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Density function of random differential equations via finite difference schemes: a theoretical analysis of a random diffusion-reaction Poisson-type problem

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ABSTRACT
A computational approach to approximate the probability density function of random differential equations is based on transformation of random variables and finite difference schemes. The theoretical analysis of this computational method has not been performed in the extant literature. In this paper, we deal with a particular random differential equation: a random diffusion-reaction Poisson-type problem of the form $-u''(x) + \alpha u(x) = \phi(x)$, $x \in [0, 1]$, with boundary conditions $u(0) = A$, $u(1) = B$. Here, α, A and B are random variables and $\phi(x)$ is a stochastic process. The term $u(x)$ is a stochastic process that solves the random problem in the sample path sense. Via a finite difference scheme, we approximate $u(x)$ with a sequence of stochastic processes in both the almost sure and L^p senses. This allows us to find mild conditions under which the probability density function of $u(x)$ can be approximated. Illustrative examples are included.

KEYWORDS
Random diffusion-reaction Poisson-type problem; Finite difference scheme; Probability density function; Numerical methods.

1. Introduction

Random differential equations are differential equations where the input coefficients and initial/boundary conditions are random variables/stochastic processes [17, 18]. The solution is a stochastic process. To completely understand the random behaviour of the solution, one needs to find its joint finite-dimensional distributions, however, in general, this is an impracticable task. A more feasible target consists in finding, or at least approximating, its probability density function (first finite-dimensional distributions) [17, Ch. 3]. Some recent contributions dealing with the computation of the probability density function of the solution of random differential equations can be found in [6, 9]. A computational method to approximate the density function is based on finite difference schemes [7]. The theoretical analysis of this computational approach has not been done in the extant literature. In this paper, we want to perform a comprehensive theoretical analysis for a particular random differential equation: a

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randomized diffusion-reaction Poisson-type problem [15, p. 433],

$$\begin{cases} -u''(x) + \alpha u(x) = \phi(x), & x \in [0,1], \\ u(0) = A, \; u(1) = B. \end{cases} \tag{1}$$

The term $-u''$ models diffusion, the term αu models reaction and ϕ represents an external source [15, p. 432]. We assume an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space, which consists of outcomes that will be generically denoted by ω; $\mathcal{F}$ is the σ-algebra of events; and $\mathbb{P}$ is the probability measure. The term $\alpha \geq 0$ and the boundary values A and B are random variables in our probability space. The source term $\phi(x)$ is a stochastic process. We will omit the evaluation at the outcome ω, however, when necessary, we will write $\alpha(\omega)$, $A(\omega)$, $B(\omega)$ and $\phi(x,\omega)$. The term $u(x)$ is a stochastic process that solves (1) in the sample path sense. When evaluating at the outcome ω, we will write $u(x,\omega)$.

Notation 1. Given a measure space $(S, \mathcal{F}, \mu)$, where $\mathcal{F}$ is the σ-algebra and μ is the measure, we will use the notation $\mathrm{L}^p(S)$ for the p-integrable measurable mappings in the Lebesgue sense: $f : S \to \mathbb{R}$ such that $\|f\|_{\mathrm{L}^p(S)} := (\int_S |f|^p \,\mathrm{d}\mu)^{1/p} < \infty$ for $1 \leq p < \infty$ and $\|f\|_{\mathrm{L}^\infty(S)} = \inf\{\sup\{|f(x)| : x \in S \backslash N\} : \mu(N) = 0\} < \infty$ for $p = \infty$. The shorten notation a.e. and a.s. will stand for "almost every" or "almost everywhere" and "almost surely", respectively.

Given an interval $I \subseteq \mathbb{R}$, the notation $\mathfrak{C}^p(I)$, $p \in \mathbb{N} \cup \{\infty\}$, means p times continuously differentiable on I. When $p = 0$, it means continuous on I, and we will write $\mathfrak{C}(I)$. For $0 < \beta \leq 1$, the notation $C^\beta(I)$ stands for the Hölder class: $f \in C^\beta(I)$ if there exists a constant $k > 0$ such that $|f(x) - f(y)| \leq k|x - y|^\beta$, for all $x, y \in I$. Do not confound $\mathfrak{C}^1(I)$ (continuously differentiable) with $C^1(I)$ (Lipschitz continuous). For $p \in \mathbb{N}$ and $0 < \beta \leq 1$, the notation $C^{p,\beta}(I)$ means being $\mathfrak{C}^p(I)$, with the p-th derivative being in $C^\beta(I)$.

Given a matrix $\mathcal{A}$, we will denote its p-norm as $\|\mathcal{A}\|_p$, $1 \leq p \leq \infty$. The j-th column of the matrix will be denoted by $\mathcal{A}(:,j)$, and its transpose will be written as $\mathcal{A}^T$. The identity matrix of size M will be denoted by I_M.

Finally, given an absolutely continuous random variable X, its density function will be denoted by f_X.

For the sake of completeness, below we give sufficient conditions on the external source ϕ in order to guarantee that the random problem (1) has a unique solution in different stochastic senses, commonly used in the extant literature.

Proposition 1.1. *The following holds:*

(i) If ϕ has sample paths in $\mathrm{L}^2([0,1])$, then there is a unique process u with sample paths in the Sobolev space $\mathrm{H}^2(0,1)$ [3, Ch. 8] that solve (1).
(ii) If ϕ has sample paths in $\mathfrak{C}([0,1])$, then there is a unique process u with sample paths in $\mathfrak{C}^2([0,1])$ that solve (1).
(iii) If ϕ has sample paths in $C^\beta([0,1])$, for some $0 < \beta \leq 1$, then there exists a unique process u with sample paths in $C^{2,\beta}([0,1])$ that solve (1).

Proof. The three statements are direct consequence of the deterministic theory for differential equations. Part (i) is a consequence of [15, Prop. 8.1] [3, Prop. 8.16]. Part (ii) is a consequence of part (i) and [3, Remark 6, p. 204]. Part (iii) is a consequence of [10, Th. 11.3.2]. □

The main goal of this paper is to analyze when $u(x)$ is an absolutely continuous random variable, for each x, and then to compute its probability density function. For this purpose, we will use a finite difference scheme to approximate the solution stochastic process $u(x)$.

2. Random finite difference scheme

Divide $[0,1]$ into M equidistant interior points $x_1,\ldots,x_M$: $0=x_0<x_1<\ldots<x_M<x_{M+1}=1$, $x_i=x_i^M=i/(M+1)$. Denote $h=1/(M+1)$. The numerical scheme, based on discretizations of the second derivative, is $-\frac{1}{h^2}u_{i+1}^M+(\frac{2}{h^2}+\alpha)u_i^M-\frac{1}{h^2}u_{i-1}^M=\phi(x_i)$, for $1\leq i\leq M$, $u_0^M=A$ and $u_{M+1}^M=B$. One expects $u_i^M\approx u(x_i)$. In matrix form, $\mathcal{A}u^M=c$, where

$$u^M=\begin{pmatrix} u_1^M \\ \vdots \\ u_M^M \end{pmatrix}, \quad c=\begin{pmatrix} \phi(x_1)+A/h^2 \\ \phi(x_2) \\ \vdots \\ \phi(x_{M-1}) \\ \phi(x_M)+B/h^2 \end{pmatrix},$$

$$\mathcal{A}=\alpha I_M+\frac{1}{h^2}L, \quad L=\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

Although not explicitly written, $\mathcal{A}$, L, c, x_i and h depend on M. Since $\alpha\geq 0$, then $\mathcal{A}$ is invertible. Thus, $u^M=\mathcal{A}^{-1}c$.

Lemma 2.1. *The matrix $\mathcal{A}$ satisfies $\|\mathcal{A}^{-1}\|_\infty\leq 1/8$.*

Proof. The matrix $\mathcal{A}$ is an M-matrix, in the sense of [13, p. 10]: $\mathcal{A}$ has its offdiagonal entries nonpositive and, for $r=(r_i)_{i=1}^M$ with $r_i=1/4-(x_i-1/2)^2>0$, one has $(\mathcal{A}r)_i=2+\alpha r_i\geq 2$, $1\leq i\leq M$. By [13, Lemma 5.3], $\|\mathcal{A}^{-1}\|_\infty\leq\|r\|_\infty/\min_i(\mathcal{A}r)_i\leq\frac{1/4}{2}=1/8$. □

Proposition 2.2. *Suppose ϕ has sample paths in $C^\beta([0,1])$, for certain $0<\beta\leq 1$. Let $x_0\in[0,1]$. Let $\{i_M\}_{M=1}^\infty$ be a sequence of indexes, $i_M\in\{1,\ldots,M\}$, such that $\lim_{M\to\infty}i_M/(M+1)=x_0$. Then $\lim_{M\to\infty}u_{i_M}^M=u(x_0)$ a.s. Moreover, the following rate of convergence holds: $|u(x_{i_M},\omega)-u_{i_M}^M(\omega)|\leq C(\omega)/8\cdot h^\beta$, where $C(\omega)$ is the Hölder constant of $u''(\cdot,\omega)$ on $[0,1]$.*

Proof. By Proposition 1.1 (iii), the sample paths of u belong to $C^{2,\beta}([0,1])$: $|u''(x,\omega)-u''(y,\omega)|\leq C(\omega)|x-y|^\beta$, for all $x,y\in[0,1]$. Using Taylor's expansions,

$$u(x+h)=u(x)+u'(x)h+u''(\xi_{x,h})h^2/2, \quad u(x-h)=u(x)-u'(x)h+u''(\eta_{x,h})h^2/2,$$

where $\xi_{x,h} \in (x, x+h)$ and $\eta_{x,h} \in (x-h, x)$. The local error of the numerical scheme is given by

$$\begin{aligned} E_L(x,h) &= \frac{-u(x-h)+2u(x)-u(x+h)}{h^2} + \alpha u(x) - \phi(x) \\ &= -\frac{1}{2}(u''(\xi_{x,h}) + u''(\eta_{x,h})) + u''(x) \\ &= \frac{u''(x)-u''(\xi_{x,h})}{2} + \frac{u''(x)-u''(\eta_{x,h})}{2}. \end{aligned}$$

Using the triangular inequality and Hölder's condition,

$$|E_L(x,h)| \leq \frac{C(\omega)}{2}|x-\xi_{x,h}|^\beta + \frac{C(\omega)}{2}|x-\eta_{x,h}|^\beta \leq C(\omega)h^\beta.$$

Let $u^s = (u(x_i))_{i=1}^M$, $f^h = (E_L(x_i,h))_{i=1}^M$ and the error $e^h = u^s - u^M$. From $\mathcal{A}u^M = c$ and $\mathcal{A}u^s - c = f^h$, we derive that $\mathcal{A}e^h = f^h$. Thus, if we denote by $\|\cdot\|_\infty$ the infinity norm for matrices, $\|e^h\|_\infty \leq \|\mathcal{A}^{-1}\|_\infty \|f^h\|_\infty \leq \|\mathcal{A}^{-1}\|_\infty C(\omega)h^\beta$. By Lemma 2.1, $\|\mathcal{A}^{-1}\|_\infty \leq 1/8$, therefore $\|e^h\|_\infty \leq C(\omega)/8 \cdot h^\beta$. This implies $|u(x_{i_M},\omega) - u^M_{i_M}(\omega)| \leq \frac{C(\omega)}{8}h^\beta$. By continuity of the sample paths of u,

$$\lim_{M\to\infty} u(x_{i_M},\omega) = \lim_{M\to\infty} u(i_M/(M+1),\omega) = u(x_0,\omega),$$

so we conclude that $\lim_{M\to\infty} u^M_{i_M}(\omega) = u(x_0,\omega)$, as wanted. □

Proposition 2.3. *Suppose that ϕ has sample paths in $C^\beta([0,1])$, for certain $0 < \beta \leq 1$. Let $x_0 \in [0,1]$. Let $\{i_M\}_{M=1}^\infty$ be a sequence of indexes, $i_M \in \{1,\ldots,M\}$, such that $\lim_{M\to\infty} i_M/(M+1) = x_0$. Let $1 \leq p < \infty$. If $S := \max\{\|A\|_{\mathrm{L}^{p+\epsilon}(\Omega)}, \|B\|_{\mathrm{L}^{p+\epsilon}(\Omega)}, \sup_{x\in[0,1]} \|\phi(x)\|_{\mathrm{L}^{p+\epsilon}(\Omega)}\} < \infty$ for some $\epsilon > 0$, then $u(x_0) \in \mathrm{L}^p(\Omega)$ and $\lim_{M\to\infty} u^M_{i_M} = u(x_0)$ in $\mathrm{L}^p(\Omega)$.*

Proof. By Proposition 2.2, $\lim_{M\to\infty} u^M_{i_M} = u(x_0)$ a.s. Then, by [20, Th. 2.4], it suffices to check that

$$\sup_{M\geq 1} \|u^M_{i_M}\|_{\mathrm{L}^{p+\epsilon}(\Omega)} < \infty. \tag{2}$$

Write $\mathcal{A} = (\alpha + 2/h^2)I_M - H$, where

$$H = \frac{1}{h^2}\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

Let $T = 1/(\alpha + 2/h^2)H$. Then $\mathcal{A} = (\alpha + 2/h^2)(I_M - T)$. The eigenvalues of $\mathcal{A}$, μ_k, are well-known [16, p. 59]: $\mu_k = \alpha + 2/h^2 \cdot (1 - \cos(k\pi h))$, $k = 1,\ldots,M$. From this, the eigenvalues of T are easily computable:

$$\eta_k = \frac{\frac{2}{h^2}\cos(k\pi h)}{\alpha + \frac{2}{h^2}}, \quad k = 1, \dots, M.$$

Since $|\cos(k\pi h)| < 1$, we deduce that

$$\|T\|_2 = \max\{|\eta_k| : k = 1, \dots, M\} < \frac{2/h^2}{\alpha + 2/h^2} \le 1.$$

The inequality $\|T\|_2 < 1$ implies that the matrix $I_M - T$ is invertible, with $(I_M - T)^{-1} = \sum_{k=0}^{\infty} T^k$. As a consequence,

$$\mathcal{A}^{-1} = \frac{1}{\alpha + \frac{2}{h^2}}(I_M - T)^{-1} = \frac{1}{\alpha + \frac{2}{h^2}} \sum_{k=0}^{\infty} T^k = \frac{1}{\alpha + \frac{2}{h^2}} \sum_{k=0}^{\infty} \frac{H^k}{\left(\alpha + \frac{2}{h^2}\right)^k}.$$

We derive that each entry $(\mathcal{A}^{-1})_{ij} = \sum_{k=0}^{\infty} (H^k)_{ij}/(\alpha + 2/h^2)^{k+1}$ increases when α decreases, therefore $(\mathcal{A}^{-1})_{ij}$ takes its maximum value at $\alpha = 0$. On the other hand,

$$u^M_{i_M} = (\mathcal{A}^{-1}c)_{i_M} = \sum_{j=1}^{M} (\mathcal{A}^{-1})_{i_M j} c_j = \underbrace{\frac{A}{h^2}(\mathcal{A}^{-1})_{i_M 1}}_{V_1} + \underbrace{\frac{B}{h^2}(\mathcal{A}^{-1})_{i_M M}}_{V_2} + \underbrace{\sum_{j=1}^{M} (\mathcal{A}^{-1})_{i_M j}\phi(x_j)}_{V_3}.$$

Taking into account (2), we bound V_1, V_2 and V_3 in $\mathrm{L}^{p+\epsilon}(\Omega)$. First, we bound the third term:

$$\begin{aligned}
\|V_3\|_{\mathrm{L}^{p+\epsilon}(\Omega)} &\le \sum_{j=1}^{M} \|(\mathcal{A}^{-1})_{i_M j}\phi(x_j)\|_{\mathrm{L}^{p+\epsilon}(\Omega)} \le \sum_{j=1}^{M} (\mathcal{A}^{-1})_{i_M j}|_{\alpha=0} \|\phi(x_j)\|_{\mathrm{L}^{p+\epsilon}(\Omega)} \\
&\le S \sum_{j=1}^{M} (\mathcal{A}^{-1})_{i_M j}|_{\alpha=0} \le S\|\mathcal{A}^{-1}|_{\alpha=0}\|_{\infty} \le \frac{S}{8},
\end{aligned}$$

by Lemma 2.1. Now we bound $\|V_1\|_{\mathrm{L}^{p+\epsilon}(\Omega)}$ and $\|V_2\|_{\mathrm{L}^{p+\epsilon}(\Omega)}$. Let $x^M = (x_1^M, \dots, x_M^M)^T$, $x_i^M = i/(M+1)$. Let $\overline{x^M} = (x_M^M, \dots, x_1^M)^T$ be the vector x^M reversed. Notice that $L\overline{x^M} = (1, 0, \dots, 0)^T$ and $Lx^M = (0, \dots, 0, 1)^T$, therefore $L^{-1}(:, 1) = \overline{x^M}$ and $L^{-1}(:, M) = x^M$. Since $\mathcal{A}|_{\alpha=0} = (1/h^2)L$, we derive that $\mathcal{A}^{-1}|_{\alpha=0}(:, 1) = h^2\overline{x^M}$ and $\mathcal{A}^{-1}|_{\alpha=0}(:, M) = h^2 x^M$. Thus,

$$\begin{aligned}
\|V_1\|_{\mathrm{L}^{p+\epsilon}(\Omega)} &= \frac{1}{h^2}\|A(\mathcal{A}^{-1})_{i_M 1}\|_{\mathrm{L}^{p+\epsilon}(\Omega)} \le \frac{(\mathcal{A}^{-1})_{i_M 1}|_{\alpha=0}}{h^2}\|A\|_{\mathrm{L}^{p+\epsilon}(\Omega)} \le \frac{h^2 x^M_{M+1-i_M}}{h^2} S \\
&= \left(1 - \frac{i_M}{M+1}\right) S \le C < \infty, \qquad (3)
\end{aligned}$$

for some $C > 0$, since the sequence $\{i_M/(M+1)\}_{M=1}^{\infty}$ is bounded. Analogously, $\|V_2\|_{\mathrm{L}^{p+\epsilon}(\Omega)} < \infty$. This proves (2). □

Proposition 2.4. *Suppose that ϕ has sample paths in $C^{\beta}([0,1])$, for certain $0<\beta\leq 1$. Let $x_0\in[0,1]$. Let $\{i_M\}_{M=1}^{\infty}$ be a sequence of indexes, $i_M\in\{1,\ldots,M\}$, such that $\lim_{M\to\infty} i_M/(M+1)=x_0$. Let $1\leq p<\infty$. If $A,B\in \mathrm{L}^p(\Omega)$ and $\sup_{x\in[0,1]}|\phi(x)|\leq Y$ a.s. for some random variable $Y\in\mathrm{L}^p(\Omega)$, then $u(x_0)\in\mathrm{L}^p(\Omega)$ and $\lim_{M\to\infty} u_{i_M}^M = u(x_0)$ in $\mathrm{L}^p(\Omega)$.*

Proof. As in (3), $|V_1|\leq |A|(1-i_M/(M+1))\leq C_1|A|$, for some constant $C_1>0$. Similarly, $|V_2|\leq |B|(i_M/(M+1))\leq C_2|B|$, for some constant $C_2>0$. On the other hand,

$$|V_3|\leq \sum_{j=1}^{M}(\mathcal{A}^{-1})_{i_M j}|\phi(x_j)|\leq Y\sum_{j=1}^{M}(\mathcal{A}^{-1})_{i_M j}\leq Y\|\mathcal{A}^{-1}\|_{\infty}\leq \frac{Y}{8},$$

by Lemma 2.1. Thus, $|u_{i_M}^M|\leq |V_1|+|V_2|+|V_3|\leq C_1|A|+C_2|B|+Y/8\in\mathrm{L}^p(\Omega)$. By the Dominated Convergence Theorem [14, p. 321], $u(x_0)\in\mathrm{L}^p(\Omega)$ and $\lim_{M\to\infty}u_{i_M}^M = u(x_0)$ in $\mathrm{L}^p(\Omega)$, as wanted. □

3. Probability density function of the solution stochastic process

Write $c=Ad+g$, where $d=(1/h^2,0,\ldots,0)^T$ and $g=(\phi(x_1),\ldots,\phi(x_{M-1}),\phi(x_M)+B/h^2)^T$. Then $u_i^M=(\mathcal{A}^{-1}c)_i=(\mathcal{A}^{-1}d)_iA+(\mathcal{A}^{-1}g)_i$. Our next task is to compute the probability density function of u_i^M.

Lemma 3.1. *Let A be an absolutely continuous random variable, independent of the random vector (Z_1,Z_2), where $Z_1\neq 0$ a.s. Then Z_1A+Z_2 is absolutely continuous, with density function $f_{Z_1A+Z_2}(z)=\mathbb{E}[f_A((z-Z_2)/Z_1)/|Z_1|]$.*

Proof. Let $\mathcal{C}$ be a Borel set in $\mathbb{R}$. Then

$$\begin{aligned}
&\mathbb{P}(Z_1A+Z_2\in\mathcal{C})=\int_{\mathbb{R}^2}\mathbb{P}(Z_1A+Z_2\in\mathcal{C}|Z_1=z_1,Z_2=z_2)\mathbb{P}_{(Z_1,Z_2)}(\mathrm{d}z_1,\mathrm{d}z_2)\\
&=\int_{\mathbb{R}^2}\mathbb{P}(z_1A+z_2\in\mathcal{C})\mathbb{P}_{(Z_1,Z_2)}(\mathrm{d}z_1,\mathrm{d}z_2)=\int_{\mathbb{R}^2}\int_{(\mathcal{C}-z_2)/z_1}f_A(a)\,\mathrm{d}a\,\mathbb{P}_{(Z_1,Z_2)}(\mathrm{d}z_1,\mathrm{d}z_2)\\
&=\int_{\mathbb{R}^2}\int_{\mathcal{C}}f_A\left(\frac{a-z_2}{z_1}\right)\frac{1}{|z_1|}\,\mathrm{d}a\,\mathbb{P}_{(Z_1,Z_2)}(\mathrm{d}z_1,\mathrm{d}z_2)\\
&=\int_{\mathcal{C}}\int_{\mathbb{R}^2}f_A\left(\frac{a-z_2}{z_1}\right)\frac{1}{|z_1|}\mathbb{P}_{(Z_1,Z_2)}(\mathrm{d}z_1,\mathrm{d}z_2)\,\mathrm{d}a\\
&=\int_{\mathcal{C}}\mathbb{E}\left[f_A\left(\frac{a-z_2}{z_1}\right)\frac{1}{|z_1|}\right]\,\mathrm{d}a.
\end{aligned}$$

□

Suppose that A is absolutely continuous, and that A and (α,B,ϕ) are independent (i.e., for all $0\leq y_1,\ldots,y_m\leq 1$, $m\in\mathbb{N}$, A and $(\alpha,B,\phi(y_1),\ldots,\phi(y_m))$ are independent). By [13, Th. 5.2], $\mathcal{A}^{-1}$ has nonnegative entries, so $(\mathcal{A}^{-1}d)_i\geq 0$. In fact, $\mathcal{A}$ is an irreducible matrix, because its entries on the superdiagonal and on the subdiagonal are nonzero. By [1, Th. 2.7, p. 141], the entries of $\mathcal{A}^{-1}$ are positive, so $(\mathcal{A}^{-1}d)_i>0$.

By Lemma 3.1,

$$f_{u_i^M}(u) = \mathbb{E}\left[f_A\left(\frac{1}{(\mathcal{A}^{-1}d)_i}\left\{u-(\mathcal{A}^{-1}g)_i\right\}\right)\frac{1}{(\mathcal{A}^{-1}d)_i}\right].$$

In practice, we can use an explicit expression for $f_{u_i^M}(u)$ that does not require the computation of $\mathcal{A}^{-1}$. The set of eigenvalues, μ_k, and eigenvectors, s^k, of $\mathcal{A}$ are known [16, p. 59]: $\mu_k = \alpha + 2/h^2 \cdot (1-\cos(k\pi h))$, $s^k = (\sin(k\pi jh))_{j=1}^M$, $k = 1,\ldots,M$. Let $D = \mathrm{diag}(\mu_1,\ldots,\mu_M)$ and $P = [s^1 \ldots s^M]$. Since $s^1,\ldots,s^M$ are pairwise orthogonal and $\|s_i\|_2 = \sqrt{(M+1)/2}$, the matrix $R = \sqrt{2/(M+1)}\,P$ is orthogonal. We have the decomposition $\mathcal{A} = RDR^T$. Its inverse is given by $\mathcal{A}^{-1} = RD^{-1}R^T$. In the end, $f_{u_i^M}$ can be expressed as follows:

$$\begin{aligned} f_{u_i^M}(u) = \mathbb{E}\Bigg[f_A\Bigg(& \frac{h^2}{\frac{2}{M+1}\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi h)}{\alpha+2/h^2(1-\cos(k\pi h))}} \cdot \\ & \cdot\Bigg\{u - \frac{2}{M+1}\sum_{j=1}^M \left(\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi jh)}{\alpha+2/h^2(1-\cos(k\pi h))}\right)\phi(x_j) \\ & - \frac{2}{M+1}\frac{B}{h^2}\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi Mh)}{\alpha+2/h^2(1-\cos(k\pi h))}\Bigg\}\Bigg) \frac{h^2}{\frac{2}{M+1}\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi h)}{\alpha+2/h^2(1-\cos(k\pi h))}}\Bigg]. \end{aligned} \tag{4}$$

Theorem 3.2. *Suppose that ϕ has sample paths in $C^\beta([0,1])$ (certain $0<\beta\le 1$), A is absolutely continuous, A and (B,α,ϕ) are independent, f_A is continuous and bounded on $\mathbb{R}$ and $\|\alpha\|_{\mathrm{L}^\infty(\Omega)} < \infty$. Let $x_0 \in (0,1)$. Let $\{i_M\}_{M=1}^\infty$ be a sequence of indexes, $i_M \in \{1,\ldots,M\}$, such that $\lim_{M\to\infty} i_M/(M+1) = x_0$. Then the sequence $\{f_{u_{i_M}^M}(u)\}_{M=1}^\infty$, defined in (4), converges to a density $f_{u(x_0)}(u)$ of $u(x_0)$, for all $u\in\mathbb{R}$.*

Proof. By Proposition 1.1, let v and w be two stochastic processes with sample paths in $C^{2,\beta}([0,1])$ that solve

$$\begin{cases} -v''(x)+\alpha v(x) = 0, \; x\in[0,1], \\ v(0)=1, \; v(1)=0, \end{cases} \qquad \begin{cases} -w''(x)+\alpha w(x) = \phi(x), \; x\in[0,1], \\ w(0)=0, \; w(1)=B. \end{cases}$$

By Proposition 2.2, $(\mathcal{A}^{-1}d)_{i_M} \to v(x_0)$ and $(\mathcal{A}^{-1}g)_{i_M} \to w(x_0)$ a.s. as $M\to\infty$. Since f_A is continuous,

$$\begin{aligned} &\lim_{M\to\infty} f_A\left(\frac{1}{(\mathcal{A}^{-1}d)_{i_M}}\left\{u-(\mathcal{A}^{-1}g)_{i_M}\right\}\right)\frac{1}{(\mathcal{A}^{-1}d)_{i_M}} \\ &\quad = f_A\left(\frac{1}{v(x_0)}\{u-w(x_0)\}\right)\frac{1}{v(x_0)} \text{ a.s.} \end{aligned} \tag{5}$$

Note that it makes sense to divide by $v(x_0)$, since by the Strong maximum principle for elliptic PDEs [8, Th. 4, p. 333], $0<v(x)<1$ for all $x\in(0,1)$, a.s.

Claim: there exists M_0 (independent of ω) such that, for all $M\ge M_0$, $(\mathcal{A}^{-1}d)_{i_M} \ge v(x_0)/2$ a.s.

We prove the claim. The random boundary value problem satisfied by v can be solved analytically. Indeed, fixed $\omega \in \Omega$, we distinguish two cases according to $\alpha(\omega) > 0$ or $\alpha(\omega) = 0$.

Case $\alpha(\omega) > 0$: The solution is $v(x) = \sinh(\sqrt{\alpha}(1-x))/\sinh(\sqrt{\alpha})$. Then $v''(x) = \alpha v(x) = \alpha \sinh(\sqrt{\alpha}(1-x))/\sinh(\sqrt{\alpha})$. By the Mean Value Theorem, $|v''(x) - v''(y)| = \frac{\alpha}{\sinh(\sqrt{\alpha})}|\sinh(\sqrt{\alpha}(1-x)) - \sinh(\sqrt{\alpha}(1-y))| = \frac{\alpha^2}{\sinh(\sqrt{\alpha})}\cosh(\xi_{x,y,\omega})|x-y|$, where $\xi_{x,y,\omega} \leq \max\{\sqrt{\alpha}(1-x), \sqrt{\alpha}(1-y)\} \leq \sqrt{\alpha}$. Hence,

$$|v''(x) - v''(y)| \leq \frac{\alpha^2 \cosh(\sqrt{\alpha})}{\sinh(\sqrt{\alpha})}|x-y| \leq \frac{\|\alpha\|^2_{\mathrm{L}^\infty(\Omega)} \cosh(\sqrt{\|\alpha\|_{\mathrm{L}^\infty(\Omega)}})}{\sinh(\sqrt{\|\alpha\|_{\mathrm{L}^\infty(\Omega)}})}|x-y|.$$

Case $\alpha(\omega) = 0$: The solution is $v(x) = 1 - x$, so $|v''(x) - v''(y)| = 0 \cdot |x-y|$.

Let

$$K = \begin{cases} \frac{\|\alpha\|^2_{\mathrm{L}^\infty(\Omega)} \cosh(\sqrt{\|\alpha\|_{\mathrm{L}^\infty(\Omega)}})}{\sinh(\sqrt{\|\alpha\|_{\mathrm{L}^\infty(\Omega)}})}, & \|\alpha\|_{\mathrm{L}^\infty(\Omega)} > 0, \\ 0, & \|\alpha\|_{\mathrm{L}^\infty(\Omega)} = 0. \end{cases}$$

Thus, $|v''(x) - v''(y)| \leq K|x-y|$. Therefore, the Hölder constant $C(\omega)$ of $v''(\cdot, \omega)$ can be taken independently of ω: $C(\omega) = K$. By Proposition 2.2, $|v(x_{i_M}, \omega) - v^M_{i_M}(\omega)| \leq K/8 \cdot h$ a.s., where $v^M_{i_M} = (\mathcal{A}^{-1}d)_{i_M}$. On the other hand, using the Mean Value Theorem to estimate $|v(x_0) - v(x_{i_M})|$ as we did before, we obtain $|v(x_0, \omega) - v(x_{i_M}, \omega)| \leq L|x_0 - x_{i_M}|$ a.s., where

$$L = \begin{cases} \frac{\|\alpha\|_{\mathrm{L}^\infty(\Omega)} \cosh(\sqrt{\|\alpha\|_{\mathrm{L}^\infty(\Omega)}})}{\sinh(\sqrt{\|\alpha\|_{\mathrm{L}^\infty(\Omega)}})}, & \|\alpha\|_{\mathrm{L}^\infty(\Omega)} > 0, \\ 1, & \|\alpha\|_{\mathrm{L}^\infty(\Omega)} = 0. \end{cases}$$

By the triangular inequality,

$$\begin{aligned} |v(x_0, \omega) - v^M_{i_M}(\omega)| &\leq |v(x_0, \omega) - v(x_{i_M}, \omega)| + |v(x_{i_M}, \omega) - v^M_{i_M}(\omega)| \\ &\leq K/8 \cdot h + L|x_0 - x_{i_M}| \text{ a.s.} \end{aligned}$$

Now, if $\alpha(\omega) > 0$, we know that $v(x_0) = \sinh(\sqrt{\alpha}(1-x_0))/\sinh(\sqrt{\alpha})$. As a function of α, it has a lower bound $m > 0$. Then, $v(x_0, \omega) \geq m$ a.s. To conclude, take M_0 such that, for all $M \geq M_0$, $K/8 \cdot h + L|x_0 - x_{i_M}| \leq m/2$. This implies that $|v(x_0, \omega) - v^M_{i_M}(\omega)| \leq m/2 \leq v(x_0, \omega)/2$, therefore $v^M_{i_M}(\omega) \geq v(x_0, \omega)/2$ a.s. This concludes the proof of the claim.

Hence, for $M \geq M_0$,

$$f_A\left(\frac{1}{(\mathcal{A}^{-1}d)_{i_M}}\left\{u - (\mathcal{A}^{-1}g)_{i_M}\right\}\right)\frac{1}{(\mathcal{A}^{-1}d)_{i_M}} \leq \|f_A\|_{\mathrm{L}^\infty(\mathbb{R})}\frac{2}{v(x_0)} \leq \|f_A\|_{\mathrm{L}^\infty(\mathbb{R})}\frac{2}{m}.$$

Since $\|f_A\|_{\mathrm{L}^\infty(\mathbb{R})}2/m$ is constant, it belongs to $\mathrm{L}^1(\Omega)$. By the Dominated Convergence

Theorem [14, p. 321],

$$\lim_{M\to\infty} f_{u_{i_M}^M}(u) = \mathbb{E}\left[f_A\left(\frac{1}{v(x_0)}\{u-w(x_0)\}\right)\frac{1}{v(x_0)}\right] =: \bar{f}(u).$$

Finally, we prove that $\bar{f}$ is a density of $u(x_0)$. Let G be a random variable with density function given by $\bar{f}$. By Scheffé's Lemma [21, p. 55], $u_{i_M}^M \to G$ in law as $M\to\infty$. On the other hand, by Proposition 2.2, $u_{i_M}^M \to u(x_0)$ a.s., so $u_{i_M}^M \to u(x_0)$ in law, as $M\to\infty$. Then $u(x_0)$ and G are equal in distribution, so $f_{u(x_0)}(u) = \bar{f}(u)$, as wanted. □

The continuity of f_A on $\mathbb{R}$ is satisfied, for instance, by the density function of the distributions Normal(μ,σ^2), $\mu\in\mathbb{R}$ and $\sigma^2>0$; Beta(a,b), $a>1$ and $b>1$; Gamma(a,b), $a>1$ and $b>0$; etc. However, it would be desirable to require only a.e. continuity for f_A, since the class of applicable density functions would be larger: Beta(a,b), $a\geq 1$ and $b\geq 1$; Uniform(a,b), $a<b$; Gamma(a,b), $a\geq 1$ and $b>0$ (in particular, Exponential(b)); truncated normal distribution; etc. This is the purpose of the following theorem.

Theorem 3.3. *Suppose that ϕ has sample paths in $C^\beta([0,1])$ (certain $0<\beta\leq 1$), A, B and α are absolutely continuous, A, B and (α,ϕ) are independent, $\|\alpha\|_{\mathrm{L}^\infty(\Omega)}<\infty$, f_A is a.e. continuous and essentially bounded on $\mathbb{R}$. Let $x_0\in(0,1)$. Let $\{i_M\}_{M=1}^\infty$ be a sequence of indexes, $i_M\in\{1,\ldots,M\}$, such that $\lim_{M\to\infty} i_M/(M+1)=x_0$. Then the sequence $\{f_{u_{i_M}^M}(u)\}_{M=1}^\infty$, defined in (4), converges to a density $f_{u(x_0)}(u)$ of $u(x_0)$, for all $u\in\mathbb{R}$.*

Proof. The proof is analogous to Theorem 3.2. Since $\alpha(\omega)>0$ a.s. (because $\mathbb{P}(\alpha=0)=0$), we have $v(x_0)=\sinh(\sqrt{\alpha}(1-x_0))/\sinh(\sqrt{\alpha})$ a.s. By the Random Variable Transformation (RVT) technique [5, Th. 1] (briefly, it is a method that consists in computing the probability density function of a transformation of an absolutely continuous random variable/vector), $v(x_0)$ is absolutely continuous. On the other hand, w can also be explicitly found using the theory of linear differential equations, and one obtains that $w(x_0)$ can be written as Z_1B+Z_2, where B and (Z_1,Z_2) are independent and $Z_1\neq 0$ a.s. By Lemma 3.1, $w(x_0)$ is absolutely continuous. By the RVT technique, $1/v(x_0)\cdot\{u-w(x_0)\}$ is absolutely continuous. Then, the probability that $1/v(x_0)\cdot\{u-w(x_0)\}$ belongs to the discontinuity set of f_A is 0. By the Continuous Mapping Theorem [19, p. 7, Th. 2.3], (5) holds. The rest of the proof is as in Theorem 3.2. □

In the following two propositions, we study whether the finite difference scheme preserves the pointwise convergence of the derivatives.

Proposition 3.4. *Assume the conditions of Theorem 3.2. If the n-th derivative $f_A^{(n)}$ exists on $\mathbb{R}$ and $f_A^{(j)}$ is bounded on $\mathbb{R}$ for each $1\leq j\leq n$, then $f_{u_{i_M}^M}$ and $f_{u(x_0)}$ have bounded n-th derivatives on $\mathbb{R}$. Moreover, if $f_A^{(n)}$ is continuous on $\mathbb{R}$, then the sequence $\{f_{u_{i_M}^M}^{(n)}(u)\}_{M=1}^\infty$ converges to $f_{u(x_0)}^{(n)}(u)$, for all $u\in\mathbb{R}$.*

Proof. Both $f_{u_{i_M}^M}$ and $f_{u(x_0)}$ possess n-th derivatives on $\mathbb{R}$ because of the differentiability of f_A and the Dominated Convergence Theorem. Indeed, fix $u \in \mathbb{R}$. We have

$$\frac{f_A\left(\frac{u+h-w(x_0)}{v(x_0)}\right)\frac{1}{v(x_0)} - f_A\left(\frac{u-w(x_0)}{v(x_0)}\right)\frac{1}{v(x_0)}}{h} \xrightarrow{h\to 0} f_A'\left(\frac{u-w(x_0)}{v(x_0)}\right)\frac{1}{v(x_0)^2}, \tag{6}$$

by definition of derivative. Now, by the deterministic Mean Value Theorem,

$$\begin{aligned}\left|\frac{f_A\left(\frac{u+h-w(x_0)}{v(x_0)}\right)\frac{1}{v(x_0)} - f_A\left(\frac{u-w(x_0)}{v(x_0)}\right)\frac{1}{v(x_0)}}{h}\right| &= \left|f_A'\left(\frac{u+\xi_h-w(x_0)}{v(x_0)}\right)\right|\frac{1}{v(x_0)^2} \\ &\leq \|f_A'\|_{\mathrm{L}^\infty(\mathbb{R})}\frac{1}{m^2},\end{aligned} \tag{7}$$

where ξ_h depends on ω and $|\xi_h| < |h|$. Notice that the inequality $v(x_0) \geq m$ a.s. from the proof of Theorem 3.2 has been utilized. The Dominated Convergence Theorem thus applies to ensure the existence of

$$f_{u(x_0)}'(u) = \mathbb{E}\left[f_A'\left(\frac{1}{v(x_0)}\{u-w(x_0)\}\right)\frac{1}{v(x_0)^2}\right]. \tag{8}$$

Analogously one justifies that

$$f_{u_{i_M}^M}'(u) = \mathbb{E}\left[f_A'\left(\frac{1}{(\mathcal{A}^{-1}d)_{i_M}}\{u-(\mathcal{A}^{-1}g)_{i_M}\}\right)\frac{1}{((\mathcal{A}^{-1}d)_{i_M})^2}\right] \tag{9}$$

exists, by using the fact that $(\mathcal{A}^{-1}d)_{i_M} \geq v(x_0)/2 \geq m/2$ a.s. For higher derivatives, the procedure works analogously and one proves

$$f_{u(x_0)}^{(n)}(u) = \mathbb{E}\left[f_A^{(n)}\left(\frac{1}{v(x_0)}\{u-w(x_0)\}\right)\frac{1}{v(x_0)^n}\right], \tag{10}$$

$$f_{u_{i_M}^M}^{(n)}(u) = \mathbb{E}\left[f_A^{(n)}\left(\frac{1}{(\mathcal{A}^{-1}d)_{i_M}}\{u-(\mathcal{A}^{-1}g)_{i_M}\}\right)\frac{1}{((\mathcal{A}^{-1}d)_{i_M})^n}\right]. \tag{11}$$

When $f_A^{(n)}$ is continuous on $\mathbb{R}$, the proof of Theorem 3.2 works with $f_A^{(n)}$ instead of f_A, so that one concludes that $\{f_{u_{i_M}^M}^{(n)}(u)\}_{M=1}^\infty$ converges to $f_{u(x_0)}^{(n)}(u)$, for all $u \in \mathbb{R}$. □

Proposition 3.5. *Assume the conditions of Theorem 3.3. If the n-th derivative $f_A^{(n)}$ exists a.e. on $\mathbb{R}$ and $\|f_A^{(j)}\|_{\mathrm{L}^\infty(\mathbb{R})} < \infty$ for each $1 \leq j \leq n$, then $f_{u_{i_M}^M}$ and $f_{u(x_0)}$ have bounded n-th derivatives on the whole $\mathbb{R}$. Moreover, if $f_A^{(n)}$ is a.e. continuous on $\mathbb{R}$, then the sequence $\{f_{u_{i_M}^M}^{(n)}(u)\}_{M=1}^\infty$ converges to $f_{u(x_0)}^{(n)}(u)$, for all $u \in \mathbb{R}$.*

Proof. As demonstrated in Theorem 3.3, $1/v(x_0)\cdot\{u-w(x_0)\}$ is absolutely continuous. Therefore, the probability that $1/v(x_0)\cdot\{u-w(x_0)\}$ belongs to the null set of non-existence of $f_A^{(n)}$ is 0. Thus, expression (6) remains being valid a.s. Bound (7) is also valid now a.s. Hence, for each $u \in \mathbb{R}$, the use of the Dominated Convergence Theorem is justified and (8)–(11) are correct.

When $f_A^{(n)}$ is a.e. continuous on $\mathbb{R}$, the proof of Theorem 3.3 is applicable with $f_A^{(n)}$ in lieu of f_A. One concludes that $\{f^{(n)}_{u^M_{i_M}}(u)\}_{M=1}^{\infty}$ converges to $f^{(n)}_{u(x_0)}(u)$, for every $u \in \mathbb{R}$. □

Remark 1. The same analysis can be performed if B is absolutely continuous and B and (A, α, ϕ) are independent. In such a case,

$$f_{u_i^M}(u) = \mathbb{E}\Bigg[f_B \Bigg(\frac{h^2}{\frac{2}{M+1}\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi Mh)}{\alpha+2/h^2(1-\cos(k\pi h))}} \cdot \left\{ u - \frac{2}{M+1}\sum_{j=1}^M \left(\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi jh)}{\alpha+2/h^2(1-\cos(k\pi h))}\right)\phi(x_j) - \frac{2}{M+1}\frac{A}{h^2}\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi h)}{\alpha+2/h^2(1-\cos(k\pi h))} \right\} \Bigg) \frac{h^2}{\frac{2}{M+1}\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi Mh)}{\alpha+2/h^2(1-\cos(k\pi h))}} \Bigg].$$

One could think of performing the same analysis by isolating $\phi(x_1)$, instead of A or B. In such a case, one would assume that $\phi(x_1)$ is absolutely continuous, and that $(\alpha, A, B, \phi(x_2), \ldots, \phi(x_M))$ and $\phi(x_1)$ are independent. To achieve this independence, one may require $\phi(y_1), \ldots \phi(y_m)$ to be independent, for every $y_1, \ldots, y_m \in [0,1]$, $m \geq 1$. A process ϕ of this type exists by Kolmogorov's Extension Theorem [2, Th. 36.2, p. 486]. However, by [11, Example 1.2.5, p. 10], this process ϕ is not jointly measurable on $[0,1] \times \Omega$. This implies that its sample paths cannot be right-continuous nor left continuous, so ϕ does not have enough regularity to apply our results.

Remark 2. The theoretical expression of $f_{u_i^M}(u)$ is

$$f_{u_i^M}(u) = \int_{\mathbb{R}^{M+2}} f_A \Bigg(\frac{h^2}{\frac{2}{M+1}\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi h)}{\alpha+2/h^2(1-\cos(k\pi h))}} \cdot \left\{ u - \frac{2}{M+1}\sum_{j=1}^M \left(\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi jh)}{\alpha+2/h^2(1-\cos(k\pi h))}\right)\phi_j - \frac{2}{M+1}\frac{b}{h^2}\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi Mh)}{\alpha+2/h^2(1-\cos(k\pi h))} \right\} \Bigg) \frac{h^2}{\frac{2}{M+1}\sum_{k=1}^M \frac{\sin(k\pi ih)\sin(k\pi h)}{\alpha+2/h^2(1-\cos(k\pi h))}} \cdot \mathbb{P}_{(\alpha,B,\phi(x_1),\ldots,\phi(x_M))}(\mathrm{d}\alpha, \mathrm{d}b, \mathrm{d}\phi_1, \ldots, \mathrm{d}\phi_M).$$

However, in practice, we use Monte Carlo simulations to compute the expectation (4), by sampling from α, B and ϕ.

4. Examples and Conclusions

Example 4.1. Consider (1) with $A \sim \text{Gamma}(2,1)$, $B \sim \text{Poisson}(3)$, $\alpha \sim \text{Uniform}(1,2)$ and $\phi(x) = \arctan(\mathrm{e}^{\cos(Dx)} + 1)$, where $D \sim \text{Binomial}(20, 0.2)$. The random variables are assumed to be independent. Notice that ϕ has sample paths in $\mathfrak{C}^\infty([0,1])$, therefore in $C^\beta([0,1])$. Then there is a solution process u in $C^{2,\beta}([0,1])$ (in fact, in $\mathfrak{C}^\infty([0,1])$). On the other hand, f_A is continuous and bounded on $\mathbb{R}$. Then Theorem 3.2 allows us to approximate the density function of $u(x)$, $x \in (0,1)$. In fact, by Proposition 3.4, we know that $f_{u(x)} \in \mathfrak{C}^\infty(\mathbb{R})$ and that its derivatives can be approximated by the finite difference scheme. Also, since A and B have moments of all orders and $|\phi(x)| \leq \pi/2$ a.s., both Proposition 2.3 and Proposition 2.4 ensure that $u(x)$ has moments of all orders and, moreover, they can be approximated. In particular, the expectation and variance of $u(x)$, $\mathbb{E}[u(x)]$ and $\mathbb{V}[u(x)]$, can be approximated.

We will do so for $x = 0.5$. Let $i_M = (M+1)/2$, for M odd. Then $\{f_{u_{i_M}^M}(u)\}_{M \text{ odd}}$, defined in (4), tends to $f_{u(0.5)}(u)$, $u \in \mathbb{R}$. In Figure 1, we show the graph of $f_{u_{i_M}^M}(u)$ for $M = 9, 11, 13$. In Table 1, we compute

$$\mathbb{E}[u_{i_M}^M] = \int_{\mathbb{R}} u f_{u_{i_M}^M}(u)\,\mathrm{d}u, \quad \mathbb{V}[u_{i_M}^M] = \int_{\mathbb{R}} u^2 f_{u_{i_M}^M}(u)\,\mathrm{d}u - \left(\mathbb{E}[u_{i_M}^M]\right)^2,$$

for $M = 9, 11, 13$. We observe that there is convergence, which agrees with our theoretical findings, and moreover it is rapid. The expectation in (4) has been computed via Monte Carlo simulation, as explained in Remark 2, with $100,000$ samples of the involved random variables for each M.

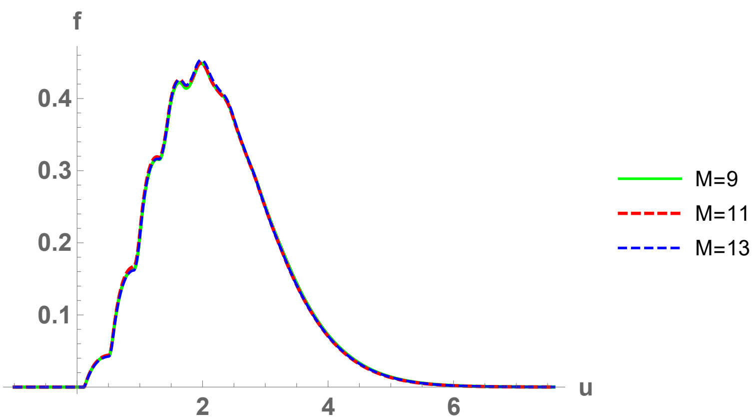

Figure 1. Graph of $f_{u_{i_M}^M}(u)$ for $M = 9$ (green), $M = 11$ (thick dashed red) and $M = 13$ (tiny dashed blue). Example 4.1.

M	9	11	13
$\mathbb{E}[u_{i_M}^M]$	2.22	2.21	2.21
$\mathbb{V}[u_{i_M}^M]$	0.89	0.88	0.88

Table 1. Expectation and variance of $u_{i_M}^M$, for $M = 9, 11, 13$. Example 4.1.

Example 4.2. We deal with (1) having as random inputs $A \sim \text{Uniform}(-1,1)$, $B \sim \text{Gamma}(4,1)$, $\alpha \sim \text{Uniform}(1,2)$ and $\phi(x)$ a standard Brownian motion on $[0,1]$ ($\phi(x) \sim \text{Normal}(0,x)$) [12, Ch. 5]. The random variables/process are assumed to be independent. Brownian motion has $C^{\beta}([0,1])$ sample paths, for $0 < \beta < 1/2$. By Proposition 1.1, there exists a solution process u in $C^{2,\beta}([0,1])$. On the other hand, f_A is a.e. continuous (two points of discontinuity, -1 and 1) and bounded on $\mathbb{R}$, and A, B and α are absolutely continuous. Then Theorem 3.3 allows us to approximate the density function of $u(x)$, $x \in (0,1)$. By Proposition 3.5, $f_{u(x)}$ is smooth and its derivatives can be approximated pointwise by the finite difference scheme. In addition, since A and B have moments of all orders and $\sup_{x\in[0,1]} \|\phi(x)\|_{\mathrm{L}^p(\Omega)} = \sup_{x\in[0,1]}(\frac{1}{\sqrt{\pi}}(2x)^{p/2}\Gamma(\frac{p+1}{2}))^{1/p} \leq (\frac{1}{\sqrt{\pi}}2^{p/2}\Gamma(\frac{p+1}{2}))^{1/p} < \infty$ for each $p \geq 1$, Proposition 2.3 entails that $u(x)$ has moments of all orders and, moreover, they can be approximated. In particular, the expectation and variance of $u(x)$, $\mathbb{E}[u(x)]$ and $\mathbb{V}[u(x)]$, can be approximated.

We work at the point $x = 0.5$ again. Let $i_M = (M+1)/2$, for M odd. Then $\{f_{u^M_{i_M}}(u)\}_{M \text{ odd}}$, defined in (4), tends to $f_{u(0.5)}(u)$, $u \in \mathbb{R}$. In Figure 2, we depict the graph of $f_{u^M_{i_M}}(u)$ for $M = 9, 11, 13$. In Table 2, we calculate the expectation and variance for $M = 9, 11, 13$. We observe convergence, which agrees with our theoretical findings, and furthermore this convergence is fast. The expectation in (4) has been determined via Monte Carlo simulation, see Remark 2, with 100,000 realizations of the involved random variables for each M. To sample from a Brownian motion, we use its Karhunen-Loève expansion on $[0,1]$ with a sufficiently large order of truncation.

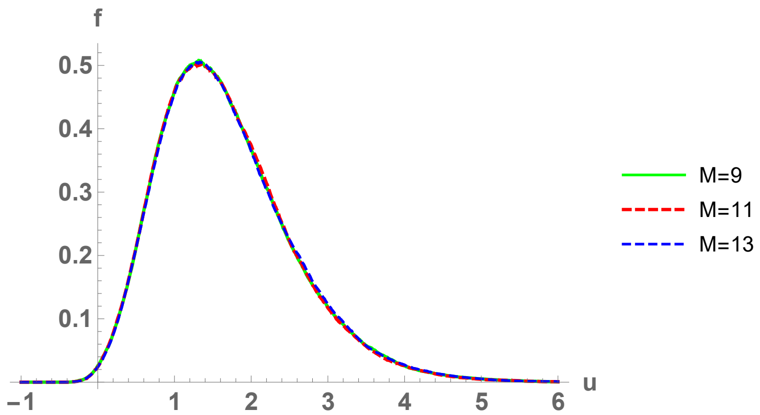

Figure 2. Graph of $f_{u^M_{i_M}}(u)$ for $M = 9$ (green), $M = 11$ (thick dashed red) and $M = 13$ (tiny dashed blue). Example 4.2.

M	9	11	13
$\mathbb{E}[u^M_{i_M}]$	1.68	1.68	1.68
$\mathbb{V}[u^M_{i_M}]$	0.77	0.77	0.77

Table 2. Expectation and variance of $u^M_{i_M}$, for $M = 9, 11, 13$. Example 4.2.

Finally, we want to point out that this study seeks to contribute to the field of random differential equations, where a main goal is to compute the mean and variance of the solution stochastic process. In this article we have shown a novel method to go beyond the computation of these statistical moments. Indeed, we have rigorously addressed the computation of the probability density function of an important random diffusion-reaction problem with random boundary conditions, by using a finite difference numerical scheme. The proposed approach can be very useful to deal with other significant random differential equations.

Research on the rate of convergence of the approximating density functions to the target density function could be conducted in the future. An issue that should be resolved in such a case is the fact that our reasoning is entirely based on existing results of convergence in Probability and Analysis (Dominated Convergence Theorem, Continuous Mapping Theorem, [20, Th. 2.4], etc.), which, at least to our knowledge, do not usually provide rates of convergence. Thus, in order to obtain optimal or at least sub-optimal bounds, we should proceed with step-by-step inequalities. We believe that this might be achievable by assuming f_A to be Lipschitz continuous on $\mathbb{R}$ and by finding the constants involved in the proof of Theorem 3.2. See our recent contribution [4], in which some theoretical rates of convergence for the approximating density functions were found in the setting of a random parabolic partial differential equation. These ideas raise new research lines for the future.

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Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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