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Domain of Existence and Uniqueness for Nonlinear Hammerstein Integral Equations †

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Abstract: In this work, we performed an study about the domain of existence and uniqueness for an efficient fifth order iterative method for solving nonlinear problems treated in their infinite dimensional form. The hypotheses for the operator and starting guess are weaker than in the previous studies. We assume omega continuity condition on second order Fréchet derivative. This fact it is motivated by showing different problems where the nonlinear operators that define the equation do not verify Lipschitz and Hölder condition; however, these operators verify the omega condition established. Then, the semilocal convergence balls are obtained and the R-order of convergence and error bounds can be obtained by following thee main theorem. Finally, we perform a numerical experience by solving a nonlinear Hammerstein integral equations in order to show the applicability of the theoretical results by obtaining the existence and uniqueness balls.

Keywords: semilocal convergence; Lipschitz condition; Hölder condition; Hammerstein integral equation; dynamical systems

MSC: 65G49; 47H99

1. Introduction

Let X and Y are Banach spaces where $G : \Omega \subseteq X \to Y$ be a nonlinear function in an open convex domain $\Omega_0 \subseteq \Omega$. We use iterative methods in order to solve the nonlinear equation:

$$G(x) = 0 \tag{1}$$

which characterizes various real life problems such as dynamical systems, boundary value problems described by ordinary differential equations, partial derivative equations and nonlinear integral equations with applications in different fields of engineering, finances, optimization costs and benefits, etc. A great variety of iterative schemes for solving these problems are obtained in [1–3]. It is well known that Newton's method is the most widely used iterative scheme to solve (1), it is defined for $k \geq 0$, by

$$x_{k+1} = x_k - \Gamma_k G(x_k) \tag{2}$$

where $\Gamma_k = G'(x_k)^{-1}$ and x_0 is the starting guess and it reaches convergence order two. Different modifications of Newton's method have been published in order to increase the order of convergence and efficiency. We center in such publications in the literature involving complete studies in the sense of local and semilocal convergence, (see, [4–19]), where authors studied the convergence of iterative methods with high order of convergence under different continuity hypotheses.

In a recent paper, Singh et al. [20] presented the semilocal convergence of an efficient fifth order method under the Lipschitz condition on second derivative for non linear operator F''. The iterative scheme can be written for $k = 0, 1, 2 \ldots$ as

$$\begin{aligned} y_k &= x_k - \Gamma_k G(x_k), \\ z_k &= y_k - \Gamma_k G(y_k), \\ x_{k+1} &= z_k - G'(y_k)^{-1} G(z_k). \end{aligned} \tag{3}$$

In real life applications, (see [21–23]), various numerical examples involving Hammerstein type integral equation [2] can be found which neither satisfy the Lipschitz nor the Hölder condition. This is the reason that motivated us to establish the semilocal convergence for the iterative method defined above in (3) under weaker conditions, which is also an efficient fifth-order method.

Consider nonlinear Hammerstein type integral equation

$$x(r) + \sum_{i=1}^{m} \int_a^b K_i(r,s) S_i(x(s)) ds = f(r),\ r \in [a,b], \tag{4}$$

where functions f, K_i and S_i for $i = 1, 2, \ldots m$ are known, the solution x is to be determined and $-\infty < a < b < +\infty$. In order to solve (4), we have to solve

$$G(x)(u) = x(u) + \sum_{i=1}^{m} \int_a^b K_i(u,v) S_i(x(v)) dv - f(u) \tag{5}$$

If $S_i'(x(u))$ is (M_i, α_i)- Hölder continuous in Ω, then, under max-norm, we have

$$\|G''(x) - G''(y)\| \leq \sum_{i=1}^{m} M_i \|x-y\|^{\alpha_i},\ M_i \geq 0,\ \alpha_i \in [0,1],\ \forall\, x, y \in \Omega. \tag{6}$$

For different α_i, G'' neither satisfies Lipschitz nor Hölder condition but satisfies the weaker ω-condition.

In this work and in Section 2, we developed the semilocal convergence analysis of an iterative method of five order of convergence; this has been done under weaker conditions for solving nonlinear equations. Moreover, theoretical results about the existence and uniqueness for the solution have been established along with error bounds for the solution. In Section 3, we developed numerical examples and obtained the radius of existence and uniqueness for the solution, showing the applicability of our study. Finally, some conclusions are included in Section 4.

2. Semilocal Convergence Analysis

In this section, we give the hypothesis for the nonlinear operator G in the starting point, so we construct the convergence ball centered at this point, that is, the ball at which all the iterates belong and converge to the solution.

2.1. Preliminaries Results

Let $x_0 \in \Omega$, such as $\Gamma_0 = G'(x_0)^{-1} \in BL(Y, X)$ exists, being a bounded linear operator from Y to X for which the following conditions hold.

(1) $\|\Gamma_0\| \leq \beta_0$
(2) $\|\Gamma_0 G(x_0)\| \leq \eta_0$
(3) $\|G''(x)\| \leq M$
(4) $\|G''(x) - G''(y)\| \leq \omega(\|x-y\|), x, y \in \Omega$, for a non-decreasing continuous real function $\omega(a)$, $a > 0, \omega(0) \geq 0$ such that, $\omega(ta) \leq t^q \omega(a)$ for $t \in [0,1], a \in (0,\infty)$ and $q \in [0,1]$.

Let $r_0 = M\beta_0\eta_0, s_0 = \beta_0\eta_0\omega(\eta_0)$ and define sequences $\{r_k\}$, $\{s_k\}$ and $\{\eta_k\}$ for $k = 0,1,2\ldots$, by

$$r_{k+1} = r_k\phi(r_k)^2\psi(r_k, s_k), \tag{7}$$
$$s_{k+1} = s_k\phi(r_k)^{2+q}\psi(r_k, s_k)^{1+q}, \tag{8}$$
$$\eta_{k+1} = \eta_k\phi(r_k)\psi(r_k, s_k), \tag{9}$$

where,

$$\phi(u) = \frac{1}{1 - ug(u)}, \tag{10}$$
$$g(u) = \left(1 + \frac{u}{2} + \frac{u^2}{2(1-u)}\left(1 + \frac{u}{4}\right)\right), \tag{11}$$

and

$$\begin{aligned}\psi(u,v) \quad &= \quad \frac{u^2}{2(1-u)}\left(1+\frac{u}{4}\right)\left[\frac{v}{1+q}\left(\frac{u^{1+q}}{2^{1+q}} + \frac{1}{2+q}\left(\frac{u^2}{2(1-u)}\left(1+\frac{u}{4}\right)\right)^{1+q}\right)\right. \\ &+ \quad \left.\frac{u}{2}\left(u + \frac{u^2}{2(1-u)}\left(1+\frac{u}{4}\right)\right)\right].\end{aligned} \tag{12}$$

Let $h(u) = g(u)\,u - 1$. Moreover, $h(0) = -1$ and $g(u)$ is an increasing function, therefore, $h(u)$ has a real root ν verifying If $u \in (0, \nu)$, we get $g(u)\,u < 1$.

Lemma 1. *Let $\phi(u)$, $g(u)$ and $\psi(u,v)$ are given by (10), (11) and (13) respectively. If $0 < r_0 < \nu$ and $\phi(r_0)^2\psi(r_0, s_0) < 1$, then*
(i) $\phi(u)$ and $g(u)$ are increasing functions verifying $\phi(u) > 1$, $g(u) > 1$ for $u \in (0, \nu)$.
(ii) $\psi(u,v)$ is an increasing function of u, for $u \in (0,\nu)$.
(iii) $\{r_k\}, \{s_k\}$ and $\{\eta_k\}$ are decreasing sequences and $r_k g(r_k) < 1$ as well as $\phi(r_k)^2\psi(r_k, s_k) < 1$ for $k \geq 0$.

Proof. The proof of (i) and (ii) are trivial. The proof of (iii) can be given in the following manner. For $k = 0$, (7) gives $r_1 = r_0\phi(r_0)^2\psi(r_0, s_0) < r_0$. Using (8) and (9), we get $s_1 = s_0\phi(r_0)^{2+q}\psi(r_0, s_0)^{1+q} < s_0(\phi(r_0)^2\psi(r_0, s_0))^{1+q} < s_0$ and $\eta_1 = \phi(r_0)\psi(r_0, s_0)\eta_0 < \eta_0$. Thus, (iii) holds for $k = 0$. Since, $\phi(u)$ and $g(u)$ are increasing functions, and therefore, by using mathematical induction Lemma 1 holds $\forall\, k \geq 0$. □

Lemma 2. *Let $\phi(u)$ and $\psi(u,v)$ be defined by (10) and (13). If $\gamma \in (0,1)$ we have $\phi(\gamma t) < \gamma\phi(t)$ and $\psi(\gamma u, \gamma^{1+q}v) < \gamma^{3+q}\psi(u,v)$.*

Proof. The proof is trivial. Since $g(\gamma t) < g(t)$, as $g(t)$ is an increasing function. Therefore, $\phi(\gamma t) < \phi(t)$. Now,

$$
\begin{aligned}
\psi(\gamma u,\gamma^{1+q}v) &= \frac{(\gamma u)^2}{2(1-\gamma u)}\left(1+\frac{\gamma u}{4}\right)\Big[\frac{(\gamma^{1+q}v)}{1+q}\left(\frac{(\gamma u)^{1+q}}{2^{1+q}}+\frac{1}{2+q}\left(\frac{(\gamma u)^2}{2(1-\gamma u)}\left(1+\frac{\gamma u}{4}\right)\right)^{1+q}\right)\\
&+ \frac{(\gamma u)}{2}\left(\gamma u+\frac{(\gamma u)^2}{2(1-\gamma u)}\left(1+\frac{\gamma u}{4}\right)\right)\Big].\\
&< \gamma^{3+q}\psi(u,v)
\end{aligned}
$$

□

Lemma 3. *Let* $\gamma = \phi(r_0)^2\psi(r_0,s_0)$, $0 < r_0 < \nu$ *and* $\delta = \frac{1}{\phi(r_0)}$. *Then,*
(i) $r_k \le \gamma^{(4+q)^{k-1}} r_{k-1} \le \gamma^{\frac{(4+q)^k-1}{3+q}} r_0$ *and* $s_k \le \left(\gamma^{(4+q)^{k-1}}\right)^{1+q} s_{k-1} \le \left(\gamma^{\frac{(4+q)^k-1}{3+q}}\right)^{1+q} s_0$.
(ii) $\phi(r_k)\psi(r_k,s_k) \le \dfrac{\gamma^{(4+q)^k}}{\phi(r_0)}\ \forall\, k \in N$.
(iii) $\eta_k \le \gamma^{\frac{(4+q)^k-1}{3+q}}\delta^k\eta_0$.

Proof. Using $k = 0$ in (7) and (8), we get $r_1 = r_0\phi(r_0)^2\psi(r_0,s_0) = \gamma r_0$ and

$$s_1 = s_0\phi(r_0)^{2+q}\psi(r_0,s_0)^{1+q} \le \gamma^{1+q}s_0.$$

Thus, Lemma holds for $k = 0$. Assume that Lemma holds for $k = n$. Using induction, we will prove for $k = n+1$. Then, we have

$$
\begin{aligned}
r_{n+1} &= r_n\phi(r_n)^2\psi(r_n,s_n)\\
&\le \gamma^{(4+q)^{n-1}} r_{n-1}\phi\left(\gamma^{(4+q)^{n-1}} r_{n-1}\right)^2\psi\left(\gamma^{(4+q)^{n-1}} r_{n-1}(\gamma^{(4+q)^{n-1}})^{1+q}s_{n-1}\right)\\
&\le \gamma^{(4+q)^{n-1}} r_{n-1}\phi\left(r_{n-1}\right)^2\left(\gamma^{(4+q)^{n-1}}\right)^{3+q}\psi(r_{n-1},s_{n-1})\\
&\le \left(\gamma^{(4+q)^{n-1}}\right)^{(4+q)} r_{n-1}\phi(r_{n-1})^2\psi(r_{n-1},s_{n-1}),\\
&\le \gamma^{(4+q)^n} r_n.
\end{aligned}
\tag{13}
$$

In a similar manner, we get

$$
\begin{aligned}
r_{n+1} &\le \gamma^{(4+q)^n} r_n \le \gamma^{(4+q)^n}\gamma^{(4+q)^{n-1}} r_{n-1}\\
&\le \cdots \le \gamma^{(4+q)^n}\gamma^{(4+q)^{n-1}}\ldots\gamma^{(4+q)^0} r_0 = \gamma^{\frac{(4+q)^{n+1}-1}{3+q}} r_0.
\end{aligned}
\tag{14}
$$

Now, we consider

$$
\begin{aligned}
s_{n+1} &= s_n\phi\left(r_n\right)^{(2+q)}\psi(r_n,s_n)^{1+q} \le s_n\left(\phi(r_n)^2\psi\left(r_n,s_n\right)\right)^{1+q}\\
&\le s_n\left(\frac{r_{n+1}}{r_n}\right)^{1+q} \le \left(\gamma^{(4+q)^n}\right)^{1+q} s_n
\end{aligned}
$$

proceeding in this way, we get

$$\begin{aligned} s_{n+1} &\leq \left(\gamma^{(4+q)^n}\right)^{1+q} s_n \leq \left(\gamma^{(4+q)^n}\right)^{1+q} \left(\gamma^{(4+q)^{n-1}}\right)^{1+q} s_{n-1} \\ &\leq \left(\gamma^{\frac{(4+q)^{n+1}-1}{3+q}}\right)^{1+q} s_0. \end{aligned} \tag{15}$$

Hence, (i) holds $\forall\, k \geq 0$ by using mathematical induction. Now, consider

$$\begin{aligned} \phi(r_k)\psi(r_k, s_k) &\leq \phi(\gamma^{\frac{(4+q)^k-1}{3+q}} r_0)\psi\left(\gamma^{\frac{(4+q)^k-1}{3+q}} r_0, (\gamma^{\frac{(4+q)^k-1}{3+q}})^{1+q} s_0\right) \\ &\leq \gamma^{(4+q)^k-1}\phi(r_0)\psi(r_0, s_0) = \gamma^{(4+q)^k}\delta. \end{aligned} \tag{16}$$

Thus, (ii) is proven. From (9), we get

$$\begin{aligned} \eta_k &= \phi(r_{k-1})\psi(r_{k-1}, s_{k-1})\eta_{k-1} \leq \prod_{n=0}^{k-1} \phi(r_n)\psi(r_n, s_n)\eta_0 \\ &\leq \prod_{n=0}^{k-1} \frac{\gamma^{(4+q)^n}}{\phi(r_0)}\eta_0 \leq \gamma^{\frac{(4+q)^k-1}{1+q}} \delta^k \eta_0. \end{aligned} \tag{17}$$

Thus, (iii) is proven. □

2.2. Main Results

In this section, we establish the recurrence relations for (3) under the assumption considered in the previous section. Consider

$$\|I - \Gamma_0 G'(y_0)\| \leq \|\Gamma_0\|\, \|G'(y_0) - G'(x_0)\| \leq M\beta_0\eta_0 = r_0, \tag{18}$$

if $r_0 < 1$, then

$$\|G'(y_0)^{-1}G'(x_0)\| \leq \frac{1}{1 - r_0} \tag{19}$$

and by substituting $k = 0$ in (3), we get

$$z_0 - x_0 = -\Gamma_0 G(x_0) - \Gamma_0 G(y_0). \tag{20}$$

By using Taylor expansion of $G(y_0)$ about x_0, we get

$$\begin{aligned} G(y_0) &= G(x_0) + G'(x_0)(y_0 - x_0) + \int_0^1 G''(x_0 + \theta(y_0 - x_0))(y_0 - x_0)^2(1 - \theta)d\theta \\ &= \int_0^1 G''(x_0 + \theta(y_0 - x_0))(y_0 - x_0)^2\theta d\theta. \end{aligned} \tag{21}$$

Then, by using (21) in (20) and taking norm, we get

$$\begin{aligned} \|z_0 - x_0\| &\leq \|\Gamma_0 G(x_0)\| + \|\Gamma\|\frac{M}{2}\|y_0 - x_0\|^2 \\ &\leq \left(1 + \frac{r_0}{2}\right)\eta_0. \end{aligned} \tag{22}$$

Now,

$$\|z_0 - y_0\| = \|\Gamma_0 G(y_0)\| \leq \frac{r_0}{2}\eta_0,$$

and by substituting $k = 0$ in (3) and taking norm, we get

$$\begin{aligned} \|x_1 - z_0\| &\leq \|G^{'}(y_0)^{-1}G(z_0)\| \leq \|G^{'}(y_0)^{-1}G^{'}(x_0)\|\|\Gamma_0 G(z_0)\| \\ &\leq \frac{r_0^2}{2(1-r_0)}\left(1+\frac{r_0}{4}\right)\eta_0. \end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned} \|x_1 - x_0\| &\leq \|x_1 - z_0\| + \|z_0 - x_0\| \\ &\leq \left(1 + \frac{r_0}{2} + \frac{r_0^2}{2(1-r_0)}\left(1+\frac{r_0}{4}\right)\right)\eta_0 = g(r_0)\eta_0. \end{aligned} \tag{24}$$

So, we have

$$\begin{aligned} \|I - \Gamma_0 G^{'}(x_1)\| &\leq \|\Gamma_0\|\|G^{'}(x_1) - G^{'}(x_0)\| \leq \beta_0 M\|x_1 - x_0\| \\ &\leq M\beta_0\eta_0 g(r_0) = r_0 g(r_0) < 1, \end{aligned}$$

therefore, by Banach Lemma, we get

$$\|\Gamma_1\| \leq \frac{\|\Gamma_0\|}{1 - r_0 g(r_0)} = \|\Gamma_0\|\phi(r_0). \tag{25}$$

Moreover,

$$\|\Gamma_0\|\|y_0 - x_0\|\omega\left(\|y_0 - x_0\|\right) \leq \beta_0\eta_0\omega(\eta_0) = s_0.$$

Using Taylor expansion of $G(x_1)$ about z_0, we get

$$\begin{aligned} G(x_1) &= \int_0^1 \left(G^{''}\left(y_0 + t(z_0 - y_0)\right) - G^{''}(y_0)\right)(z_0 - y_0)(x_1 - z_0)dt + G^{''}(y_0)(z_0 - y_0)(x_1 - z_0) \\ &+ \int_0^1 G^{''}(z_0 + t(x_1 - z_0))(x_1 - z_0)^2(1-t)dt + \frac{1}{2}G^{''}(z_0)\left(x_1 - z_0\right)^2. \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned} \|\Gamma_1 G(x_1)\| &\leq \phi(r_0)\|\Gamma_0\|\|G(x_1)\|, \\ &\leq \phi(r_0)\frac{{r_0}^2}{2(1-r_0)}\left(1+\frac{r_0}{4}\right)\left[\frac{s_0}{1+q}\left(\frac{{r_0}^{1+q}}{2^{1+q}} + \frac{1}{2+q}\left(\frac{{r_0}^2}{2(1-r_0)}\left(1+\frac{r_0}{4}\right)\right)^{1+q}\right)\right. \\ &+ \left.\frac{r_0}{2}\left(r_0 + \frac{{r_0}^2}{2(1-r_0)}\left(1+\frac{r_0}{4}\right)\right)\right] \\ &= \phi(r_0)\psi(r_0, s_0)\eta_0 = \eta_1. \end{aligned} \tag{27}$$

Using (27), we get

$$\begin{aligned} M\|\Gamma_1\|\|\Gamma_1 G(x_1)\| &\leq M\phi(r_0)\|\Gamma_0\|\phi(r_0)\psi(r_0, s_0)\eta_0 \\ &\leq r_0\phi(r_0)^2\psi(r_0, s_0) = r_1 \end{aligned} \tag{28}$$

and

$$\begin{aligned} \|\Gamma_1\|\|\Gamma_1 G(x_1)\|\omega\left(\|\Gamma_1 G(x_1)\|\right) &\leq \beta_0\eta_0\omega(\eta_0)\phi(r_0)^{1+q}\psi\left(r_0, s_0\right)^{1+q} \\ &\leq s_0\phi(r_0)^{2+q}\psi\left(r_0, s_0\right)^{1+q} = s_1. \end{aligned} \tag{29}$$

The following recurrence relations are established for $k \geq 1$ using mathematical induction.

(I) $\|\Gamma_k\| \leq \phi(r_{k-1})\|\Gamma_{k-1}\|$,
(II) $\|\Gamma_k G(x_k)\| \leq \phi(r_{k-1})\psi(r_{k-1}, s_{k-1})\eta_{k-1}$,
(III) $M\|\Gamma_k\|\|\Gamma_k G(x_k)\| \leq r_k$,
(IV) $\|\Gamma_k\|\|\Gamma_k G(x_k)\|\omega\left(\|\Gamma_k G(x_k)\|\right) \leq s_k$,
(V) $\|x_k - x_{k-1}\| \leq g(r_{k-1})\eta_{k-1}$,

Hence, the recurrence relations (I)–(IV) for $k = 1$ follow from (25), (27), (28) and (29) respectively. The recurrence relation (V) is proved for $k = 1$ in (24). Using mathematical induction, these recurrence relations hold $\forall\ k \geq 1$.

2.3. Convergence Theorem

Theorem 1. *Let $r_0 = M\beta_0\eta_0 < \nu$, $s_0 = \beta_0\eta_0\omega(\eta_0)$ and assumptions (1)-(4) hold. Then for $\overline{B}(x_0, R\eta_0) \subseteq \Omega$, where $R = \dfrac{g(r_0)}{1-\delta\gamma}$, the sequence $\{x_k\}$ generated by (3) converges to the solution of (1). Moreover, $y_k, z_k, x_{k+1}, x^* \in \overline{B}(x_0, R\eta_0)$ and x^* is the unique solution in $B\left(x_0, \frac{2}{L_1\beta_0} - R\eta_0\right) \cap \Omega$. Then the error bound for iterative scheme verifies:*

$$\|x_k - x^*\| \leq g(r_0)\delta^k \frac{\gamma^{\frac{(4+q)^k-1}{3+q}}}{1-\delta\gamma^{(4+q)^k}}\eta_0.$$

Proof. To prove the convergence theorem, we prove that $\{x_k\}$ is a Cauchy sequence. Using (V), we get

$$\|x_{k+1} - x_k\| \leq g(r_k)\eta_k \leq g(r_0)\eta_k \leq g(r_0)\prod_{j=0}^{k-1}\phi(r_j)\psi(r_j, s_j)\eta_0. \tag{30}$$

Now, we consider

$$\begin{aligned}
\|x_{k+m} - x_k\| &\leq \|x_{k+m} - x_{k+m-1}\| + \|x_{k+m-1} - x_{k+m-2}\| + \ldots + \|x_{k+1} - x_k\| \\
&\leq g(r_0)\prod_{j=0}^{k+m-2}\phi(r_j)\psi(r_j, s_j)\eta_0 + g(r_0)\prod_{j=0}^{k+m-1}\phi(r_j)\psi(r_j, s_j)\eta_0 + \ldots \\
&+ g(r_0)\prod_{j=0}^{k-1}\phi(r_j)\psi(r_j, s_j)\eta_0 \\
&\leq g(r_0)\sum_{l=0}^{m-1}\left(\prod_{j=0}^{k+l-1}\phi(r_j)\psi(r_j, s_j)\eta_0\right)
\end{aligned} \tag{31}$$

by using Lemma 3 (iii), we get

$$\begin{aligned}
\|x_{k+m} - x_k\| &\leq g(r_0)\sum_{l=0}^{m-1}\delta^{k+l}\left(\gamma^{\frac{(4+q)^{k+l}-1}{3+q}}\right)\eta_0 \\
&\leq g(r_0)\delta^k\left(\gamma^{\frac{(4+q)^k-1}{3+q}}\right)\sum_{l=0}^{m-1}\left(\delta\gamma^{(4+q)^k}\right)^l \\
&\leq g(r_0)\delta^k\left(\gamma^{\frac{(4+q)^k-1}{3+q}}\right)\frac{1-(\delta\gamma^{(4+q)^k})^m}{1-\delta\gamma^{(4+q)^k}}\eta_0.
\end{aligned} \tag{32}$$

Hence, $\{x_k\}$ is a Cauchy sequence which converges to x^* as $k \to \infty$. Taking $m \to \infty$ in (32), we get

$$\|x_k - x^*\| \leq g(r_0)\delta^k \gamma^{\frac{(4+q)^k-1}{3+q}} \frac{1}{1-\delta\gamma^{(4+q)^k}}\eta_0. \tag{33}$$

Taking $k = 0$ in (33), we get

$$\|x^* - x_0\| \leq \frac{g(r_0)}{1-\delta\gamma}\eta_0 \leq R\eta_0. \tag{34}$$

Hence, $x^* \in \overline{B}(x_0, R\eta_0)$. Now,

$$\|x_{k+1} - x_0\| \leq \sum_{i=0}^{k} \|x_{i+1} - x_i\| \leq \sum_{i=0}^{k} g(r_i)\eta_i \leq g(r_0)\sum_{i=0}^{k}\eta_i \leq R\eta_0,$$

and

$$\|y_k - x_0\| \leq \|y_k - x_k\| + \|x_k - x_0\| \leq \eta_k + \sum_{i=0}^{k-1} g(r_i)\eta_i \leq g(r_0)\sum_{i=0}^{k}\eta_i \leq R\eta_0.$$

Using (22), we get

$$\|z_k - x_0\| \leq \|z_k - x_k\| + \|x_k - x_0\| \leq \left(1 + \frac{r_0}{2}\right)\eta_k + \sum_{i=0}^{k-1} g(r_i)\eta_i \leq g(r_0)\sum_{i=0}^{k}\eta_i \leq R\eta_0.$$

Hence, $y_k, z_k, x_{k+1} \in \overline{B}(x_0, R\eta_0)$.

To show the uniqueness of x^*, let $z^* \in B\left(x_0, \frac{2}{M\beta} - R\eta_0\right) \cap \Omega$ be such that $G(z^*) = 0$, $z^* \neq x^*$. Then $0 = G(z^*) - G(x^*) = \int_0^1 G'(x^* + t(z^* - x^*))dt(z^* - x^*) = P(z^* - x^*)$, where, $P = \int_0^1 G'(x^* + t(z^* - x^*))dt$. Now,

$$\begin{aligned}
\|I - \Gamma_0 P\| &\leq \|\Gamma_0\| \int_0^1 \left\| \left(G'(x^* + t(z^* - x^*)) - G'(x_0)\right) \right\| dt \\
&\leq \frac{M\beta}{2}\left(\|x^* - x_0\| + \|z^* - x_0\|\right) \\
&< \frac{M\beta}{2}\left(R\eta_0 + \frac{2}{M\beta} - R\eta_0\right) \\
&= 1
\end{aligned}$$

Therefore, $\|I - \Gamma_0 P\| < 1$. Thus, P^{-1} exists by Banach Lemma and hence $z^* = x^*$. □

3. Numerical Experience

In this section, different numerical examples are solved in order to corroborate the theoretical results obtained and the efficiency of our approach.

Example 1. *Consider nonlinear integral equation*

$$G(x)(s) = x(s) - 1 + \int_0^1 H(s,t)\left(\frac{3}{5}x(t)^{7/3} + \frac{6}{15}x(t)^3\right)dt, \tag{35}$$

where $s \in [0,1]$, $x \in \Omega = B(0,2) \subset X$.

Clearly,

$$\|G''(x) - G''(y)\| \leq \frac{7}{30}\|x-y\|^{1/3} + \frac{3}{10}\|x-y\|.$$

where $\omega(\mu) = \frac{7}{30}\mu^{1/3} + \frac{3}{10}\mu$ and $q = \frac{1}{3}$. Therefore, neither the Lipschitz nor the Hölder condition hold but the ω-condition holds. Taking $x_0(t) = 1$, all the assumptions are satisfied. Therefore, the existence and uniqueness balls for integral equation are given by $\overline{B}(x_0, 0.21621)$ and $B(x_0, 1.2939)$, respectively. The values of the sequences $\{r_k\}$, $\{s_k\}$ and $\{\eta_k\}$ are given in Table 1.

Table 1. The values of r_k, s_k and η_k.

k	r_k	s_k	η_k
0	0.24527	5.1729×10^{-2}	0.18519
1	7.7009×10^{-4}	2.126×10^{-5}	4.1532×10^{-4}
2	6.8009×10^{-17}	8.359×10^{-23}	3.665×10^{-17}
3	3.6373×10^{-82}	7.8181×10^{-110}	1.9601×10^{-82}
4	1.5916×10^{-408}	5.5956×10^{-545}	8.5771×10^{-409}
5	2.5535×10^{-2040}	1.0509×10^{-2720}	1.3761×10^{-2040}

The error bounds for x^* are presented in Table 2.

Table 2. Error bounds.

k	$\|x_k - x^*\|$
0	4.8381×10^{-4}
1	4.9181×10^{-15}
2	3.4921×10^{-62}
3	5.2372×10^{-266}
4	9.3095×10^{-1149}
5	1.6043×10^{-4973}

Example 2. *Consider nonlinear integral equation*

$$G(x)(s) = x(s) - f(s) - \lambda \int_0^1 \frac{s}{s+t} x(t)^{2+q} dt, \tag{36}$$

where, $x, f \in C[0,1]$, $\lambda \in \mathbb{R}$ and $s \in [0,1]$.

Clearly,

$$\|G''(x) - G''(y)\| \leq |\lambda| \log 2(1+q)(2+q)\|x-y\|^q.$$

Here $\omega(\eta) = |\lambda| \log 2(1+q)(2+q)\eta^q$ Clearly, Lipschitz condition fails for $q \in (0,1)$ but Hölder condition holds. Taking $x_0 = x_0(s) = 1$, $q = \frac{1}{5}$, $\lambda = \frac{1}{4}$, and $f(s) = 1$, all the assumptions are satisfied. Therefore the existence and uniqueness balls for integral equation is given by $\overline{B}(x_0, 0.3174)$ and $B(x_0, 2.3879)$ respectively. The values of $\{r_k\}$, $\{s_k\}$ and $\{\eta_k\}$ are given in Table 3.

Table 3. The values of r_k, s_k and η_k.

k	c_k	d_k	η_k
0	0.20704	0.16052	0.28005
1	3.4155×10^{-4}	6.9698×10^{-5}	3.5373×10^{-4}
2	1.1985×10^{-18}	3.1433×10^{-22}	1.2408×10^{-18}
3	6.1808×10^{-91}	5.6532×10^{-109}	6.399×10^{-91}
4	2.255×10^{-452}	1.0637×10^{-542}	2.3347×10^{-452}
5	1.4579×10^{-2259}	2.509×10^{-2711}	1.5094×10^{-2259}

The error bounds for x^* are presented in Table 4.

Table 4. Error bounds.

k	$\|x_k - x^*\|$
0	4.004×10^{-4}
1	6.3045×10^{-16}
2	3.9701×10^{-65}
3	2.1201×10^{-271}
4	2.2551×10^{-1137}
5	4.3337×10^{-4774}

4. Conclusions

In this study, we present the semilocal convergence for an iterative scheme that reach order of convergence five. We obtained the theoretical results by constructing the recurrence relations that describe this algorithm that is proven to have a very efficient behavior. The hypotheses we set are under weaker conditions than the used in previous studies and allow us to obtain competitive error bounds. Finally, applied problems are solved involving nonlinear integral equations and big size nonlinear systems. The convergence balls defining the existence domain were obtained for the considered examples.

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