CONVERGENT DISCRETE NUMERICAL SOLUTIONS OF STRONGLY COUPLED MIXED PARABOLIC SYSTEMS [*]

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Abstract

This paper deals with the construction of convergent discrete numerical solutions of strongly coupled parabolic partial differential systems. The proposed method is based on the application of a discrete separation of variables technique to the discretized problem and its further exact solution which avoids the solution of large algebraic systems.

Keywords: Difference schemes, strongly coupled system.

1 Introduction

Coupled partial differential systems with coupled boundary value conditions are frequent in quantum mechanical scattering problems [2, 14], chemical physics, thermoelastoplastic modelling, diffusion problems [8], nerve conduction problems [13], mechanics [16] and other fields. This paper deals with coupled parabolic systems of the form

$$\begin{aligned} u_t(x,t) - Au_{xx}(x,t) - Bu(x,t) &= 0, \qquad 0 < x < 1, \quad t > 0, & (1)\\ A_1\, u(0,t) + B_1\, u_x(0,t) &= 0, \qquad t > 0, & (2)\\ A_2\, u(1,t) + B_2\, u_x(1,t) &= 0, \qquad t > 0, & (3)\\ u(x,0) &= F(x), \quad 0 \leqslant x \leq 1, & (4)\end{aligned}$$

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where $u = (u_1, \dots, u_s)^T$ y $F = (f_1, \dots, f_s)^T$ are s-dimensional vectors, elements of $\mathbb{C}^s$, and A_i, B_i, for $i = 1, 2$ are $s \times s$ complex matrices, elements of $\mathbb{C}^{s\times s}$.

We assume that

$$\mathcal{A} = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \text{ and } A_1 \text{ are invertible matrices} . \tag{5}$$

Strongly coupled partial differential systems of the type (1)–(4) appear in Geomechanics [18], the study of the Hodgkin-Huxley nerve conduction equation [7, 13], in ignition of a single component nonreacting gas in a closed cylindrical vessel with with conservation of mass [12], or in the study of sudden cardiac death as a consequence of ventricular fibrillation [20].

Analytic-numerical solutions of problem (1)–(4) have been given in [9] for the case where $B = 0$, and in [17] for the case where $B_2 = B_1 = 0$ and A_2 is invertible. In this paper convergent discrete numerical solutions of problem (1)–(5) are constructed using difference schemes, a discrete separation of variables method and solving explicitely the mixed partial difference discretized problem. Particular cases of the above problem have been recently treated in [8, 11]. It is important to point out that method proposed here avoids the solution of large algebraic systems as it occurs using standard difference methods.

This paper is organized as follows. Section 2 deals with the study of the boundary partial difference problem resulting from the discretization of problem (1)–(3) using forward difference schemes under hypothesis (5). Section 3 deals with the construction of convergent discrete solutions of problem (1)–(5) using a discrete separation of variables method and results of section 2. Finally section 4 includes an illustrative example.

Throughout this paper, the set of all eigenvalues of a matrix D in $\mathbb{C}^{s\times s}$ is denoted by $\sigma(D)$. The spectral radius of D denoted by $\rho(D)$ is the maximum of the set $\{|z|;\, z \in \sigma(D)\}$. We denote by D^H the conjugate transpose of D and by $D^\dagger$ the Moore-Penrose pseudoinverse of D. The kernel of D, denoted by $\ker D$ coincides with the image of the matrix $I - D^\dagger D$ denoted by $\operatorname{Im}\left(I - D^\dagger D\right)$, see [4]. We say that a subspace E of $\mathbb{C}^s$ is invariant by the matrix A of $\mathbb{C}^{s\times s}$ si $A(E) \subset E$. Hence, property $A\left(\ker G\right) \subset \ker G$ is equivalent to the condition $G\,A\left(I - G^\dagger G\right) = 0$. The 2-norm of D will be denoted by

$$\|D\| = \sup_{v\neq 0} \frac{\|Dv\|_2}{\|v\|_2} ,$$

where for a vector v in $\mathbb{C}^s$, $\|v\|_2 = \left(v^H v\right)^{1/2}$ is the Euclidean norm of v, see [6]. If $D = D^H$ is an Hermitian matrix.

2 The discretized partial difference boundary problem

Let us divide the domain $[0,1]\times[0,\infty[$ into equal rectangles of sides $\Delta x = h$ and $\Delta t = k$, introduce coordinates of a typical mesh point (mh, nk) and let us represent $U(m,n) = u(mh, nk)$. Approximating the partial derivatives appearing in (1) by the forward difference approximations

$$\left.\begin{array}{rcl} u_t(mh,nk) & \approx & \frac{U(m,n+1)-U(m,n)}{k}\,; \\ u_{xx}(mh,nk) & \approx & \frac{U(m+1,n)-2U(m,n)+U(m,n-1)}{h^2} \end{array}\right] , \tag{6}$$

substituting (6) into (1)-(4) and denoting

$$r = \frac{k}{h^2}\,, \qquad h = \frac{1}{M}\,, \tag{7}$$

one gets the partial difference system:

$$\left.\begin{array}{c} U(m,n+1) \\ = rA\left[U(m+1,n)+U(m-1,n)\right] + \left(I + \frac{rB}{M^2} - 2rA\right) U(m,n) \\ 1 \le m \le M-1,\ n \ge 0, \end{array}\right] , \tag{8}$$

$$A_1 U(0,n) + MB_1\left[U(1,n) - U(0,n)\right] = 0\ , \qquad n \ge 0 \tag{9}$$
$$A_2 U(M,n) + MB_2\left[U(M,n) - U(M-1,n)\right] = 0\ , \qquad n \ge 0 \tag{10}$$
$$U(m,0) = F(mh) = f(m)\ , \qquad 0 \le m \le M. \tag{11}$$

The difference scheme (8) is consistent with equation (1) in the sense of [19, p.19], see section 3 of [11]. Let us seek nontrivial solutions $\{U(m,n)\}$ of the boundary problem (8)–(10) of the form

$$U(m,n) = G(n)\,H(m)\,, \quad G(n) \in \mathbb{C}^{s\times s}\,, \quad H(m) \in \mathbb{C}^s\,. \tag{12}$$

Substituting (12) into (8) and taking into account section 3 of [11] one gets that $\{U(m,n)\}$ given by (12) satisfies (8) if $\{G(n)\}$, $\{H(m)\}$ satisfy

$$G(n+1) - \left(I + \frac{r\,B}{M^2} + \rho\,A\right) G(n) = 0\,, \quad n \ge 0\,, \tag{13}$$

$$H(m+1) - \left(\frac{2r+\rho}{r}\right) H(m) + H(m-1) = 0\,, \quad 1 \le m \le M-1\,, \tag{14}$$

where ρ is a real number. Note that the solution of (13) satisfying $G(0) = I$, is given by

$$G(n) = \left(I + \frac{r\,B}{M^2} + \rho\,A\right)^n\,, \quad n \ge 0\,. \tag{15}$$

If ρ satisfies

$$-4r < \rho < 0\,, \tag{16}$$

then the algebraic equation

$$z^2 - \left(\frac{2r+\rho}{r}\right) z + 1 = 0\,, \tag{17}$$

has two different solutions z_0, z_1 given by

$$\left.\begin{array}{c} z_0 = \frac{2r+\rho}{2r} + i\left(1 - \left(\frac{2r+\rho}{2r}\right)^2\right)^{\frac{1}{2}} = e^{i\theta}\,, \\[2ex] z_1 = \frac{2r+\rho}{2r} - i\left(1 - \left(\frac{2r+\rho}{2r}\right)^2\right)^{\frac{1}{2}} = e^{-i\theta}\,, \\[2ex] 0 < \theta < \pi\,, \qquad \cos\theta = \frac{2r+\rho}{2r}\,, \qquad \rho = -4r\sin^2\left(\frac{\theta}{2}\right)\,, \qquad i^2 = -1 \end{array}\right]\,. \tag{18}$$

Since vector equation (14) has scalar coefficients, its solution can be written in the form

$$H(m) = \cos(m\theta)\,c + \sin(m\theta)\,d\,, \qquad c, d \in \mathbb{C}^s, \quad 1 \le m \le M-1. \tag{19}$$

Under hypothesis (5), premultiplying the boundary condition (2) by A_1^{-1} one gets a new condition where matrix appearing in the left upper block is the identity matrix. Thus we assume that $A_1 = I$. Using (12), the boundary condition (9) takes the form

$$G(n)\,H(0) + M\,B_1\,G(n)\,\left[H(1) - H(0)\right] = 0\,, \quad n \ge 0\,. \tag{20}$$

By (19) one gets $H(0) = c$ and considering (20) for $n = 0$, it follows that

$$\left[I - (1-\cos\theta)\,M\,B_1\right]\,c = -(M\,\sin\theta)\,B_1\,d\,. \tag{21}$$

Premultiplying (19) by $[I - (1-\cos\theta)\,M\,B_1]$ and taking into account (21) one gets

$$\left.\begin{array}{c}[I-(1-\cos\theta)\,M\,B_1]\;H(m)\\ \\ =-M\,B_1\cos(m\theta)\sin\theta\,d+\sin(m\theta)\,[I-(1-\cos\theta)\,M\,B_1]\;d\\ \\ =\left[\sin(m\theta)I-2M\,B_1\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2m-1}{2}\right)\theta\right)\right]d\\ \\ 1\le m\le M-1\;.\end{array}\right]\;. \qquad (22)$$

By the *spectral mapping theorem* [5, p.569] the eigenvalues of matrix $I-(1-\cos\theta)\,M\,B_1$ are $\{1-(1-\cos\theta)\,M\,w\,;\;\; w\in\sigma(B_1)\}$ and the real part of these eigenvalues are

$$1-(1-\cos\theta)\,M\,w_1\,;\qquad w=w_1+i\,w_2\in\sigma(B_1)\,.$$

If $w_1\le 0$ then $1-(1-\cos\theta)\,M\,w_1\neq 0$. If $w_1>0$, taking

$$M>\frac{1}{(1-\cos\theta)w_1}\,,$$

one gets $1-(1-\cos\theta)\,M\,w_1<0$. Thus, taking M large enough so that

$$M>\frac{1}{(1-\cos\theta)\,\gamma(B_1)}\,, \qquad (23)$$

where

$$\gamma(B_1)=\begin{cases}\min\{w_1;w=w_1+iw_2\in\sigma(B_1)\;,\;w_1>0\}\;,\text{ if}\\ \exists w\in\sigma(B_1)\;,\;\mathrm{Re}(w)>0\\ \\ (1-\cos\theta)^{-1}\;,\text{ if }\mathrm{Re}(w)\le 0\;\;\forall w\in\sigma(B_1)\,,\end{cases} \qquad (24)$$

one gets that

$$I-(1-\cos\theta)\,M\,B_1 \qquad \text{is invertible}\,, \qquad (25)$$

and then for $1\le m\le M-1$

$$H(m)=\left[\sin(m\theta)\,I-2M\,\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2m-1}{2}\right)\theta\right)B_1\right]d\;, \qquad (26)$$

is also a solution set of equation (14) for every vector $d\in\mathbb{C}^s$. Taking into account (14) for $m=1$, (26) for $m=1,2$, that $\cos\theta=\frac{2r+\rho}{2r}$ together with (20), one gets

$$H(0)=-(M\sin\theta)\,B_1 d\,. \qquad (27)$$

Substituting (15), (26) and (27) into (20), for $n > 0$ one gets

$$-M\sin\theta\left[\left(I+\frac{rB}{M^2}+\rho\,A\right)^n B_1 - B_1\left(I+\frac{rB}{M^2}+\rho\,A\right)^n\right]d = 0\,,\;\; n>0. \tag{28}$$

Since $\sin\theta \neq 0$ because $\theta \in]0,\pi[$, by (18) we have

$$w = \frac{r}{M^2\rho} = \frac{-1}{M^2\sin^2\left(\frac{\theta}{2}\right)} \neq 0\,, \tag{29}$$

and (28) can be written in the form

$$\left[\left(I+\rho\,(A+w\,B)\right)^n B_1 - B_1\left(I+\rho\,(A+w\,B)\right)^n\right]d = 0\,,\quad d\in\mathbb{C}^s\,,\;\; n>0\,. \tag{30}$$

Considering (14) for $m = M-1$, one gets

$$H(M) = \left[\sin(M\theta)\,I - 2M\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2M-1}{2}\right)\theta\right)B_1\right]d\,,\;\; d\in\mathbb{C}^s\,. \tag{31}$$

By imposing to $U(m,n)$, given by (12), the boundary condition (10) for $n \geq 0$ and using (15), (26) and (31) one gets

$$\begin{aligned}
&\{A_2\,(I+\rho(A+wB))^n\sin(M\theta)\\
&-2M\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2M-1}{2}\right)\theta\right)A_2\,(I+\rho(A+wB))^n\,B_1\\
&+2M\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2M-1}{2}\right)\theta\right)B_2\,(I+\rho(A+wB))^n\\
&+4M^2\sin^2\left(\frac{\theta}{2}\right)\sin((M-1)\theta)B_2\,(I+\rho(A+wB))^n\,B_1\Big\}\,d = 0\,,\quad n\geq 0.
\end{aligned} \tag{32}$$

Substituting (30) into (32) for $n > 0$ and using (32) for $n = 0$, it follows that for $n \geq 0$

$$\begin{aligned}
&\Big\{A_2\sin(M\theta) - 2M\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2M-1}{2}\right)\theta\right)A_2B_1\\
&\quad+2M\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2M-1}{2}\right)\theta\right)B_2\\
&\quad+4M^2\sin^2\left(\frac{\theta}{2}\right)\sin((M-1)\theta)B_2B_1\Big\}\,(I+\rho(A+wB))^n\,d = 0.
\end{aligned} \tag{33}$$

Let p be the degree of the minimal polynomial of the matrix $A + w\,B$, then by *Cayley-Hamilton theorem*, see [15, p. 206], for $n \geq p$ the powers $(A +$

$w\,B)^n$ are expressed in terms of I, $A+w\,B$, $(A+w\,B)^2, \ldots, (A+w\,B)^{p-1}$. Since $w \neq 0$, condition (33) holds if:

$$\Big\{ 2M \sin\left(\frac{\theta}{2}\right) \cos\left(\left(\frac{2M-1}{2}\right)\theta\right)(B_2 - A_2B_1) + A_2\sin(M\theta) \\ +4M^2\sin^2\left(\tfrac{\theta}{2}\right)\sin((M-1)\theta)B_2B_1\Big\}\,(A+wB)^n\,d = 0\,, \quad 0 \le n < p\,. \tag{34}$$

In order to guarantee that $\{U(m,n)\}$ is a nontrivial solution, vectors d appearing in (34) must be nonzero. By (34), there are nonzero vectors d satisfying (34) if

$$L(\theta) = 2M\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2M-1}{2}\right)\theta\right)(B_2 - A_2B_1) + A_2\sin(M\theta) \\ +4M^2\sin^2\left(\frac{\theta}{2}\right)\sin((M-1)\theta)\,B_2B_1 \text{ is singular, } 0<\theta<\pi. \tag{35}$$

Note that $L(\theta)$ can be written in the form:

$$L(\theta) = 2M\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2M-1}{2}\right)\theta\right)\left[(B_2 - A_2B_1) + \frac{A_2}{M}\right] \\ +\sin((M-1)\theta)\left[A_2 + 4M^2\sin^2\left(\frac{\theta}{2}\right)B_2B_1\right]. \tag{36}$$

By the properties of the *Schur complement* of a matrix, see [3], together with hypothesis (5) with $A_1 = I$, it follows that

$$B_2 - A_2B_1 \quad \text{is invertible.} \tag{37}$$

By (37) and the *Banach lemma*, see [6], it follows that

$$(B_2 - A_2B_1) + \frac{A_2}{M} \qquad \text{is invertible if} \quad M > \|A_2\| \left\|(B_2 - A_2B_1)^{-1}\right\|. \tag{38}$$

If M satisfies (38) and $0 < \theta < \pi$ makes that $L(\theta)$ defined by (36) is singular, then we obtain that $\sin((M-1)\theta) \neq 0$. Thus $L(\theta)$ is singular if and only if

$$A_2 + 4M^2\sin^2\left(\frac{\theta}{2}\right)B_2B_1 + \\ +\frac{2M\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2M-1}{2}\right)\theta\right)}{\sin((M-1)\theta)}\left[(B_2 - A_2B_1) + \frac{A_2}{M}\right] \text{ is singular,} \tag{39}$$

or the equivalent condition

$$\frac{\sin(M\theta)}{\sin((M-1)\theta)}\left(B_2 - A_2 B_1\right)^{-1} A_2 + 4M^2 \sin^2\left(\frac{\theta}{2}\right)\left(B_2 - A_2 B_1\right)^{-1} B_2 B_1 +$$

$$\frac{2M\sin\left(\frac{\theta}{2}\right)\cos\left(\left(\frac{2M-1}{2}\right)\theta\right)}{\sin\left((M-1)\theta\right)} I\,, \quad \text{is singular}, \ 0 < \theta < \pi. \tag{40}$$

Let us introduce the matrices

$$\widehat{A}_2 = \left(A_2 B_1 - B_2\right)^{-1} A_2\,, \qquad \widehat{B}_2 = \left(A_2 B_1 - B_2\right)^{-1} B_2 = \widehat{A}_2 B_1 - I\,. \tag{41}$$

Using matrices $\widehat{A}_2$, $\widehat{B}_2$ defined in (41) and the *spectral mapping theorem* condition (39) means that

$$\left.\begin{array}{l} M\left(\dfrac{\sin(M\theta)}{\sin\left((M-1)\theta\right)} - 1\right) \quad \text{is an eigenvalue of the matrix} \\ \dfrac{\sin(M\theta)}{\sin\left((M-1)\theta\right)}\widehat{A}_2 + 4M^2\sin^2\left(\frac{\theta}{2}\right)\left(\widehat{A}_2 B_1^2 - B_1\right)\,, \quad 0 < \theta < \pi \end{array}\right] . \tag{42}$$

Let us assume that

$$\left.\begin{array}{l} \text{There exist } \alpha \in \sigma\left(\widehat{A}_2\right) \cap \mathbb{R}\,; \ \beta \in \sigma(B_1) \cap \mathbb{R} \ \text{ and } \ v \in \mathbb{C}^s \sim \{0\} \\ \text{such that } \left(\widehat{A}_2 - \alpha\, I\right) v = \left(B_1 - \beta\, I\right) v = 0 \end{array}\right] . \tag{43}$$

By (43) it follows that

$$\left[\frac{\sin(M\theta)}{\sin\left((M-1)\theta\right)}\widehat{A}_2 + 4M^2\sin^2\left(\frac{\theta}{2}\right)\left(\widehat{A}_2 B_1^2 - B_1\right)\right] v =$$
$$= \left[\frac{\sin(M\theta)}{\sin\left((M-1)\theta\right)}\alpha + 4M^2\sin^2\left(\frac{\theta}{2}\right)\left(\alpha\beta^2 - \beta\right)\right] v\,, \quad 0 < \theta < \pi\,,$$

or

$$\left.\begin{array}{c} v \ \text{ is an eigenvector of the matrix} \\ \dfrac{\sin(M\theta)}{\sin\left((M-1)\theta\right)}\widehat{A}_2 + 4M^2\sin^2\left(\dfrac{\theta}{2}\right)\left(\widehat{A}_2 B_1^2 - B_1\right) \\ \text{associated to the real eigenvalue} \\ \dfrac{\sin(M\theta)}{\sin\left((M-1)\theta\right)}\alpha + 4M^2\sin^2\left(\dfrac{\theta}{2}\right)\left(\alpha\beta^2 - \beta\right) \end{array}\right] . \tag{44}$$

Taking M large enough so that

$$M > \alpha\,,$$

condition (42) and (44) makes possible to find solutions of the scalar equation

$$\frac{\sin(M\theta)}{\sin((M-1)\theta)} = \frac{M}{M-\alpha} + 4M^2 \sin^2\left(\frac{\theta}{2}\right) \frac{\left(\alpha\beta^2 - \beta\right)}{M-\alpha}\,, \quad 0 < \theta < \pi\,,$$

or

$$\cot\left(\left(M-1\right)\theta\right) = -\cot\theta + \frac{M}{M-\alpha}\left[\frac{1}{\sin\theta} + 2M\left(\alpha\beta^2 - \beta\right)\tan\left(\frac{\theta}{2}\right)\right] \qquad 0 < \theta < \pi. \tag{45}$$

For each integer δ with $1 \leq \delta \leq M-1$, in the interval $J_\delta = \left]\frac{(\delta-1)\pi}{M-1}, \frac{\delta\pi}{M-1}\right[$ one satisfies

$$\left.\begin{array}{c} \lim_{\theta\to\frac{(\delta-1)\pi}{M-1}+} \cot((M-1)\theta) = +\infty\,; \\ \\ \lim_{\theta\to\frac{\delta\pi}{M-1}-} \cot((M-1)\theta) = -\infty\,;\ \cot((M-1)\theta) \text{ decreases in } J_\delta, \end{array}\right] \tag{46}$$

because

$$\frac{d}{d\theta}\left(\cot((M-1)\theta)\right) = -\frac{M-1}{\sin^2((M-1)\theta)} < 0\,.$$

Furthermore the function $e_M(\theta)$ describing the right hand side of (45) is continuous and increasing in $]0,\pi[$ if

$$M > \max\left\{\frac{\alpha}{1-\cos\theta}, \alpha\right\}\,, \quad \theta \in\,]0,\pi[\,, \tag{47}$$

and some of the following conditions are satisfied

$$\left.\begin{array}{l} \beta = 0\,, \\ \alpha\beta = 1\,, \\ \beta > 0 \text{ and } \alpha\beta > 1\,, \\ \beta < 0 \text{ and } \alpha\beta < 1\,. \end{array}\right]\,. \tag{48}$$

Then by (46)–(48) there exists only one solution θ_δ of (45) in the interval J_δ, satisfying

$$\left.\begin{array}{c} \cot\left((M-1)\,\theta_\delta\right) \\ = -\cot\theta_\delta + \dfrac{M}{M-\alpha}\left[\dfrac{1}{\sin\theta_\delta} + 2M\left(\alpha\beta^2-\beta\right)\tan\left(\dfrac{\theta_\delta}{2}\right)\right] \\ \\ 1 \le \delta \le M-1\,, \quad \theta_\delta \in J_\delta \end{array}\right] . \tag{49}$$

Hence condition (34) can be written in the form

$$\begin{array}{c} S(\alpha,\beta,\theta_\delta)\,(A+w_\delta\,B)^n\;d_\delta = 0\,, \\ 0 \le n \le p(\delta)-1\,,\;\; 1 \le \delta \le M-1\,, \end{array} \tag{50}$$

where

$$\begin{aligned} S(\alpha,\beta,\theta_\delta) = {} & \frac{\sin(M\theta_\delta)}{\sin\left((M-1)\theta_\delta\right)}\widehat{A}_2 + 4M^2\sin^2\left(\frac{\theta_\delta}{2}\right)\left(\widehat{A}_2B_1^2 - B_1\right) + \\ & -\left[\frac{\sin(M\theta_\delta)}{\sin\left((M-1)\theta_\delta\right)}\alpha + 4M^2\sin^2\left(\frac{\theta_\delta}{2}\right)\left(\alpha\beta^2-\beta\right)\right] I\,, \end{aligned} \tag{51}$$

$p(\delta)$ is the degree of the minimal polynomial of the matrix $A + w_\delta\,B$, being θ_δ the solution of (49) and

$$w_\delta = \frac{-1}{4M^2\sin^2\left(\dfrac{\theta_\delta}{2}\right)}\,, \quad 1 \le \delta \le M-1\,. \tag{52}$$

Let us introduce the block matrix defined by

$$T(\alpha,\beta,\theta_\delta) = \left[\begin{array}{c} B_1(A+w_\delta B) - (A+w_\delta B)B_1 \\ B_1(A+w_\delta B)^2 - (A+w_\delta B)^2B_1 \\ \vdots \\ B_1(A+w_\delta B)^{p(\delta)-1} - (A+w_\delta B)^{p(\delta)-1}B_1 \\ S(\alpha,\beta,\theta_\delta) \\ S(\alpha,\beta,\theta_\delta)(A+w_\delta B) \\ S(\alpha,\beta,\theta_\delta)(A+w_\delta B)^2 \\ \vdots \\ S(\alpha,\beta,\theta_\delta)(A+w_\delta B)^{p(\delta)-1} \end{array}\right]\,, \tag{53}$$

Then vectors d_δ satisfy (50) and the corresponding to (30), i.e.,

$$\left[(A+w_\delta\,B)^n\,B_1 - B_1\,(A+w_\delta\,B)^n\right]\,d_\delta = 0\,, \quad 0 < n < p(\delta)\,, \tag{54}$$

if and only if

$$T(\alpha,\beta,\theta_\delta)d_\delta = 0\,, \qquad 1 \le \delta \le M-1\,, \quad d_\delta \in \mathbb{C}^s \sim \{0\}\,. \tag{55}$$

Note that if vectors $\{d_\delta\}_{\delta=1}^{M-1}$ are chosen so that

$$(B_1 - \beta I)\, d_\delta = \left(\widehat{A}_2 - \alpha I\right) d_\delta = 0\,, \quad d_\delta \in \mathbb{C}^s \sim \{0\}\,, \;\; 1 \le \delta \le M-1\,, \tag{56}$$

and

$$\{(A + w_\delta B)^n\, d_\delta\,;\; 1 \le n \le p(\delta) - 1\} \subset \ker\left(\widehat{A}_2 - \alpha I\right) \cap \ker\left(B_1 - \beta I\right) \tag{57}$$
$$1 \le \delta \le M-1\,,$$

then vectors d_δ satisfy (50) and (54), or equivalently (55). Replacing θ by θ_δ into (15) and (26), by (12) it follows that

$$\begin{aligned} U_\delta(m,n) = {} & \left[I - r\left(4\sin^2\left(\frac{\theta_\delta}{2}\right)A - \frac{B}{M^2}\right)\right]^n \cdot \\ & \cdot \left[\sin\left(m\theta_\delta\right) - 2M\beta\sin\left(\frac{\theta_\delta}{2}\right)\cos\left(\left(\frac{2m-1}{2}\right)\theta_\delta\right)\right] d_\delta, \end{aligned} \tag{58}$$

for $1 \le m \le M-1$, $n \ge 0$, define nonzero solutions of problem (8)–(10).

Summarizing the following result has been established:

Theorem 2.1 *Let us consider the boundary value problem (8)–(10) under hypothesis (5) with $A_1 = I$, let $\widehat{A}_2 = (A_2B_1 - B_2)^{-1} A_2$ and let $M > 0$ be a large enough positive integer so that (23) and (38) hold.*

- *(i) Assume condition (43) and take M satisfying (47). Then there exist solutions θ_δ of (49), $\theta_\delta \in \left]\frac{(\delta-1)\pi}{M-1}, \frac{\delta\pi}{M-1}\right[= J_\delta$, $1 \le \delta \le M-1$, making the matrix $L(\theta_\delta)$ defined by (36) singular.*
- *(ii) Under hypothesis of (i), let d_δ be vectors in $\mathbb{C}^s$ satisfying (56) and (57) for $1 \le \delta \le M-1$, then $\{U_\delta(m,n)\}$ given by (58) defines nontrivial solutions of problem (8)–(10).*

Remark 2.1 *The case where apart from the invertibility of $\mathcal{A}$ one has $B_1 = I$ can be treated in an analogous way taking into account the properties of the Schur complement, see [3]. Considering the change $m \to M-m$, the cases where $A_2 = I$ or $B_2 = I$ can be transformed into the previous cases.*

3 The mixed problem

This section deals with the construction of exact solutions of the mixed difference problem (8)–(10). Assume the notation and hypotheses of *theorem 2.1-(i) and (ii)*. By superposition of solutions of the boundary problem (8)–(10) one gets

$$\left.\begin{array}{c} U(m,n) \\ = \sum_{\delta=1}^{M-1} \left[I - r\left(4\,\sin^2\left(\dfrac{\theta_\delta}{2}\right) A - \dfrac{B}{M^2}\right)\right]^n \cdot \\ \cdot \left[\left(1-\dfrac{\beta M\rho_\delta}{2r}\right)\sin(m\theta_\delta) - \beta M\cos(m\theta_\delta)\sin(\theta_\delta)\right] d_\delta \,, \\ \rho_\delta = -4r\sin^2\left(\dfrac{\theta_\delta}{2}\right) \,, \quad 1 \le \delta \le M-1 \end{array}\right] . \qquad (59)$$

By imposing to $\{U(m,n)\}$ given by (59) that satisfies the initial condition (11), implies that vectors d_δ appearing in (59) must verify

$$f(m) = \sum_{\delta=1}^{M-1} \left[\left(1-\frac{\beta M\rho_\delta}{2r}\right)\sin(m\theta_\delta) - \beta M\cos(m\theta_\delta)\sin(\theta_\delta)\right] d_\delta \,. \qquad (60)$$

Let $f_q(m)$ and $d_{\delta,q}$ be the q-th component of vectors $f(m)$ and d_δ respectively. Consider the scalar Sturm-Liouville problem

$$\left.\begin{array}{r} -h(m+1)+2h(m)-h(m-1) = -\dfrac{\rho}{r}h(m) \\ h(0) = \dfrac{\beta M}{\beta M-1}h(1) \\ h(M) = \dfrac{M(\alpha\beta-1)}{\alpha+M(\alpha\beta-1)}h(M-1) \end{array}\right\} , \quad 1 \le m \le M-1 \,. \qquad (61)$$

By [1, chap. 11] problem (61) has exactly $M-1$ eigenvalues given by $\left\{\frac{-\rho_\delta}{r}\right\}_{\delta=1}^{M-1}$, where $\rho_\delta = -4r\sin^2\left(\frac{\theta_\delta}{2}\right)$ and θ_δ satisfies (49). For each eigenvalue $\frac{-\rho_\delta}{r}$ there exists one eigenfunction sequence

$$\{h_\delta(m)\} = \left\{\left(1-\frac{\beta M\rho_\delta}{2r}\right)\sin(m\theta_\delta) - \beta M\cos(m\theta_\delta)\sin(\theta_\delta)\right\} , \qquad (62)$$

and these eigenfunctions are orthogonal with respect to the weight function $w(m) = 1$, for $1 \leq m \leq M-1$. The q-th component of equation (60) takes the form

$$f_q(m) = \sum_{\delta=1}^{M-1} \left\{ \left(1 - \frac{\beta M \rho_\delta}{2r}\right) \sin(m\theta_\delta) - \beta M \cos(m\theta_\delta) \sin(\theta_\delta) \right\} d_{\delta,q} \,. \tag{63}$$

By the orthogonality of eigenfunctions $\{h_\delta(m)\}$ appearing in (60) and the theory of discrete Fourier series, see [1, chap. 11], it follows that

$$d_{\delta,q} = \frac{\sum\limits_{\nu=1}^{M-1} \left\{ \left(1 - \frac{\beta M \rho_\delta}{2r}\right) \sin(\nu\theta_\delta) - \beta M \cos(\nu\theta_\delta) \sin(\theta_\delta) \right\} f_q(\nu)}{\sum\limits_{\nu=1}^{M-1} \left\{ \left(1 - \frac{\beta M \rho_\delta}{2r}\right) \sin(\nu\theta_\delta) - \beta M \cos(\nu\theta_\delta) \sin(\theta_\delta) \right\}^2} \,, \tag{64}$$

$$1 \leq \delta \leq M-1 \,, \quad 1 \leq q \leq s \,,$$

or in vectorial form

$$d_{\delta} = \frac{\sum\limits_{\nu=1}^{M-1} \left\{ \left(1 - \frac{\beta M \rho_\delta}{2r}\right) \sin(\nu\theta_\delta) - \beta M \cos(\nu\theta_\delta) \sin(\theta_\delta) \right\} f(\nu)}{\sum\limits_{\nu=1}^{M-1} \left\{ \left(1 - \frac{\beta M \rho_\delta}{2r}\right) \sin(\nu\theta_\delta) - \beta M \cos(\nu\theta_\delta) \sin(\theta_\delta) \right\}^2} \,, \tag{65}$$

$$1 \leq \delta \leq M-1 \,.$$

Expression (65) for vectors d_δ must be compatible with conditions (56), (57). This means that $\{f(m)\}$ must satisfy

$$(B_1 - \beta I)\, f(m) = \left(\widehat{A}_2 - \alpha I\right) f(m) = 0 \,, \quad 1 \leq m \leq M-1 \,, \tag{66}$$

and if w_δ is given by (52),

$$\left\{ (A + w_\delta B)^n f(m) \,,\ 1 \leq n \leq p(\delta) - 1 \right\} \subset \ker\left(\widehat{A}_2 - \alpha I\right) \cap \ker\left(B_1 - \beta I\right) \,, \tag{67}$$

for $1 \leq m \leq M-1$, $1 \leq \delta \leq M-1$.

If $\{f(m)\}_{m=1}^{M-1}$ satisfies (66), (67) then $\{U(m,n)\}$ defined by (59) where d_δ is given by (65) is a solution of problem (8)–(11). Note that conditions (66) and (67) are satisfied if

$$f(m) \in \ker\left(\widehat{A}_2 - \alpha\, I\right) \cap \ker\left(B_1 - \beta\, I\right)\,, \quad 1 \le m \le M-1\,, \tag{68}$$

and

$$\left.\begin{array}{l} \ker\left(\widehat{A}_2 - \alpha\, I\right) \cap \ker\left(B_1 - \beta\, I\right) \text{ is an invariant subspace} \\ \text{by the matrix } A + w_\delta\, B\,, \quad 1 \le \delta \le M-1\,. \end{array}\right] . \tag{69}$$

Using *lemma 1* of [9], conditions (68) and (69) can be written in the form

$$f(m) \in \operatorname{Im} L(\alpha,\beta)\,, \quad 1 \le m \le M-1\,, \tag{70}$$

$$\left(I - L(\alpha,\beta)L(\alpha,\beta)^\dagger\right)\left(A + w_\delta\, B\right) L(\alpha,\beta) = 0\,, \quad 1 \le \delta \le M-1\,, \tag{71}$$

where

$$\left.\begin{array}{l} L(\alpha,\beta) = \left(I - P_\alpha^\dagger P_\alpha\right)\left\{I - \left[Q_\beta\left(I - P_\alpha^\dagger P_\alpha\right)\right]^\dagger \left[Q_\beta\left(I - P_\alpha^\dagger P_\alpha\right)\right]\right\} \\ \\ P_\alpha = \widehat{A}_2 - \alpha\, I\,, \quad Q_\beta = B_1 - \beta\, I\,, \end{array}\right] . \tag{72}$$

Note that condition (71) means that $\operatorname{Im} L(\alpha,\beta)$ is an invariant subspace by the matrix $A + w_\delta\, B$, for $1 \le \delta \le M-1$. The solution $\{U(m,n)\}$ of the mixed problem (8)–(11), defined by (59), (65), is stable, i.e. remains bounded as $n \to \infty$ if $\{f(m)\}$ is bounded and matrices

$$I - r\left(4\, A \sin^2\left(\frac{\theta_\delta}{2}\right) - \frac{B}{M^2}\right)\,, \quad 1 \le \delta \le M-1\,,$$

are convergent. By *theorem 2.1* of [10] this occurs if

$$x > 0 \quad \text{for all} \quad x \in \sigma\left(\frac{A + A^H}{2}\right)\,, \tag{73}$$

$$y \le 0 \quad \text{for all} \quad y \in \sigma\left(\frac{B + B^H}{2}\right)\,, \tag{74}$$

and if $\widetilde{A}_1 = \frac{A+A^H}{2}$, $\widetilde{B}_1 = \frac{B+B^H}{2}$, $\widetilde{A}_2 = \frac{A-A^H}{2i}$, $\widetilde{B}_2 = \frac{B-B^H}{2i}$ and θ_1 is the unique solution of (49) in $\left]0, \frac{\pi}{M-1}\right[$, r satisfies

$$r < \frac{M^2\left[\left(2M\sin\left(\frac{\theta_1}{2}\right)\right)^2\lambda_{\min}\left(\widetilde{A}_1\right)-\lambda_{max}\left(\widetilde{B}_1\right)\right]}{\left[4M^2\lambda_{max}\left(\widetilde{A}_1\right)+\rho\left(\widetilde{B}_1\right)\right]^2+\left[4M^2\lambda_{max}\left(\widetilde{A}_2\right)+\rho\left(\widetilde{B}_2\right)\right]^2}\,. \quad (75)$$

Summarizing the following result has been established:

Theorem 3.1 *Consider the mixed problem (8)-(11) under hypothesis (43) and (5) with* $A_1 = I$. *Let* $\widehat{A}_2 = (A_2B_1 - B_2)^{-1}A_2$ *and let* $M > 0$ *integer large enough so that (23),(38) and (47) hold. Let* θ_δ *be the solution of (49) and* w_δ *be defined by (52) for* $1 \leq \delta \leq M-1$. *Suppose that* $\{f(m)\}$ *satisfies conditions (70) and (71) where* $L(\alpha,\beta)$ *is defined by (72). Then* $\{U(m,n)\}$ *defined by (59) where* d_δ *is given by (65) is a solution of problem (8)–(11). Furthermore, if matrices* A, B *satisfy conditions (73)–(74),* $\{f(m)\}$ *is bounded and* r *is small enough so that (75) holds, then* $\{U(m,n)\}$ *is stable.*

Now we study conditions more general than those considered in *theorem 3.1.* Let us assume that

$$\Lambda = \{\alpha(1),\ldots,\alpha(t)\} \subset \mathbb{R}\cap\sigma\left(\widehat{A}_2\right)\,, \quad (76)$$

$$\Omega = \{\beta(1),\ldots,\beta(q)\} \subset \mathbb{R}\cap\sigma(B_1)\,. \quad (77)$$

By *lemma 1* of [9] condition

$$L\left(\alpha(i),\beta(j)\right) \neq 0\,,\quad 1\leq i\leq t\,,\ 1\leq j\leq q\,, \quad (78)$$

is equivalent to

$$\ker\left(\widehat{A}_2-\alpha(i)\,I\right)\cap\ker\left(B_1-\beta(j)\,I\right)\neq\emptyset\,,\quad 1\leq i\leq t\,,\ 1\leq j\leq q\,. \quad (79)$$

Consider the set $\mathcal{F}\subset\Lambda\times\Omega$ defined by

$$\mathcal{F}=\left\{\begin{array}{l}(\alpha(i_\ell),\beta(j_\ell))\in\Lambda\times\Omega \text{ satisfying some of the conditions of } (48)\,,\\ \left(\widehat{A}_2-\alpha(i_\ell)\,I\right)\,v_\ell=(B_1-\beta(i_\ell)\,I)\;v_\ell=0\,,\ v_\ell\in\mathbb{C}^s\sim\{0\}\,,\\ L\left(\alpha(i_\ell),\beta(j_\ell)\right)\neq 0\end{array}\right\} \quad (80)$$

and the block matrix

$$\mathcal{L} = \left[L\left(\alpha(i_1),\beta(j_1)\right), L\left(\alpha(i_2),\beta(j_2)\right), \ldots, L\left(\alpha(i_p),\beta(j_p)\right)\right] \in \mathbb{C}^{s\times ps} . \quad (81)$$

and suppose that $f(m) \in \operatorname{Im}\mathcal{L}$ for $0 \le m \le M$, or equivalently

$$\left(I - \mathcal{L}\mathcal{L}^{\dagger}\right) f(m) = 0 , \quad 0 \le m \le M , \quad (82)$$

because $\operatorname{Im}\mathcal{L} = \ker\left(I - \mathcal{L}\mathcal{L}^{\dagger}\right)$. By *lemma 1* of [9] one gets

$$\mathcal{S}_\ell = \operatorname{Im} L\left(\alpha(i_\ell),\beta(j_\ell)\right) = \ker\left(\widehat{A}_2 - \alpha(i_\ell)\, I\right) \cap \ker\left(B_1 - \beta(j_\ell)\, I\right) , \quad (83)$$

and by (81), (83), the subspace $\operatorname{Im}\mathcal{L}$ is the direct sum of the subspaces $\mathcal{S}_\ell$,

$$\operatorname{Im}\mathcal{L} = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_p . \quad (84)$$

Let $\left\{\widehat{f_\ell}(m)\right\}_{m=0}^{M}$ be the projection sequence of $\{f(m)\}_{m=0}^{M}$ on the subspace S_ℓ, defined by:

$$\widehat{f_\ell}(m) = [0,\ldots,0, L\left(\alpha(i_\ell),\beta(j_\ell)\right), 0,\ldots,0]\, \mathcal{L}^{\dagger} f(m) , \quad (85)$$
$$1 \le \ell \le p ,\ 0 \le m \le M .$$

Since $\widehat{f_\ell}(m)$ lies in $\mathcal{S}_\ell$, by (82) it follows that:

$$\sum_{\ell=1}^{p} \widehat{f_\ell}(m) = \mathcal{L}\mathcal{L}^{\dagger} f(m) = f(m) , \quad 0 \le m \le M . \quad (86)$$

Let us suppose that $\operatorname{Im}\mathcal{L}\left(\alpha(i_\ell),\beta(j_\ell)\right)$ in an invariant subspace by the matrix $A + w_\delta^{(\ell)} B$, i.e.:

$$\left.\begin{array}{c}\left[I - L\left(\alpha(i_\ell),\beta(j_\ell)\right) L\left(\alpha(i_\ell),\beta(j_\ell)\right)^{\dagger}\right] \left(A + w_\delta^{(\ell)} B\right) L\left(\alpha(i_\ell),\beta(j_\ell)\right) = 0 , \\ w_\delta^{(\ell)} = \dfrac{-1}{4M^2 \sin^2\left(\dfrac{\theta_\delta^{(\ell)}}{2}\right)} , \quad 1 \le \delta \le M-1, \end{array}\right] \quad (87)$$

where $\theta_\delta^{(\ell)}$ is the solution of (49) associated to the pair $(\alpha(i_\ell),\beta(j_\ell))$ in J_δ. Consider problem (P_ℓ) defined by (8)–(10) together with the initial condition

$$U(m,0) = \widehat{f_\ell}(m) , \quad 0 \le m \le M ,\ 1 \le \ell \le p , \quad (88)$$

and note that solution $\{U_\ell(m,n)\}$ of problem (P_ℓ) is defined by (59) where $d_\delta^{(\ell)}$ is given by

$$d_\delta^{(\ell)} = \frac{\sum\limits_{\nu=1}^{M-1} \left\{ \left(1 - \frac{\beta(j_\ell)M\rho_\delta^{(\ell)}}{2r}\right) \sin\left(\nu\theta_\delta^{(\ell)}\right) - \beta(j_\ell)M\cos\left(\nu\theta_\delta^{(\ell)}\right)\sin\left(\theta_\delta^{(\ell)}\right)\right\} \widehat{f}_\ell(\nu)}{\sum\limits_{\nu=1}^{M-1} \left\{ \left(1 - \frac{\beta(j_\ell)M\rho_\delta^{(\ell)}}{2r}\right) \sin\left(\nu\theta_\delta^{(\ell)}\right) - \beta(j_\ell)M\cos\left(\nu\theta_\delta^{(\ell)}\right)\sin\left(\theta_\delta^{(\ell)}\right)\right\}^2}, \tag{89}$$

for $1 \le \delta \le M-1\,,\ \ 1 \le \ell \le p\,,\ \ 1 \le j \le q$.

$$U_\ell(m,n) = \sum_{\delta=1}^{M-1} \left[I - r\left(4A\sin^2\left(\frac{\theta_\delta^{(\ell)}}{2}\right) - \frac{B}{M^2}\right)\right]^n \cdot$$

$$\cdot \left[\left(1 - \frac{\beta(j_\ell)M\rho_\delta^{(\ell)}}{2r}\right)\sin(m\theta_\delta^{(\ell)}) - \beta(j_\ell)M\cos(m\theta_\delta^{(\ell)})\sin(\theta_\delta^{(\ell)})\right] d_\delta^{(\ell)}. \tag{90}$$

By linearity and (86), (90) it follows that

$$U(m,n) = \sum_{\ell=1}^{p} U_\ell(m,n)\,, \quad 1 \le m \le M-1\,,\ \ n \ge 0\,, \tag{91}$$

is a solution of problem (8)–(11). Furthermore (91) is a stable solution if (73)–(74) hold and the parameter r verifies

$$r < \min_{1\le\ell\le p} \left\{ \frac{M^2\left[\left(2M\sin\left(\frac{\theta_1^{(\ell)}}{2}\right)\right)^2 \lambda_{\min}\left(\widetilde{A}_1\right) - \lambda_{max}\left(\widetilde{B}_1\right)\right]}{\left[4M^2\lambda_{max}\left(\widetilde{A}_1\right) + \rho\left(\widetilde{B}_1\right)\right]^2 + \left[4M^2\lambda_{max}\left(\widetilde{A}_2\right) + \rho\left(\widetilde{B}_2\right)\right]^2} \right\}. \tag{92}$$

Summarizing the following result is a consequence of *theorem 3.1.*

Theorem 3.2 *Consider problem (8)–(11) under hypothesis (5) with* $A_1 = I$*, assume (76) and (77) and let* M *be an integer satisfying (23), (38) and*

$$M > \max_{1\le\ell\le p} \left\{ \frac{\alpha(i_\ell)}{1 - \cos\left(\theta^{(\ell)}\right)}, \alpha(i_\ell) \right\}. \tag{93}$$

Let $\mathcal{F}$ and $\mathcal{L}$ be defined by (80) and (81) respectively, assume that $\{f(m)\}$ is bounded, conditions (73)–(74) are satisfied and r is small enough so that (92) holds. Let $\left\{\widehat{f}_\ell(m)\right\}_{m=0}^{M}$ be defined by (85), let $w_\delta^{(\ell)}$ be defined by (87) and assume that condition (87) holds. If $\{U_\ell(m,n)\}$ is given by (90) then $\{U(m,n)\}$ defined by (91) is a stable solution of problem (8)–(11).

4 Example

Consider the problem (1)–(4) with data:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -3 & 2 & 0 \\ 0 & -8 & 0 \\ 0 & 5 & -3 \end{bmatrix}, \; A_1 = I \,,$$

$$B_1 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 2 & -\frac{1}{2} \end{bmatrix}, \; A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & \frac{1}{2} & -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 5 & \frac{3}{2} & 0 \\ 0 & -2 & 0 \\ 0 & -\frac{9}{4} & -\frac{1}{2} \end{bmatrix},$$

$f(m) = F(mh) = (f_1(m), f_2(m), f_3(m))^T$, and $h = \frac{1}{M}$, $1 \le m \le M-1$. Hypothesis (5) is satisfied, $\widehat{A}_2 = (A_2 B_1 - B_2)^{-1} A_2 = A_2$ with

$$\sigma\left(\widehat{A}_2\right) = \{-1, 2\} \,, \qquad \sigma\left(B_1\right) = \left\{-\frac{1}{2}, 3\right\} \,.$$

Let $\alpha(1) = -1$, $\alpha(2) = 2$, $\beta(1) = -\frac{1}{2}$, $\beta(2) = 3$ and note that both pairs $(\alpha(1), \beta(1))$, $(\alpha(2), \beta(2))$ satisfy (48) and

$$v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \qquad \left(\widehat{A}_2 - \alpha(1)\, I\right) v = (B_1 - \beta(1)\, I)\, v = 0 \,,$$

$$w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \qquad \left(\widehat{A}_2 - \alpha(2)\, I\right) w = (B_1 - \beta(2)\, I)\, w = 0 \,.$$

For the pair $(\alpha(1), \beta(1)) = (-1, -1/2)$ the matrix $L(\alpha(1), \beta(1))$ defined by (72) takes the value

$$\left.\begin{aligned} L\,(-1,-1/2) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq 0\,; \\ I - L\,(-1,-1/2)\,L\,(-1,-1/2)^{\dagger} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}\right] \tag{94}$$

Let $\left\{\theta_\delta^{(1)}\right\}_{\delta=1}^{M-1}$ be the solutions of (49) corresponding to the pair $(-1,-1/2)$ and let

$$w_\delta^{(1)} = \frac{-1}{4M^2\sin^2\left(\dfrac{\theta_\delta^{(1)}}{2}\right)}\,, \quad 1 \leq \delta \leq M-1\,.$$

Hence

$$A + w_\delta^{(1)}B = \begin{bmatrix} 1-3w_\delta^{(1)} & -1+2w_\delta^{(1)} & 0 \\ 0 & 2-8w_\delta^{(1)} & 0 \\ 0 & 2+5w_\delta^{(1)} & 1-3w_\delta^{(1)} \end{bmatrix}. \tag{95}$$

By (94) and (95) it follows that

$$\left[I - L\,(-1,-1/2)\,L\,(-1,-1/2)^{\dagger}\right]\left(A + w_\delta^{(1)}B\right)L\,(-1,-1/2) = 0\,, \quad 1 \leq \delta \leq M-1\,, \tag{96}$$

Let us consider now the pair $(\alpha(2),\beta(2)) = (2,3)$. Computing one gets

$$L(2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0 \text{ and } I - L(2,3)L(2,3)^{\dagger} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{97}$$

Let $\left\{\theta_\delta^{(2)}\right\}_{\delta=1}^{M-1}$ be the solutions of (49) corresponding to the pair $(2,3)$ and let

$$w_\delta^{(2)} = \frac{-1}{4M^2\sin^2\left(\dfrac{\theta_\delta^{(2)}}{2}\right)}\,, \quad 1 \leq \delta \leq M-1\,.$$

Note that

$$A + w_\delta^{(2)} B = \begin{bmatrix} 1 - 3w_\delta^{(2)} & -1 + 2w_\delta^{(2)} & 0 \\ 0 & 2 - 8w_\delta^{(2)} & 0 \\ 0 & 2 + 5w_\delta^{(2)} & 1 - 3w_\delta^{(2)} \end{bmatrix} .$$

Computing the matrix $\mathcal{L} = [L(\alpha(1), \beta(1)), L(\alpha(2), \beta(2))]$ one gets

$$\mathcal{L} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

Condition (82) is satisfied by any vector function $\{f(m)\}$ of the form

$$f(m) = (f_1(m), 0, f_3(m))^T .$$

The projections $\left\{\widehat{f}_1(m)\right\}$, $\left\{\widehat{f}_2(m)\right\}$ defined by (85) take the form

$$\widehat{f}_1(m) = [L(\alpha(1), \beta(1)), 0]\, \mathcal{L}^\dagger f(m) = \begin{bmatrix} 0 \\ 0 \\ f_3(m) \end{bmatrix} , \tag{98}$$

$$\widehat{f}_2(m) = [0, L(\alpha(2), \beta(2))]\, \mathcal{L}^\dagger f(m) = \begin{bmatrix} f_1(m) \\ 0 \\ 0 \end{bmatrix} . \tag{99}$$

Note that

$$\frac{A + A^H}{2} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} , \quad \sigma\left(\frac{A + A^H}{2}\right) = \left\{\frac{1079}{396}, \frac{297}{1079}, 1\right\} ;$$

$$\frac{B + B^H}{2} = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -8 & \frac{5}{2} \\ 0 & \frac{5}{2} & -3 \end{bmatrix} , \quad \sigma\left(\frac{B + B^H}{2}\right) = \left\{-\frac{1211}{132}, -3, -\frac{1729}{947}\right\} ,$$

and thus the stability conditions (73), (74) are satisfied. Taking small enough values of r satisfying (92), M verifying (23), (38) and (92) by *theorem 3.2* the vector function

$$U(m,n) = \sum_{\ell=1}^{2} U_\ell(m,n)\,,$$

where $\{U_\ell(m,n)\}$ are defined by (89), (90) and $\left\{\widehat{f}_\ell(m)\right\}$ by (98)–(99) is a stable solution of the mixed problem (8)–(11) with the above data.

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