Omega)$. Additionally, notice that
 50 from the fractional differential equation with $\alpha = 1$, since $f(X(t), t)$ is mean square continuous
 51 for $t \in [a, b]$, we deduce that $X'(t)$ is of class $C^1([a, b])$ in the mean square sense, thus $X(t)$
 52 is mean square uniformly continuous in $t \in [a, b]$, which guarantees the integral operator $J_{a^+}^\alpha(X(t))$
 53 is well-defined. Using this operator and defining $Z(t) := f(X(t), t) \in L^2(\Omega)$ for each, $t \in [a, b]$,
 54 the integral equation in (2.1) reads

$$X(t) = X_0 + J_{a^+}^\alpha(Z(t)), \quad t \in [a, b], \quad 0 < \alpha \leq 1. \quad (2.2)$$

55 The next result establishes a relationship between the solutions of the integral equation given
 56 by (2.2) and the IVP given by (1.1). To prove it, first we need the following auxiliary result.

Lemma 2.1. *Let $\alpha, \beta \geq 0$. If $Y(t)$ is mean square continuous on $[a, b]$, then for each $t \in [a, b]$,*

$$J_{a^+}^\alpha(J_{a^+}^\beta(Y(t))) = J_{a^+}^{\alpha+\beta}(Y(t)).$$

Proof. If α or β are zero, the identity is evident by the definition of the operator $J_{a^+}^\alpha$. Now, we
 assume that $\alpha, \beta > 0$. By definition of the operator $J_{a^+}^\beta$, it follows that $J_{a^+}^\beta(Y(s)) := \frac{1}{\Gamma(\beta)} \int_a^s (s-r)^{\beta-1} Y(r) dr$, for each $s \in [a, b]$. Now, for each t in $[a, b]$,

$$\begin{aligned} J_{a^+}^\alpha(J_{a^+}^\beta(Y(t))) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} J_{a^+}^\beta(Y(s)) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_a^s (s-r)^{\beta-1} Y(r) dr \right) ds \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t \left(\int_a^s (t-s)^{\alpha-1} (s-r)^{\beta-1} Y(r) dr \right) ds. \end{aligned} \quad (2.3)$$

57 As for each $t \in [a, b]$ fixed, the triangle $\Delta = \{(s, r) : a \leq s \leq t, a \leq r \leq s\}$ can also be written
 58 as $\Delta = \{(s, r) : r \leq s \leq t, a \leq r \leq t\}$, the above double mean square integral over Δ on the
 59 right-hand side of (2.3) is

$$\frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t \int_r^t (t-s)^{\alpha-1} (s-r)^{\beta-1} Y(r) ds dr = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t Y(r) \left(\int_r^t (t-s)^{\alpha-1} (s-r)^{\beta-1} ds \right) dr.$$

60 The substitution $\tau = t - s$ yields

$$J_{a^+}^\alpha(J_{a^+}^\beta(Y(t))) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t Y(r) \left(\int_0^{t-r} \tau^{\alpha-1} (t - \tau - r)^{\beta-1} d\tau \right) dr.$$

A second substitution $v = \frac{\tau}{t-r}$ implies

$$J_{a^+}^\alpha(J_{a^+}^\beta(Y(t))) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t \left(\int_0^1 v^{\alpha-1} (t-r)^{\alpha+\beta-1} (1-v)^{\beta-1} dv \right) Y(r) dr.$$

61 Since $\int_0^1 v^{\alpha-1} (1-v)^{\beta-1} dv = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ one gets,

$$J_{a^+}^\alpha(J_{a^+}^\beta(Y(t))) = \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-r)^{\alpha+\beta-1} Y(r) dr = J_{a^+}^{\alpha+\beta}(Y(t)).$$

62

63 **Theorem 2.2.** Let $X(t)$ be a mean square solution of the integral equation given by (2.1), where
 64 $f(X(t), t)$ satisfies hypotheses H1 and H2. Then, $X(t)$ is a mean square solution of the random
 65 fractional IVP (1.1).

66 *Proof.* As above, define $Z(t) := f(X(t), t)$. Hence, from the integral equation in terms of the
 67 operator $J_{a^+}^\alpha$, we derive

$$X(t) - X_0 = J_{a^+}^\alpha(Z(t)), \quad t \in [a, b], \quad 0 < \alpha \leq 1. \quad (2.4)$$

68 As $0 < \alpha \leq 1$, $\beta := 1 - \alpha \geq 0$. By applying the operator $J_{a^+}^{1-\alpha}$ to both sides of the equation (2.4)
 69 and using Lemma 2.1 we obtain

$$J_{a^+}^{1-\alpha}(X(t) - X_0) = J_{a^+}^{1-\alpha}(J_{a^+}^\alpha(Z(t))) = J_{a^+}^1(Z(t)). \quad (2.5)$$

70 Notice that, in view of hypotheses, we have used that $Z(t)$ is mean square continuous on $[a, b]$.
 71 Furthermore, by property (5) of mean square integrals in [19, p. 103], this also entails that
 72 $J_{a^+}^1(Z(t))$ is mean square differentiable on (a, b) . Hence, using (2.5) one gets

$$\frac{d}{dt} (J_{a^+}^{1-\alpha}(X(t) - X_0)) = \frac{d}{dt} (J_{a^+}^{1-\alpha}(X(t))) - \frac{d}{dt} (J_{a^+}^{1-\alpha}(X_0)) = \frac{d}{dt} (J_{a^+}^1(Z(t))). \quad (2.6)$$

Now, we compute the mean square derivatives $\frac{d}{dt} (J_{a^+}^{1-\alpha}(X_0))$ and $\frac{d}{dt} (J_{a^+}^{1-\alpha}(X(t)))$. First, note that

$$J_{a^+}^{1-\alpha}(X_0) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} X_0 ds = \frac{X_0(t-a)^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)},$$

73 SO

$$\frac{d}{dt} (J_{a^+}^{1-\alpha}(X_0)) = \frac{X_0(t-a)^{-\alpha}}{\Gamma(1-\alpha)}. \quad (2.7)$$

Next, $J_{a^+}^{1-\alpha}(X(t)) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} X(s) ds$. Now, since $X(t)$ is mean square integrable and
 $f(t, s) = (t-s)^{-\alpha}$ is almost everywhere continuous in $(t, s) \in [a, b] \times [a, b]$, the mean square
 Leibniz rule yields

$$\frac{d}{dt} (J_{a^+}^{1-\alpha}(X(t))) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (-\alpha)(t-s)^{-\alpha-1} X(s) ds.$$

By using the mean square integration by parts formula, one gets

$$\frac{d}{dt} \left(J_{a^+}^{1-\alpha} (X(t)) \right) = \frac{1}{\Gamma(1-\alpha)} \left(X(a)(t-a)^{-\alpha} + \int_a^t (t-s)^{-\alpha} X'(s) ds \right). \quad (2.8)$$

By definition, $({}^C D_{a^+}^\alpha X)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} X'(s) ds$. Taking into account (2.7), (2.8) and the initial condition $X(a) = X_0$ of IVP (1.1), we deduce $\frac{d}{dt} \left(J_{a^+}^{1-\alpha} (X(t)) \right) = \frac{d}{dt} \left(J_{a^+}^{1-\alpha} (X_0) \right) + ({}^C D_{a^+}^\alpha X)(t)$, which implies

$$\frac{d}{dt} \left(J_{a^+}^{1-\alpha} (X(t)) \right) - \frac{d}{dt} \left(J_{a^+}^{1-\alpha} (X_0) \right) = ({}^C D_{a^+}^\alpha X)(t). \quad (2.9)$$

On the other hand, by the right-hand side of equation (2.6) and applying of [19, property (5), p.103], since $Z(t) = f(X(t), t)$ is mean square continuous, one gets

$$\frac{d}{dt} \left(J_{a^+}^1 (Z(t)) \right) = \frac{d}{dt} \left(\int_a^t Z(s) ds \right) = Z(t) = f(X(t), t). \quad (2.10)$$

74 Finally, $\frac{d}{dt} \left(J_{a^+}^{1-\alpha} (X(t)) \right) - \frac{d}{dt} \left(J_{a^+}^{1-\alpha} (X_0) \right) = \frac{d}{dt} \left(J_{a^+}^1 (Z(t)) \right)$, from (2.9) and (2.10) it follows that
 75 $({}^C D_{a^+}^\alpha X)(t) = f(X(t), t)$, which means that $X(t)$ is a mean square solution of the random fractional
 76 IVP (1.1). ■

77 3. Numerical approximations to the random fractional IVP

This section is devoted to construct reliable discrete approximations to the solution stochastic process of the random fractional IVP (1.1). Suppose that $X(t)$ is a second-order stochastic process such that satisfies the random integral equation given by (2.1), hence, according to Th. 2.2, it also satisfies IVP (1.1). Define the mesh $\{t_n : n \geq 0\}$ as $t_0 := a$ and $t_n := t_0 + nh$, being $h = (b-a)/M > 0$, for a positive integer M that satisfies that $t_M = b$. Let n be any positive integer such that $0 \leq n \leq M$. Evaluating (2.1) at t_n , one gets

$$\begin{aligned} X(t_n) &= X_0 + \frac{1}{\Gamma(\alpha)} \int_a^{t_n} (t_n - s)^{\alpha-1} f(X(s), s) ds \\ &= X_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} f(X(s), s) ds, \quad t_n \in [a, b], \quad 0 < \alpha \leq 1. \end{aligned} \quad (3.1)$$

Now, taking the following approximation

$$\int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} f(X(s), s) ds \approx \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} f(X(t_j), t_j) ds, \quad j = 0, 1, \dots, n-1$$

in (3.1), we obtain an approximation X_n to the second-order random variable $X(t_n)$, formally representing the true solution of the random fractional IVP (1.1) at the time $t = t_n$, defined as

$$\begin{aligned} X_n &= X_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} f(X_j, t_j) ds \\ &= X_0 + \frac{h^\alpha}{\alpha \Gamma(\alpha)} \sum_{j=0}^{n-1} [(n-j)^\alpha - (n-(j+1))^\alpha] f(X_j, t_j), \quad t_n \in [a, b], \quad 0 < \alpha \leq 1, \end{aligned} \quad (3.2)$$

78 where in the last step we have substituted the value of $\int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} ds$, which is well-defined
 79 since the resulting exponent is $\alpha > 0$.

80 4. Error analysis

81 To study the mean square convergence of the random numerical scheme given by equation
 82 (3.2), we introduce the sequence of errors e_n defined as $e_0 = 0$ and $e_n = X_n - X(t_n)$, $n = 1, \dots, M$.
 83 We will prove that for each $t = t_n$ fixed, the $\lim_{h \rightarrow 0} \|e_n\|_2 = 0$, which means that the scheme
 84 given by equation (3.2) is mean square convergent for every $t := t_n$ in $[a, b]$. Now, consider the
 85 last expression of $X(t_n)$ in equation (3.1) and the first expression of X_n in equation (3.2). By
 86 subtracting $X(t_n)$ from X_n , we find

$$e_n = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} [f(X_j, t_j) - f(X(s), s)] (t_n - s)^{\alpha-1} ds.$$

87 Hence, applying [19, p.102] one gets

$$\|e_n\|_2 \leq \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|f(X_j, t_j) - f(X(s), s)\|_2 (t_n - s)^{\alpha-1} ds. \quad (4.1)$$

To accomplish our task, we will find out a bound of the quantity $\|f(X_j, t_j) - f(X(s), s)\|_2$ by using the hypotheses H1 and H2 on f . Indeed,

$$\begin{aligned} \|f(X_j, t_j) - f(X(s), s)\|_2 &\leq \|f(X_j, t_j) - f(X(t_j), t_j)\|_2 \\ &\quad + \|f(X(t_j), t_j) - f(X(s), t_j)\|_2 \\ &\quad + \|f(X(s), t_j) - f(X(s), s)\|_2 \\ &\leq \kappa \|e_j\|_2 + \kappa \|X(t_j) - X(s)\| + W(S, h). \end{aligned}$$

88 The fundamental theorem of mean square calculus [19, p.104] together with property (3) in [19,
 89 p.102], imply $\|X(t_j) - X(s)\| = \left\| \int_s^{t_j} X'(r) dr \right\| \leq h \max_{r \in [a, b]} X'(r)$. Notice that here we have applied
 90 that $f(X(t), t)$ is mean square continuous, hence $X'(t)$ is mean square continuous, so mean square
 91 integrable. Therefore,

$$\|f(X_j, t_j) - f(X(s), s)\|_2 \leq \kappa \|e_j\|_2 + \gamma(h), \quad (4.2)$$

where $\gamma(h) := \kappa h \max_{r \in [a, b]} X'(r) + W(S, h)$. Observe that $\lim_{h \rightarrow 0} \gamma(h) = 0$, since by hypothesis H2, $\lim_{h \rightarrow 0} W(S, h) = 0$. Next, using the inequalities (4.1) and (4.2) we find

$$\begin{aligned} \|e_n\|_2 &\leq \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (\kappa \|e_j\|_2 + \gamma(h)) (t_n - s)^{\alpha-1} ds \\ &= \frac{\kappa}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \|e_j\|_2 \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} ds \\ &\quad + \frac{\gamma(h)}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} ds. \end{aligned} \quad (4.3)$$

Observe that $\int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} ds = \frac{h^\alpha}{\alpha} [(n-j)^\alpha - (n-(j+1))^\alpha]$ and $(n-j)^\alpha - (n-(j+1))^\alpha \leq 1$, $j = 0, 1, \dots, n-1$, and so $\int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} ds \leq \frac{h^\alpha}{\alpha}$. We will use the last inequality and the equality, respectively, on the right-hand side of the last expression in the inequality (4.3). Indeed,

$$\|e_n\|_2 \leq \frac{\kappa h^\alpha}{\alpha \Gamma(\alpha)} \sum_{j=0}^{n-1} \|e_j\|_2 + \frac{\gamma(h) h^\alpha}{\alpha \Gamma(\alpha)} \sum_{j=0}^{n-1} [(n-j)^\alpha - (n-(j+1))^\alpha]. \quad (4.4)$$

Taking into account the value of the following finite telescopic sum $\sum_{j=0}^{n-1} [(n-j)^\alpha - (n-(j+1))^\alpha] = n^\alpha$ and $(t_n - t_0)^\alpha = (nh)^\alpha$, from (4.4) it follows that

$$\|e_n\|_2 \leq \frac{\kappa h^\alpha}{\alpha \Gamma(\alpha)} \sum_{j=0}^{n-1} \|e_j\|_2 + \frac{\gamma(h)}{\alpha \Gamma(\alpha)} (t_n - t_0)^\alpha.$$

Defining $A(h) := \frac{\kappa h^\alpha}{\alpha \Gamma(\alpha)}$ and $B(h) := \frac{\gamma(h)}{\alpha \Gamma(\alpha)} (t_n - t_0)^\alpha$, the above inequality reads

$$\|e_n\|_2 \leq A(h) \sum_{j=0}^{n-1} \|e_j\|_2 + B(h), \quad n = 1, 2, \dots \quad (4.5)$$

In the following deduction, the inequality given by (4.5) will be repeatedly used. Indeed,

$$\begin{aligned} \|e_1\|_2 &\leq A(h)\|e_0\|_2 + B(h), \\ \|e_2\|_2 &\leq A(h)\|e_0\|_2 + B(h) + A(h)\|e_1\|_2 \\ &\leq (1 + A(h))(A(h)\|e_0\|_2 + B(h)). \end{aligned}$$

Let k be an integer such that $2 \leq k < M$. Suppose that

$$\|e_j\|_2 \leq (1 + A(h))^{j-1} (A(h)\|e_0\|_2 + B(h)) \quad (4.6)$$

holds for all integer j such that $1 \leq j \leq k$. We will show that the inequality in (4.6) fulfils for $j = k+1$, that is

$$\|e_{k+1}\|_2 \leq (1 + A(h))^k (A(h)\|e_0\|_2 + B(h)).$$

By using the inequality given by (4.5) and the induction hypothesis given by (4.6), one gets

$$\begin{aligned} \|e_{k+1}\|_2 &\leq A(h)\|e_0\|_2 + B(h) + A(h) \sum_{j=1}^k \|e_j\|_2 \\ &\leq (A(h)\|e_0\|_2 + B(h)) + A(h) \sum_{j=1}^k (1 + A(h))^{j-1} (A(h)\|e_0\|_2 + B(h)) \\ &= (A(h)\|e_0\|_2 + B(h)) \left[1 + A(h) \sum_{j=1}^k (1 + A(h))^{j-1} \right]. \end{aligned}$$

Since $(1 + A(h))^k = 1 + A(h) \sum_{j=1}^k (1 + A(h))^{j-1}$,

$$\|e_{k+1}\|_2 \leq (A(h)\|e_0\|_2 + B(h))(1 + A(h))^k.$$

Then, it follows that

$$\|e_n\|_2 \leq (1 + A(h))^{n-1}(A(h)\|e_0\|_2 + B(h)), \quad (4.7)$$

for all integer n such that $1 \leq n \leq M$. As $\lim_{h \rightarrow 0} A(h) = \lim_{h \rightarrow 0} B(h) = 0$, it follows from the inequality (4.7) that $\lim_{h \rightarrow 0} \|e_n\|_2 = 0$. This means that the scheme given by (3.2) is mean square convergent at the fixed number $t = t_n$. In this case we say that the scheme is mean square convergent in the fixed station sense. The next result summarizes our findings.

Theorem 4.1. *With the previous notation, if the function f on the right-hand side of the random fractional IVP (1.1) satisfies conditions H1 and H2, then the random fractional forward Euler-like scheme given by (3.2) is mean square convergent to the solution of (1.1), for every $t = t_n := a + nh \in [a, b]$.*

5. Statistical moments of the numerical approximations

So far we have established sufficient conditions in order to guarantee the mean square convergence of the random fractional numerical scheme (3.2). In practice, apart from constructing approximations of random fractional differential equations it is also important to provide reliable information related to the main statistical properties of such approximations. In particular, a main goal is the computation of the mean and the variance of the approximations to the solution stochastic process. This section is addressed to accomplish this target in the context of problem IVP (1.1).

Taking the expectation operator, $\mathbb{E}[\cdot]$, in expression (3.2) and using its linearity, we deduce that the mean of the X_n is given by

$$\mathbb{E}[X_n] = \mathbb{E}[X_0] + C(\alpha) \sum_{j=0}^{n-1} a_{nj}(\alpha) \mathbb{E}[f(X_j, t_j)], \quad (5.1)$$

where $C(\alpha) = \frac{h^\alpha}{\Gamma(\alpha+1)}$ and $a_{nj}(\alpha) = (n-j)^\alpha - (n-(j+1))^\alpha$.

Similarly, taking the covariance operator, $\mathbb{Cov}[\cdot, \cdot]$, and using that it is bilinear, one gets that the covariance of X_n and X_m is given by

$$\begin{aligned} \mathbb{Cov}[X_n, X_m] &= \mathbb{Cov}\left[X_0 + C(\alpha) \sum_{j=0}^{n-1} a_{nj}(\alpha) f(X_j, t_j), X_0 + C(\alpha) \sum_{i=0}^{m-1} a_{mi}(\alpha) f(X_i, t_i)\right] \\ &= \mathbb{V}[X_0] + C(\alpha) \sum_{i=0}^{m-1} a_{mi}(\alpha) \mathbb{Cov}[X_0, f(X_i, t_i)] \\ &\quad + C(\alpha) \sum_{j=0}^{n-1} a_{nj}(\alpha) \mathbb{Cov}[f(X_j, t_j), X_0] \\ &\quad + (C(\alpha))^2 \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} a_{nj}(\alpha) a_{mi}(\alpha) \mathbb{Cov}[f(X_j, t_j), f(X_i, t_i)]. \end{aligned} \quad (5.2)$$

111 Letting $m = n$ in (5.2), we obtain the variance of X_n ,

$$\begin{aligned} \mathbb{V}[X_n] &= \mathbb{V}[X_0] + 2C(\alpha) \sum_{j=0}^{n-1} a_{nj}(\alpha) \text{Cov}[X_0, f(X_j, t_j)] \\ &\quad + (C(\alpha))^2 \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} a_{nj}(\alpha) a_{ni}(\alpha) \text{Cov}[f(X_j, t_j), f(X_i, t_i)], \end{aligned} \quad (5.3)$$

112 where

$$\text{Cov}[X_0, f(X_j, t_j)] = \mathbb{E}[X_0 f(X_j, t_j)] - \mathbb{E}[X_0] \mathbb{E}[f(X_j, t_j)], \quad (5.4)$$

113 and

$$\text{Cov}[f(X_j, t_j), f(X_i, t_i)] = \mathbb{E}[f(X_j, t_j) f(X_i, t_i)] - \mathbb{E}[f(X_j, t_j)] \mathbb{E}[f(X_i, t_i)]. \quad (5.5)$$

114 6. First probability density function of the approximate solution stochastic process

115

In the previous section, we have constructed approximations to the mean ($\mathbb{E}[X_n]$) and the variance ($\mathbb{V}[X_n]$) of the solution stochastic process, $X(t)$, to the random fractional IVP (1.1) at any point $t_n \in [a, b]$ prefixed. Taking advantage of this minimum probability information (position and variability), in this section we will show how to apply the Maximum Entropy Principle (MEP) [20] in order to calculate an approximation of the probability density function (PDF), say $f_{X_n}(x)$, of the random variable X_n . Since this can be done for any point $t_n \in [a, b]$, the set $\{f_{X_n}(x) : n \geq 0\}$ approximates what is usually termed the first probability density function (1-PDF), $f_1(x, t)$, of $X(t)$, $t \in [a, b]$. We point out that computing approximations to $f_1(x, t)$ is advantageous, since from its integration one can calculate any one-dimensional moments, as well as to evaluate the probability that an event lies within any interval of specific interest, say $[c_1, c_2]$,

$$\mathbb{E}[(X(t))^m] = \int_{-\infty}^{\infty} x^m f_1(x, t) dx, \quad m = 1, 2, \dots, \quad \mathbb{P}[\{\omega \in \Omega : c_1 \leq X(t)(\omega) \leq c_2\}] = \int_{c_1}^{c_2} f_1(x, t) dx.$$

116 So, via the computation of the 1-PDF, further information of the solution is achieved.

117 Entropy is said to be a measure of uncertainty constructed on the basis of incomplete data
118 (for instance, the knowledge of a few moments as the mean and the variance). Here, we will use
119 the so-called *differential entropy* or *Shannon's entropy*, S_A , which, for a given random variable,
120 say A , is defined by

$$S_A = - \int_{\mathcal{D}(A)} f_A(a) \log(f_A(a)) da,$$

121 where $\mathcal{D}(A)$ denotes the domain of A . The larger, the uncertainty of A , the larger its entropy [20,
122 Section 2.2]. In our setting, entropy will be used to construct approximations of the PDF, $f_{X_n}(x)$,
123 of approximation the X_n from the information provided by its available probabilistic information,
124 namely, $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$ (or equivalently $\mathbb{E}[(X_n)^2] = \mathbb{V}[X_n] + \mathbb{E}[X_n]^2$) calculated by (5.1) and
125 (5.3)–(5.5), respectively. Specifically, let X_n be the continuous random variable, defined by the
126 numerical scheme (3.2), that approximates the solution $X(t)$ at $t = t_n$. Then, the MEP consists
127 of seeking $f_{X_n}(x)$ such that, satisfying the condition to be a PDF (i.e., its integral on the domain

128 of X_n is the unit), the two first theoretical moments of X_n , calculated via the PDF, also match the
 129 corresponding values of $\mathbb{E}[X_n]$ and $\mathbb{E}[(X_n)^2]$ (obtained via the numerical approximations), i.e.,

$$\int_{\mathcal{D}(X_n)} f_{X_n}(x) dx = 1, \quad \int_{\mathcal{D}(X_n)} x f_{X_n}(x) dx = \mathbb{E}[X_n], \quad \int_{\mathcal{D}(X_n)} x^2 f_{X_n}(x) dx = \mathbb{E}[(X_n)^2],$$

where $\mathcal{D}(X_n)$ denotes the domain of the random variable X_n , which is in practice unknown. To overcome this lack of information in the domain of integration, here we use the Bienaymé-Chebyshev's inequality [21, p. 122] to approximate $\mathcal{D}(X_n)$. According to this important result, we can assure that the interval $[a_1, a_2]$, with $a_1 = \mathbb{E}[X_n] - 10\sqrt{\mathbb{V}[X_n]}$ and $a_2 = \mathbb{E}[X_n] + 10\sqrt{\mathbb{V}[X_n]}$, will contain all the outcomes of X_n with a probability of 99.9%, regardless the distribution of X_n . So, in our subsequent numerical computations we will take $\mathcal{D}(X_n) \approx [a_1, a_2]$. The PDF $f_{X_n}(x)$ is calculated by maximizing Shannon entropy of the random variable X_n . To this end, we apply the variational extension of classical Lagrange multiplier method [20]. Hence, we search a function, $f_{X_n} : [a_1, a_2] \rightarrow \mathbb{R}$, such that

$$\begin{aligned} \mathcal{L}(f_{X_n}, \lambda_0, \lambda_1, \lambda_2) &= - \int_{a_1}^{a_2} f_{X_n}(x) \log(f_{X_n}(x)) dx + \lambda_0 \left(1 - \int_{a_1}^{a_2} f_{X_n}(x) dx \right) \\ &\quad + \lambda_1 \left(\mathbb{E}[X_n] - \int_{a_1}^{a_2} x f_{X_n}(x) dx \right) + \lambda_2 \left(\mathbb{E}[(X_n)^2] - \int_{a_1}^{a_2} x^2 f_{X_n}(x) dx \right) \\ &= - \int_{a_1}^{a_2} f_{X_n}(x) \left(\log(f_{X_n}(x)) + \sum_{i=0}^2 \lambda_i x^i \right) dx + \lambda_0 + \lambda_1 \mathbb{E}[X_n] + \lambda_2 \mathbb{E}[(X_n)^2], \end{aligned}$$

130 where λ_i , $i = 0, 1, 2$, are the so-called Lagrange multipliers. Using variational calculus, we
 131 impose the four conditions:

$$\frac{\partial \mathcal{L}(f_{X_n}, \lambda_0, \lambda_1, \lambda_2)}{\partial f_{X_n}} = 0, \quad \frac{\partial \mathcal{L}(f_{X_n}, \lambda_0, \lambda_1, \lambda_2)}{\partial \lambda_0} = 0, \quad \frac{\partial \mathcal{L}(f_{X_n}, \lambda_0, \lambda_1, \lambda_2)}{\partial \lambda_1} = 0, \quad \frac{\partial \mathcal{L}(f_{X_n}, \lambda_0, \lambda_1, \lambda_2)}{\partial \lambda_2} = 0. \quad (6.1)$$

The first condition yields

$$\frac{\partial \mathcal{L}(f_{X_n}, \lambda_0, \lambda_1, \lambda_2)}{\partial f_{X_n}} = - \int_{a_1}^{a_2} \left(1 + \log(f_{X_n}(x)) + \sum_{i=0}^2 \lambda_i x^i \right) dx = 0.$$

132 Obviously, this condition holds when $1 + \log(f_{X_n}(x)) + \sum_{i=0}^2 \lambda_i x^i = 0$. This yields

$$f_{X_n}(x) = \mathbb{1}_{[a_1, a_2]} e^{-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2}, \quad (6.2)$$

133 where $\mathbb{1}_{[a_1, a_2]}$ denotes the characteristic function on the interval $[a_1, a_2] \approx \mathcal{D}(X_n)$. On the other
 134 hand, with the remaining conditions of (6.1), we can compute the values of Lagrange multipliers
 135 $\lambda_0, \lambda_1, \lambda_2$, solving the nonlinear system

$$\begin{cases} \frac{\partial \mathcal{L}(f_{X_n}, \lambda_0, \lambda_1, \lambda_2)}{\partial \lambda_0} = 0 \longrightarrow \int_{a_1}^{a_2} e^{-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2} dx = 1, \\ \frac{\partial \mathcal{L}(f_{X_n}, \lambda_0, \lambda_1, \lambda_2)}{\partial \lambda_1} = 0 \longrightarrow \int_{a_1}^{a_2} x e^{-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2} dx = \mathbb{E}[X_n], \\ \frac{\partial \mathcal{L}(f_{X_n}, \lambda_0, \lambda_1, \lambda_2)}{\partial \lambda_2} = 0 \longrightarrow \int_{a_1}^{a_2} x^2 e^{-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2} dx = \mathbb{E}[(X_n)^2]. \end{cases} \quad (6.3)$$

136 **7. Numerical examples**

137 The aim of this section is to present some examples in order to illustrate the random fractional
 138 forward Euler-like method developed in the previous sections. To check its accuracy, the first ex-
 139 ample have been chosen so that the exact values for the mean and the variance can be determined,
 140 and then we can compare them against the ones provide by approximations obtained via the ran-
 141 dom fractional numerical scheme. Additionally, a similar comparative analysis is performed for
 142 the 1-PDF, which is calculated in two ways, namely, via the Random Variable Transformation
 143 technique [22], which is exact, and by the approximate method exhibited in Section 6, which
 144 combines the MEP and the random fractional numerical scheme. To analyse the accuracy of the
 145 approximations for the mean and the variance, we will use of the absolute error (AE) for the
 146 mean and for the variance,

$$\begin{aligned} \text{AE(Mean)}(t, n) &= |\mathbb{E}[X_n] - \mathbb{E}[X(t_n)]|, \\ \text{AE(Variance)}(t, n) &= |\mathbb{V}[X_n] - \mathbb{V}[X(t_n)]|, \end{aligned} \quad (7.1)$$

147 where $\mathbb{E}[X(t_n)]$ and $\mathbb{V}[X(t_n)]$ are the exact mean and variance of the solution at time t_n , re-
 148 spectively, and $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$ denote their corresponding approximations given by (5.1) and
 149 (5.3)–(5.5), respectively. As we will see later, we have chosen the absolute error as error measure
 150 since in our examples the mean values of the solutions and their variances are close to zero.

151 **Example 7.1.** *Let us consider the following random fractional IVP*

$$\begin{cases} ({}^C D_{0^+}^\alpha X)(t) &= -X(t) + At^2 + X_0 + \frac{2A}{\Gamma(3-\alpha)}t^{2-\alpha}, \quad t \in [0, 0.5], \quad 0 < \alpha \leq 1, \\ X(0) &= X_0, \end{cases} \quad (7.2)$$

152 where A and X_0 are independent second-order random variables. According to the IVP (1.1),
 153 $f(X, t) = -X + At^2 + X_0 + \frac{2A}{\Gamma(3-\alpha)}t^{2-\alpha}$ and is straightforward to check that $f(X, t)$ verifies H1 and
 154 H2. Thus, the numerical scheme (3.2) is given by

$$X_n = X_0 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} ((n-j)^\alpha - (n-j-1)^\alpha) \left(-X_j + At_j^2 + X_0 + \frac{2A}{\Gamma(3-\alpha)}t_j^{2-\alpha} \right).$$

155 To calculate approximations of the mean, we apply expression (5.1), taking into account that

$$\mathbb{E}[f(X_j, t_j)] = -\mathbb{E}[X_j] + t_j^2 \mathbb{E}[A] + \mathbb{E}[X_0] + \frac{2t_j^{2-\alpha}}{\Gamma(3-\alpha)} \mathbb{E}[A], \quad (7.3)$$

156 while to determine approximations of the variance, we use expression (5.3)–(5.5), taking into
 157 account that

$$\begin{aligned} \text{Cov}[X_0, f(X_j, t_j)] &= -\text{Cov}[X_0, X_j] + \mathbb{V}[X_0], \\ \text{Cov}[f(X_j, t_j), f(X_i, t_i)] &= \mathbb{V}[X_0] + \mathbb{V}[A] \left(t_j^2 t_i^2 + \frac{2t_j^2 t_i^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t_j^{2-\alpha} t_i^2}{\Gamma(3-\alpha)} + 4 \frac{t_j^{2-\alpha} t_i^{2-\alpha}}{\Gamma(3-\alpha)^2} \right) \\ &\quad + \text{Cov}[X_j, A] \left(-t_i^2 - 2 \frac{t_i^{2-\alpha}}{\Gamma(3-\alpha)} \right) \\ &\quad + \text{Cov}[X_i, A] \left(-t_j^2 - 2 \frac{t_j^{2-\alpha}}{\Gamma(3-\alpha)} \right) \\ &\quad + \text{Cov}[X_j, X_i] + \text{Cov}[X_j, X_0] + \text{Cov}[X_i, X_0]. \end{aligned} \quad (7.4)$$

	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
$n = 100$	4.77208e-03	1.95806e-02	4.44101e-02	7.92553e-02
$n = 200$	4.88730e-03	1.97919e-02	4.47069e-02	7.96296e-02
$n = 400$	4.94406e-03	1.98965e-02	4.48540e-02	7.98154e-02
$n = 800$	4.97217e-03	1.99484e-02	4.49272e-02	7.99079e-02
$n = 1600$	4.98613e-03	1.99743e-02	4.49637e-02	7.99540e-02
$n = 3200$	4.99308e-03	1.99871e-02	4.49818e-02	7.99770e-02

Table 1: Approximations of the mean calculated by expressions (5.1) and (7.3) for different values of n and different time instants t . Example 7.1.

	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
$n = 100$	1.00323e-02	1.00596e-02	1.00990e-02	1.01698e-02
$n = 200$	1.00197e-02	1.00364e-02	1.00633e-02	1.01186e-02
$n = 400$	1.00074e-02	1.00161e-02	1.00373e-02	1.00897e-02
$n = 800$	1.00670e-02	1.01184e-02	1.01790e-02	1.02686e-02
$n = 1600$	9.96446e-03	9.94269e-03	9.93635e-03	9.96339e-03
$n = 3200$	9.99630e-03	9.99561e-03	1.00056e-02	1.00436e-02

Table 2: Approximations of the variance calculated by expressions (5.3) and (7.4) for different values of n and different time instants t . Example 7.1.

158 *It is easy to check that the solution stochastic process to the random fractional IVP (7.2) is given*
159 *by*

$$X(t) = At^2 + X_0. \quad (7.5)$$

Consequently, since A and X_0 are independent, the mean and the variance of $X(t)$ is given by

$$\begin{aligned} \mathbb{E}[X] &= t^2\mathbb{E}[A] + \mathbb{E}[X_0], \\ \mathbb{V}[X] &= t^4\mathbb{V}[A] + \mathbb{V}[X_0]. \end{aligned}$$

160 *To carry out computations, let us consider that A has a beta distribution $A \sim \text{Be}(80; 80)$ and X_0*
161 *has a Gaussian distribution with zero mean and standard deviation 0.1, $X_0 \sim N(0; 0.1^2)$. In this*
162 *example, we will take as fractional order $\alpha = 0.7 \in (0, 1]$. Tables 1 and 2 collect the values of*
163 *the approximations for the mean and for the variance, respectively, of the solution computed via*
164 *(5.1) and (7.3) for the mean, and via (5.3) and (7.4) for the variance, at the final times instants $t =$*
165 *0.1, 0.2, 0.3, 0.4 and different nodes of discretization $M = 100, 200, 400, 800, 1600, 3200$. So, for*
166 *t fixed, the step size $h = t/M$ decreases as N increases. In Tables 3 and 4, we show the absolute*
167 *errors, defined in (7.1), for the approximations of the mean and the variance, respectively. From*
168 *the figures collected in these tables, we can observe that for t fixed, the errors decrease as M*
169 *increases as expected.*

170 *So far, we have computed approximations for the main statistical moments, namely the mean*
171 *and the variance, of the solution $X(t)$ via the random numerical scheme, and they have been*
172 *compared with their exact values. On the one hand, using the MEP described in Section 6*
173 *together with these approximations for the mean and for the variance, we can approximate the*
174 *1-PDF of $X(t)$ according to (6.2) where the parameters λ_i , $i = 0, 1, 2$ solve (6.3). In Figure 1,*
175 *we show the approximation of 1-PDF at the time interval $[0, 0.4]$. The values of λ_0 , λ_1 and λ_2 ,*
176 *corresponding to the time instants $t = 0.1, 0.2, 0.3, 0.4$ are collected in Table 5.*

	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
$n = 100$	2.27923e-04	4.19409e-04	5.89888e-04	7.44651e-04
$n = 200$	1.12701e-04	2.08060e-04	2.93103e-04	3.70370e-04
$n = 400$	5.59350e-05	1.03507e-04	1.45974e-04	1.84577e-04
$n = 800$	2.78325e-05	5.15880e-05	7.28065e-05	9.20999e-05
$n = 1600$	1.38729e-05	2.57419e-05	3.63469e-05	4.59913e-05
$n = 3200$	6.92263e-06	1.28546e-05	1.81559e-05	2.29775e-05

Table 3: Absolute errors for the approximations of the mean collected in Table 1. Example 7.1.

	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
$n = 100$	3.16109e-05	5.48771e-05	8.13128e-05	1.21075e-04
$n = 200$	1.89957e-05	3.16587e-05	4.56261e-05	6.98054e-05
$n = 400$	6.65378e-06	1.13590e-05	1.96601e-05	4.09081e-05
$n = 800$	6.62481e-05	1.13694e-04	1.61319e-04	2.19851e-04
$n = 1600$	3.62636e-05	6.20514e-05	8.12956e-05	8.53784e-05
$n = 3200$	4.41900e-06	9.12639e-06	1.20779e-05	5.20279e-06

Table 4: Absolute errors for the approximations of the variance collected in Table 2. Example 7.1.

	λ_0	λ_1	λ_2
$t = 0.1$	-2.38238e+00	-4.99982e-01	4.99982e+01
$t = 0.2$	-2.36349e+00	-1.99934e+00	4.99835e+01
$t = 0.3$	-2.28182e+00	-4.49353e+00	4.99281e+01
$t = 0.4$	-2.06287e+00	-7.96575e+00	4.97859e+01

Table 5: Values of parameters λ_0 , λ_1 and λ_2 that, according to (6.2), determine the approximate 1-PDF using MEP to the random fractional IVP (1.1) at different time instants $t = 0.1, 0.2, 0.3, 0.4$. Example 7.1.

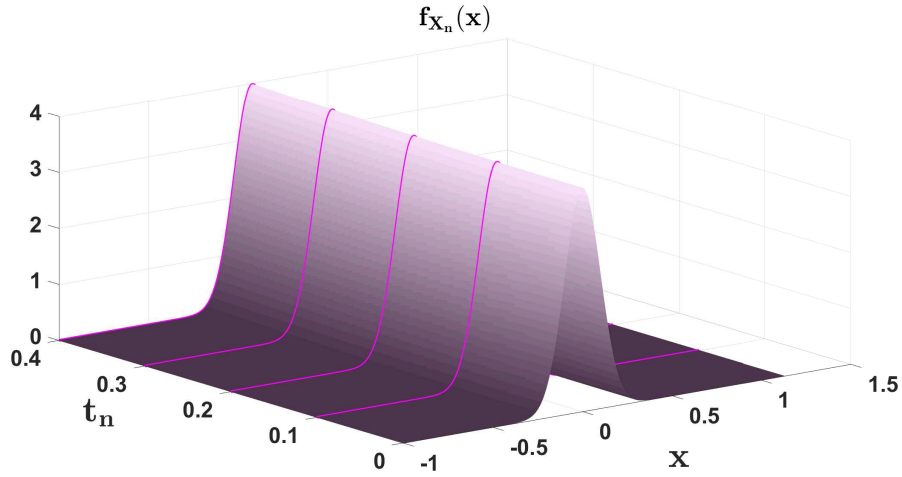


Figure 1: Approximate 1-PDF for the random fractional IVP (1.1) obtained via the combination of the MEP and the random numerical scheme. Example 7.1.

	Error Norm
$t = 0.1$	1.47402e-05
$t = 0.2$	5.89410e-05
$t = 0.3$	1.32434e-04
$t = 0.4$	2.34549e-04

Table 6: Values of the error (via the 2-norm) between the exact 1-PDF, given in (7.6), and its approximations obtained via MEP at different times $t = 0.1, 0.2, 0.3, 0.4$. Example 7.1.

177 On the other hand, as the solution stochastic process of the random fractional IVP (7.2) is
178 given by (7.5), its 1-PDF can be exactly calculated by applying the Random Variable Transform-
179 tion technique [22]. To this end, let us fix t and define the injective mapping $r : \mathbb{R}^2 \rightarrow \mathbb{R}^2$
180 whose components are

$$Y_1 = r_1(X_0, A) = At^2 + X_0, \quad Y_2 = r_2(X_0, A) = A.$$

181 Its inverse mapping, $s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, is given by

$$X_0 = s_1(Y_1, Y_2) = Y_1 - Y_2 t^2, \quad A = s_2(Y_1, Y_2) = Y_2.$$

182 The Jacobian of this transformation is 1. Therefore, taking into account that X_0 and A are
183 independent random variables, the joint PDF of the random vector $\mathbf{Y} = (Y_1, Y_2)$ is given by

$$f_{\mathbf{Y}}(y_1, y_2) = \left(\frac{1}{\sqrt{2\pi}0.1^2} e^{-\frac{1}{2} \left(\frac{y_1 - y_2 t^2}{0.1} \right)^2} \right) \frac{(y_2)^{79} (1 - y_2)^{79}}{\text{Be}(80, 80)},$$

184 where $\text{Be}(\beta_1, \beta_2)$ denotes the deterministic beta special function of parameters $\beta_1, \beta_2 > 0$. Since
185 the solution $X(t)$ is the first component of vector \mathbf{Y} , to obtain the 1-PDF of $X(t)$ we marginalize
186 $f_{\mathbf{Y}}(y_1, y_2)$ with respect to Y_2 . This yields

$$f_{X(t)}(x) = \frac{1}{0.1 \sqrt{2\pi} \text{Be}(80, 80)} \int_0^1 a^{79} (1 - a)^{79} e^{-\frac{1}{2} \left(\frac{x - at^2}{0.1} \right)^2} da. \quad (7.6)$$

187 Figure 2 shows a graphical comparison between the exact 1-PDF, $f_{X(t)}(x)$, and its approximation
188 at the times $t = 0.1, 0.2, 0.3, 0.4$. We can observe that approximations are very good. As a
189 measure of the accuracy of these approximations, in Table 6 we show values of the 2-norm of the
190 difference between $f_{X(t)}(x)$ and approximations $f_{X_n}(x)$ at the above-mentioned time instants. We
191 observe that these figures increase at time increases as expected.

192 **Example 7.2.** Let us consider the following random fractional IVP

$$\begin{cases} ({}^C D_{0^+}^\alpha X)(t) &= \lambda X(t) + A \quad t \in [0, 1], \quad 0 < \alpha \leq 1, \\ X(0) &= X_0, \end{cases} \quad (7.7)$$

193 where $\lambda \in \mathbb{R}$ and X_0 and A are independent second-order random variables. It is easy to prove
194 that $f(X, t) = \lambda X + A$ fulfils hypotheses H1 and H2. In [23], the solution of this particular IVP
195 is obtained using a generalized version of the Frobenius method. This approach leads to the
196 following generalized power series

$$X(t) = X_0 \sum_{m=0}^{\infty} \frac{\lambda^m}{\Gamma(\alpha m + 1)} t^{\alpha m} + A \sum_{m=1}^{\infty} \frac{\lambda^{m-1}}{\Gamma(\alpha m + 1)} t^{\alpha m},$$

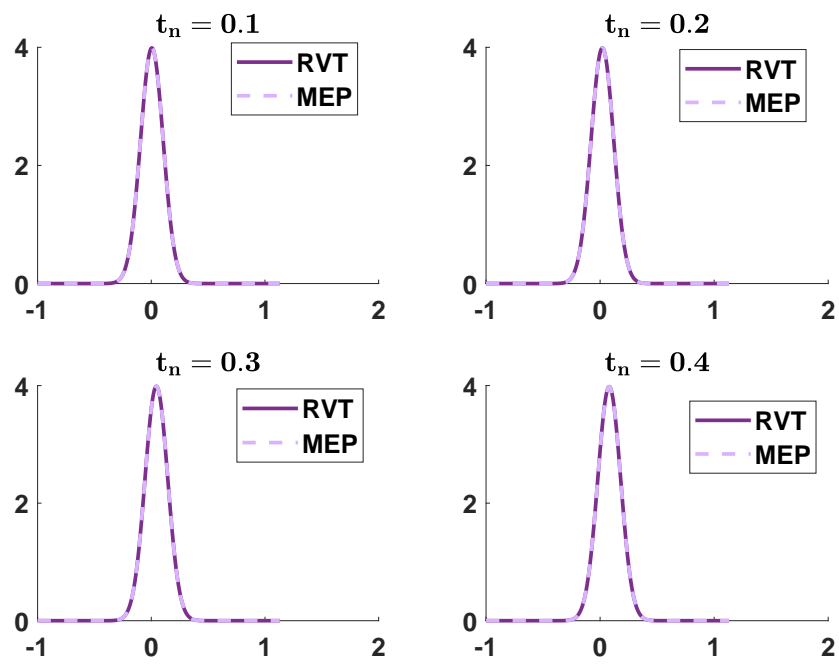


Figure 2: Graphical comparison between the exact 1-PDF given in (7.6) and its approximations obtained via MEP at different times $t = 0.1, 0.2, 0.3, 0.4$. Example 7.1.

consequently its mean and its second order moment are given by ([23, Eq. 21,23])

$$\begin{aligned}
\mathbb{E}[X(t)] &= \mathbb{E}[X_0] \sum_{m=0}^{\infty} \frac{\lambda^m}{\Gamma(\alpha m + 1)} t^{\alpha m} + \mathbb{E}[A] \sum_{m=1}^{\infty} \frac{\lambda^{m-1}}{\Gamma(\alpha m + 1)} t^{\alpha m}, \\
\mathbb{E}[X(t)^2] &= \mathbb{E}[X_0^2] \sum_{m=0}^{\infty} \frac{\lambda^{2m}}{\Gamma(\alpha m + 1)^2} t^{2\alpha m} \\
&\quad + 2\mathbb{E}[X_0^2] \sum_{m=0}^{\infty} \sum_{n=0}^{m-1} \frac{\lambda^{m+n}}{\Gamma(\alpha m + 1)\Gamma(\alpha n + 1)} t^{\alpha(m+n)} \\
&\quad + \mathbb{E}[A^2] \sum_{m=1}^{\infty} \frac{\lambda^{2(m-1)}}{\Gamma(\alpha m + 1)^2} t^{2\alpha m} \\
&\quad + 2\mathbb{E}[A^2] \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} \frac{\lambda^{m+n-2}}{\Gamma(\alpha m + 1)\Gamma(\alpha n + 1)} t^{\alpha(m+n)} \\
&\quad + 2\mathbb{E}[X_0] \mathbb{E}[A] \sum_{m=0}^{\infty} \sum_{n=1}^{m-1} \frac{\lambda^{m+n-1}}{\Gamma(\alpha m + 1)\Gamma(\alpha n + 1)} t^{\alpha(m+n)}.
\end{aligned}$$

197 Taking into account that $\mathbb{V}[X(t)] = \mathbb{E}[X(t)^2] - \mathbb{E}[X(t)]^2$, the variance is easily obtained from the
198 two previous expressions.

199 On the other hand, the numerical scheme (3.2) is given by the following expression

$$X_n = X_0 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} [(n-j)^\alpha - (n-(j+1))^\alpha] (\lambda X_j + A).$$

200 To compute the approximations for the mean and for the variance of the numerical solution given
201 by expressions (5.1) and (5.3)–(5.5), respectively, we need the following expressions

$$\mathbb{E}[f(X_j, t_j)] = \lambda \mathbb{E}[X_j] + \mathbb{E}[A], \quad (7.8)$$

202 and

$$\begin{aligned}
\mathbb{Cov}[X_0, f(X_j, t_j)] &= \lambda \mathbb{Cov}[X_0, X_j], \\
\mathbb{Cov}[f(X_j, t_j), f(X_i, t_i)] &= \lambda^2 \mathbb{Cov}[X_j, X_i] + \lambda \mathbb{Cov}[X_j, A] + \lambda \mathbb{Cov}[X_i, A] + \mathbb{Var}[A],
\end{aligned} \quad (7.9)$$

203 where in the second identity we have used that $\mathbb{Cov}[X_0, A] = 0$, since X_0 and A are assumed
204 to be independent. To carry out the numerical example, let us consider that $A \sim \text{Ga}(1, 1/2)$,
205 $B_0 \sim \text{Exp}(2)$, $\alpha = 0.7$ and $\lambda = 0.75$. In Tables 7 and 8, the values of the approximations for the
206 mean and for the variance, respectively, for different $t = 0.2, 0.4, 0.6, 0.8$ and for different nodes
207 of discretization $n = 100, 200, 400, 800, 1600, 3200$ are shown. In Tables 9 and 10, we collect
208 the values of the absolute errors for the mean and for the variance, respectively. These figures
209 show strong agreement between both approaches, validating the approximations obtained by the
210 numerical scheme.

211 Once reliable approximations for the mean and for the variance have been computed, we can
212 take advantage for the MEP explained in Section 6, to compute approximations of the 1-PDF of
213 the solution for the IVP (7.7). The values of λ_0, λ_1 and λ_2 for $t = 0.2, 0.4, 0.6, 0.8$, are collected
214 in Table 11. In Figure 3, we have plotted the approximate 1-PDF on the interval $0 \leq t \leq 0.8$.

	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
$n = 100$	8.79545e-01	1.19617e+00	1.53429e+00	1.90780e+00
$n = 200$	8.79084e-01	1.19710e+00	1.53707e+00	1.91297e+00
$n = 400$	8.73584e-01	1.19045e+00	1.52909e+00	1.90346e+00
$n = 800$	8.85815e-01	1.20441e+00	1.54454e+00	1.92030e+00
$n = 1600$	8.73489e-01	1.18991e+00	1.52785e+00	1.90127e+00
$n = 3200$	8.76741e-01	1.19452e+00	1.53416e+00	1.90966e+00

Table 7: Approximations for the mean calculated by expressions (5.1) and (7.8) for different values of n at different time instants. Example 7.2.

	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
$n = 100$	4.73540e-01	7.32315e-01	1.07587e+00	1.53917e+00
$n = 200$	4.83486e-01	7.48728e-01	1.10215e+00	1.58059e+00
$n = 400$	4.58941e-01	7.13128e-01	1.05240e+00	1.51228e+00
$n = 800$	4.76056e-01	7.39889e-01	1.09176e+00	1.56789e+00
$n = 1600$	4.60671e-01	7.19230e-01	1.06591e+00	1.53717e+00
$n = 3200$	4.77874e-01	7.42555e-01	1.09631e+00	1.57625e+00

Table 8: Approximations for the variance calculated by expressions (5.3) and (7.9) for different values of n at different time instants. Example 7.2.

	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
$M = 100$	3.86683e-03	4.05048e-03	4.38730e-03	4.79608e-03
$M = 200$	3.40606e-03	4.97333e-03	7.16724e-03	9.97200e-03
$M = 400$	2.09392e-03	1.67543e-03	8.12368e-04	4.57895e-04
$M = 800$	1.01376e-02	1.22837e-02	1.46425e-02	1.72940e-02
$M = 1600$	2.18919e-03	2.21026e-03	2.05022e-03	1.73005e-03
$M = 3200$	1.06352e-03	2.39790e-03	4.25688e-03	6.65941e-03

Table 9: Absolute errors for the mean. Example 7.2.

	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
$M = 100$	1.00640e-02	1.36111e-02	1.77728e-02	2.26666e-02
$M = 200$	2.00101e-02	3.00239e-02	4.40562e-02	6.40841e-02
$M = 400$	4.53483e-03	5.57630e-03	5.69099e-03	4.21865e-03
$M = 800$	1.25798e-02	2.11850e-02	3.36613e-02	5.13880e-02
$M = 1600$	2.80493e-03	5.26002e-04	7.81413e-03	2.06665e-02
$M = 3200$	1.43977e-02	2.38508e-02	3.82108e-02	5.97441e-02

Table 10: Absolute error for the variance. Example 7.2.

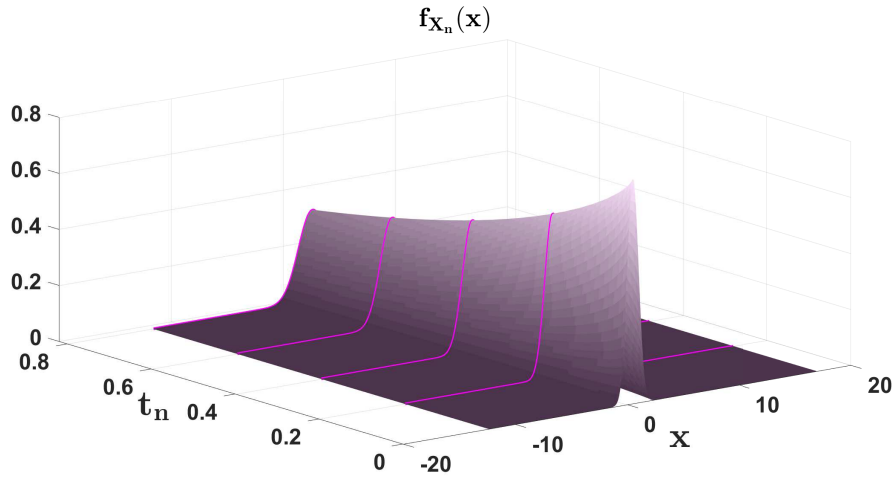


Figure 3: Approximate 1-PDF for the IVP (7.7) using MEP combined with the approximations of the mean and the variance obtained via the random fractional numerical scheme.

	λ_0	λ_1	λ_2
t=0.2	3.61678e-01	-1.88937e+00	1.07880e+00
t=0.4	7.42481e-01	-1.65871e+00	6.95697e-01
t=0.6	1.05322e+00	-1.44590e+00	4.72548e-01
t=0.8	1.32114e+00	-1.25486e+00	3.29706e-01

Table 11: Values of the parameters λ_0 , λ_1 and λ_2 to construct the approximate 1-PDF using MEP for the IVP (7.7) at different $t = 0.2, 0.4, 0.6, 0.8$.

215 8. Conclusions

216 In this paper we have studied the fractional forward Euler-like numerical method to random
217 fractional differential equations. The study has been conducted by means of the so-called mean
218 square calculus. We have given mild sufficient condition in order to guarantee the mean square
219 convergence. This type of stochastic convergence guarantees the mean and the variance of the
220 approximations will converge to the corresponding exact values. This results very useful since
221 these probabilistic moments are not known in practice. This key probabilistic information has
222 been utilized to go further and to calculate reliable approximations of the first probability density
223 function of the solution stochastic process of random fractional differential equations by applying
224 the Maximum Entropy Principle (MEP). Our numerical examples show very satisfactory results.
225 This contribution provides a new approach to approximate the density of random fractional dif-
226 ferential equations via the combination of numerical schemes and MEP. We plan to extend our
227 ideas to other types of numerical schemes.

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