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Nondifferentiable Multiobjective Programming Problem under Strongly K-G_f-Pseudoinvexity Assumptions

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Abstract: In this paper we consider the introduction of the concept of (strongly) K-G_f-pseudoinvex functions which enable to study a pair of nondifferentiable K-G- Mond-Weir type symmetric multiobjective programming model under such assumptions.

Keywords: symmetric duality; multiobjective; nondifferentiable; K-G_f-pseudoinvex functions; K-G-Mond-Weir model

1. Introduction

Duality mathematical programming is used in Economics, Control Theory, Business and other diverse fields. In mathematical programming, a pair of primal and dual problems are said to be symmetric when the dual of the dual is the primal problem, i.e. when the dual problem is expressed in the form of the primal problem, then it does happen that its dual is the primal problem. This type of dual problem was introduced by Dorn [1], later on Mond and Weir [2] studying them under weaker convexity assumptions.

Antczak [3] introduced the notion of G-invex function obtaining some optimality conditions which he himself [4] comprehends to be a G_f -invex function, deriving optimality conditions for a multiobjective nonlinear programming problem. Ferrara and Stefaneseu [5] also discussed the conditions of optimality and duality for multiobjective programming problem, and Chen [6] considered multiobjective fractional problems and its duality theorems under higher-order (F, α, ρ, d)- convexity.

In recent years, several definitions such as nonsmooth univex, nonsmooth quasiunivex, and nonsmooth pseudoinvex functions have been introduced by Xianjun [7]. By introducing these new concepts, sufficient optimality conditions for a nonsmooth multiobjective problem were obtained and, a fortiori, weak and strong duality results were established for a Mond-Weir type multiobjective dual program.

Jiao [8] introduced new concepts of nonsmooth $K - \alpha - d_I$ -invex and generalized type I univex functions over cones by using Clarke's generalized directional derivative and d_I -invexity for a nonsmooth vector optimization problem with cone constraints. Op. cit. also established sufficient optimality conditions and Mond-Weir type duality results under $K-\alpha-d_I$ -invexity and type I cone-univexity assumptions. Very recently Dubey et al. [9] studied further Mond-Weir type dual model multiobjective programming problems over arbitrary cones.

Pitea and Postolache [10] developed the study of a new class of multi-time multiobjective variational problems of minimizing a vector of functionals of curvilinear integral type by means of which they were able to obtain results concerning duals of Mond-Weir type, generalized Mond-Weir-Zalmai type and under some assumptions of (ρ, b)-quasi-invexity, proving that the value of the objective function of the primal cannot exceed the value of the dual. And Pitea and Antczak [11] provided additional duality Mond-Weir type results and in the sense of Wolfe for multi-time multiobjective variational problems with univex functionals.

In the present paper we consider a pair of K-G-Mond-Weir type multiobjective symmetric dual program for which we establish the weak duality theorem, as well as the corresponding strong, and converse ones under K-G_f-pseudo-invexity/strongly K-G_f-pseudo-invexity assumptions. In the process we construct a lemma that enables us to prove the strength and converse duality theorems under K-G_f-pseudo-invexity/strongly K-G_f-pseudo-invexity assumptions.

2. Preliminaries and Definitions

As usual, throughout this paper, R^n will stand for the n-dimensional Euclidean space and R^n_+ for its non-negative orthant. Let $f = (f_1, f_2, ..., f_k) : X \to R^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subset R^n$ and $I_{f_i}(X)$ be the range of f_i, that is, the image of X under f_i, $i = 1, 2, ..., k$. $G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to R$ is strictly increasing on the range of $I_{f_i}(X)$, $i = 1, 2, ..., k$.

Definition 1. *Let S be a cone in R^s, the positive polar cone S^* of S is defined by*

$$S^* = \{y \in R^s : \ x^T y \geq 0, \ \forall x \in S\}.$$

Given two closed convex pointed cones K and Q with nonempty interiors in R^k and R^m, respectively, we consider two vector minimization problems, each of them accompanied by a natural weak minimum definition.

(KMP) K-minimize $f(x)$
Subject to $X^0 = \{x \in X \subset R^n : g_j(x) \in \ Q\}$

where $f : X \to R^k$ and $g : X \to R^m$ are differentiable functions defined on X.

Definition 2. *A point $\bar{x} \in X^0$ is said to be a weak minimum of (KMP) if there exists no other $x \in X^0$ such that $f(\bar{x}) - f(x) \in intK$.*

Lemma 1. *If $\bar{y} \in X^0$ is a weak minimum of (KMP), then there exist $\alpha \in K^*$, $\beta \in Q^*$ which are not simultaneously zero such that*

$$\left(\alpha^T \nabla f(\bar{y}) + \beta^T \nabla g(\bar{y})\right)^T (y - \bar{y}) \geq 0, \ \forall \, y \in Q,$$

$$\beta^T \nabla g(\bar{y}) = 0.$$

The second vector minimization problem that we consider is the following one.

(KGMP) K-minimize $G_f(f(x))$
Subject to $X^0 = \{x \in X \subset R^n : G_{g_j}(g_j(x)) \in \ Q\}$

Remark 1. *If $G_f(t) = t$ and $G_{g_j}(t_j) = t_j,\ j = 1,2,3,...,m$, then the vector minimization problem (KGMP) reduces to vector minimization problem (KMP).*

Definition 3. *A point $\bar{x} \in X^0$ is said to be a weak minimum of (KGMP) if there exists no other $x \in X^0$ such that $G_f(f(\bar{x})) - G_f(f(x)) \in intK$.*

Lemma 2. *If $\bar{y} \in X^0$ is a weak minimum of (KGMP), then there exist $\alpha \in K^*$, $\beta \in Q^*$ which are not simultaneously zero such that*

$$\left(\alpha^T G'_f(f(\bar{y}))\nabla f(\bar{y}) + \beta^T G'_g(g(\bar{y}))\nabla g(\bar{y})\right)^T (y - \bar{y}) \geq 0,\ \forall\, y \in Q,$$

$$\beta^T G'_g(g(\bar{y}))\nabla g(\bar{y}) = 0.$$

Let $C_1 \subseteq R^n$ and $C_2 \subseteq R^m$ be two closed convex cones with non-empty interiors, and let S_1 and S_2 be two non-empty open sets in R^n and R^m, respectively, so that $C_1 \times C_2 \subseteq S_1 \times S_2$. Given a vector valued differentiable function $f = (f_1, f_2, ..., f_k) : S_1 \times S_2 \to R^k$ we consider the following definitions.

Definition 4. *The function f is said to be K-η-pseudoinvex at $u \in S_1$, if $\forall\ x \in S_1$ and for fixed $v \in S_2$, we have*

$$\begin{aligned} &-\eta^T(x,u)\{\nabla_x f_1(u,v), \nabla_x f_2(u,v), ..., \nabla_x f_k(u,v)\} \notin intK \\ &\qquad \Rightarrow \{-f_1(x,v) + f_1(u,v), -f_2(x,v) + f_2(u,v), ..., -f_k(x,v) + f_k(u,v)\} \notin intK. \end{aligned}$$

Definition 5. *The function f is said to be strongly $K - \eta$-pseudoinvex at $u \in S_1$, if $\forall\ x \in S_1$ and for fixed $v \in S_2$, we have*

$$\begin{aligned} &-\eta^T(x,u)\left\{\nabla_x f_1(u,v), \nabla_x f_2(u,v), ..., \nabla_x f_k(u,v)\right\} \notin intK \\ &\qquad \Rightarrow \left(f_1(x,v) - f_1(u,v), f_2(x,v) - f_2(u,v), ..., f_k(x,v) - f_k(u,v)\right) \in K. \end{aligned}$$

Definition 6. *The function f is said to be K-G_f-pseudoinvex at $u \in S_1$ (with respect to η) if $\forall\ x \in S_1$ and for fixed $v \in S_2$, we have*

$$\begin{aligned} &-\eta^T(x,u)\left\{G_{f_1}(f_1(u,v))\nabla_x f_1(u,v), ..., G_{f_k}(f_k(u,v))\nabla_x f_k(u,v)\right\} \notin intK \\ &\qquad \Rightarrow \left(-G_{f_1}(f_1(x,v)) + G_{f_1}(f_1(u,v)), ..., -G_{f_k}(f_k(x,v)) + G_{f_k}(f_k(u,v))\right) \notin intK. \end{aligned}$$

Remark 2. *If $G_{f_i}(t) = t,\ i = 1,2,3,...,k$, then Definition 2.6 becomes $K - \eta$-pseudoinvex (Definition 4).*

Definition 7. *The function f is said to be strongly $K - G_f$-pseudoinvex at $u \in S_1$ (with respect to η) if $\forall\ x \in S_1$ and for fixed $v \in S_2$, we have*

$$\begin{aligned} &-\eta^T(x,u)\left\{G_{f_1}(f_1(u,v))\nabla_x f_1(u,v), ..., G_{f_k}(f_k(u,v))\nabla_x f_k(u,v)\right\} \notin intK \\ &\qquad \Rightarrow \left(G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(u,v)), ..., G_{f_k}(f_k(x,v)) - G_{f_k}(f_k(u,v))\right) \in K. \end{aligned}$$

Remark 3. *If $G_{f_i}(t) = t,\ i = 1,2,3,...,k$, then Definition 7 reduces in $K - \eta$-pseudoinvex (see, Definition 5).*

Finally we recall that [12] given a compact convex set C in R^n, the support function of C is defined by

$$s\,(x|C) = \max\{x^T y : y \in C\}.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For any convex set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leq 0 \text{ for all } z \in S\}.$$

It is readily verified that for a compact convex set S, $y \in N_S(x)$ if and only if

$$s(y|S) = x^T y.$$

3. *K*- *G*-Mond-Weir Type Primal Dual Model

In this section, we consider a multiobjective *K*-*G*-Mond–Weir type primal-dual model over arbitrary cones:

(KGMPP) *K*-minimize $U(x,y) = \left(G_{f_1}(f_1(x,y)),\ G_{f_2}(f_2(x,y)), ..., G_{f_k}(f_k(x,y))\right)$

Subject to

$$-\sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)] \in C_2^*, \tag{1}$$

$$y^T\left(\sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)]\right) \geq 0, \tag{2}$$

$$\lambda \in \text{int}K^*,\ x \in C_1,\ i = 1,2,...,k. \tag{3}$$

(KGMDP) *K*-maximize $V(u,v) = \left(G_{f_1}(f_1(u,v)),\ G_{f_2}(f_2(u,v)), ..., G_{f_k}(f_k(u,v))\right)$

Subject to

$$\sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)] \in C_1^*, \tag{4}$$

$$u^T\left(\sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)]\right) \leq 0, \tag{5}$$

$$\lambda \in \text{int}K^*,\ v \in C_2,\ i = 1,2,...,k, \tag{6}$$

where, for $i = 1,2,3,...,k$, it holds that:

(I) K^*, C_1^* and C_2^* are the positive polar cones of K, C_1 and C_2, respectively.

(II) Given $f_i : S_1 \times S_2 \to R, if\ G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$ has any of its components $G_{f_i} : I_{f_i}(S_1 \times S_2) \to R$ as a strictly increasing function on its domain, G_f is a differentiable function.

Next we prove weak, strong and converse duality theorems for (KGMPP) and (KGMDP), respectively. Let Z^0 and W^0 be the set of feasible solutions of (KGMPP) and (KGMDP), respectively.

Theorem 1 (Weak duality theorem). *Let $(x,y,\lambda) \in Z^0$ and $(u,v,\lambda) \in W^0$. Let*

(i) $\{f_1(.,v), f_2(.,v), ..., f_k(.,v)\}$ *be strongly K-G_f-pseudoinvex at u with respect to η_1,*

(ii) $\{f_1(x,.), f_2(x,.), ..., f_k(x,.)\}$ *be K-*G_f*-pseudoincave at y with respect to* η_2,
(iii) $\eta_1(x,u) + u \in C_1,\ \forall\ x \in C_1$,
(iv) $\eta_2(v,y) + y \in C_2,,\ \forall\ y \in C_2$.

Then,

$$\begin{aligned}(G_{f_1}(f_1(u,v)),\ G_{f_2}(f_2(u,v)),...,G_{f_k}(f_k(u,v)))\\ - (G_{f_1}(f_1(x,y)),\ G_{f_2}(f_2(x,y)),...,G_{f_k}(f_k(x,y))) \notin int\ K.\end{aligned} \tag{7}$$

Proof. The proof is given by contradiction. Suppose that (7) does not hold. Then,

$$\begin{aligned}(G_{f_1}(f_1(u,v)), G_{f_2}(f_2(u,v)),...,G_{f_k}(f_k(u,v)))\\ - (G_{f_1}(f_1(x,y)), G_{f_2}(f_2(x,y)),...,G_{f_k}(f_k(x,y))) \in int\ K.\end{aligned} \tag{8}$$

For the dual constraint (4) and assumption (iii), we get

$$(\eta_1(x,u)+u)^T \sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)] \geq 0.$$

Using the dual constraint (5) in the above inequality, we deduce that

$$\eta_1(x,u)^T \sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)] \geq 0,$$

or equivalently,

$$\sum_{i=1}^{k} \lambda_i [\eta_1(x,u)^T \{G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)\}] \geq 0.$$

Taking into account that $\lambda \in intK^*$,

$$\begin{aligned}-\eta_1(x,u)^T\{G'_{f_1}(f_1(u,v))\nabla_x f_1(u,v), G'_{f_2}(f_2(u,v))\nabla_x f_2(u,v)\\ ,...,G'_{f_k}(f_k(u,v))\nabla_x f_k(u,v)\} \notin intK.\end{aligned}$$

By hypothesis (i), it holds that

$$\begin{aligned}\{G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(u,v)), G_{f_2}(f_2(x,v)) - G_{f_2}(f_2(u,v))\\ ,....,G_{f_k}(f_k(x,v)) - G_{f_k}(f_k(u,v))\} \in K.\end{aligned}$$

Having in mind (8), we obtain

$$\begin{aligned}\{-G_{f_1}(f_1(x,y)) + G_{f_1}(f_1(x,v), - G_{f_2}(f_2(x,y)) + G_{f_2}(f_2(x,v)\\ ,...,-G_{f_1}(f_1(x,y)) + G_{f_1}(f_1(x,v)\} \in K + \mathrm{int}K \subset intK.\end{aligned} \tag{9}$$

On the similar lines to the above proof, we have

$$\begin{aligned}-\eta_2(v,y)^T\{G'_{f_1}(f_1(x,y))\nabla_y f_1(x,y), G'_{f_2}(f_2(x,y))\nabla_y f_2(x,y)\\ ,...,G'_{f_k}(f_k(x,y))\nabla_y f_k(x,y)\} \notin intK.\end{aligned}$$

By using now generalized convexity assumptions, it follows that

$$\begin{aligned}\{-G_{f_1}(f_1(x,y)) + G_{f_1}(f_1(x,v), -G_{f_2}(f_2(x,y)) + G_{f_2}(f_2(x,v)\\ ,...,-G_{f_1}(f_1(x,y)) + G_{f_1}(f_1(x,v)\} \notin intK,\end{aligned}$$

a contradiction with (9). Hence, the conclusion follows. □

Theorem 2 (Strong duality theorem). Let $(\bar{x}, \bar{y}, \bar{\lambda})$ be a weak efficient solution of (KGMPP); fix $\lambda = \bar{\lambda}$ in (KGMDP) and suppose that

(i) $\{G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})\}_{i=1}^k$ *is linearly independent,*

(ii) $R_+^k \subset K$.

Then, $(\bar{x}, \bar{y}, \bar{\lambda}) \in W^0$ and the objective values of (KGMPP) and (KGMDP) coincide. Moreover, if the assumptions of Theorem 1 are satisfied for all feasible solutions of (KGMPP) and (KGMDP), then $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weak efficient solution of (KGMDP).

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weak efficient solution of (KGMPP), by Lemma 2, then there exist $\alpha \in K^*$, $\beta \in C_2$, $\gamma \in R_+$ and $\delta \in K$ such that

$$\Bigg(\sum_{i=1}^k \alpha_i[G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y})] + (\beta - \gamma\bar{y})^T \sum_{i=1}^k \bar{\lambda}_i[G''_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y})\nabla_y f_i(\bar{x}, \bar{y})$$
$$+G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_{xy} f_i(\bar{x}, \bar{y})]\Bigg)^T (x - \bar{x})^T + \Bigg[\sum_{i=1}^k \alpha_i(G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})$$
$$+\sum_{i=1}^k (\beta - \gamma\bar{y})^T\bar{\lambda}_i(G''_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})(\nabla_y f_i(\bar{x}, \bar{y}))^T + G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_{yy} f_i(\bar{x}, \bar{y}))$$
$$-\sum_{i=1}^k \gamma\bar{\lambda}_i G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})\Bigg]^T (y - \bar{y}) + [(\beta - \gamma\bar{y})^T$$

$$G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y}) - \delta]^T(\lambda - \bar{\lambda}) \geq 0,\ \forall\, x \in C_1,\ y \in R^m,\ \lambda \in \text{int}K^*, \tag{10}$$

$$\beta^T \sum_{i=1}^k \bar{\lambda}_i[G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})] = 0, \tag{11}$$

$$\gamma\bar{y}^T \sum_{i=1}^k \bar{\lambda}_i[G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})] = 0, \tag{12}$$

$$\delta^T\bar{\lambda} = 0, \tag{13}$$

$$(\alpha, \beta, \gamma, \delta) \neq (0, 0, 0, 0),\ \ (\alpha, \beta, \gamma, \delta) \geq (0, 0, 0, 0). \tag{14}$$

Since $\delta \in K$ and $\bar{\lambda} \in \text{int}K^*$, Equation (13) implies $\delta = 0$.
Taking $x = \bar{x},\ y = \bar{y}$ in (10), we deduce that

$$[(\beta - \gamma\bar{y})^T G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})]^T(\lambda - \bar{\lambda}) \geq 0,\ \forall\, \lambda \in \text{int}K^*.$$

This implies that $(\beta - \gamma\bar{y})^T G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y}) \in -N_{\text{int}K^*}(\bar{\lambda})$, $N_{\text{int }K^*}(\bar{\lambda})$ being the normal cone to $\text{int}K^*$ at $\bar{\lambda}$.
Since $\bar{\lambda} \in \text{int}K^* = \text{int}(\text{int}K^*)$, $N_{\text{int }K^*}(\bar{\lambda}) = \{0\}$, we obtain that

$$(\beta - \gamma\bar{y})^T G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y}) = 0. \tag{15}$$

By assumption (i), we have

$$\beta - \gamma\bar{y} = 0. \tag{16}$$

Now, we claim that $\alpha \neq 0$. Indeed, if $\alpha = 0$, then (10) becomes

$$-[\sum_{i=1}^k \gamma\bar{\lambda}_i\{G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})\}]^T(y - \bar{y}) \geq 0,\ \forall\, y \in R^m. \tag{17}$$

This gives

$$\sum_{i=1}^{k}\gamma\bar{\lambda}_i[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})](y-\bar{y})=0. \tag{18}$$

By hypothesis (i), we have $\gamma\bar{\lambda}_i=0,\ i=1,2,3,...,k$. Since $R^k_+\subset K,\ \mathrm{int}K^*\subset \mathrm{int}R^k_+$ and since $\bar{\lambda}>0$, we get $\gamma=0$. Thus, from (16), we have $\beta=0$. A contradiction with the fact that $(\alpha,\beta,\gamma,\delta)\neq 0$. Hence, $\alpha\neq 0$, i.e. $\alpha>0$.

Now, the last equation and (16) yield

$$\sum_{i=1}^{k}(\alpha_i-\gamma\bar{\lambda}_i)\big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\big]=0.$$

By independence linearity hypothesis (i), this implies that

$$\alpha_i=\gamma\bar{\lambda}_i, i=1,2,...,k. \tag{19}$$

From $\bar{\lambda}_i>0,\ i=1,2,...,k$, and $\alpha_{k_0}=0$, for some k_0, it follows that $\gamma=0$. Now from (16), (19) and $\gamma=0$, we have $\alpha_i=0,\ i=1,2,...,k$, a contradiction with (14). Hence $\alpha_i>0,\ i=1,2,3,...,k$. Therefore, $\gamma>0$.

Taking $y=\bar{y}$ in (10), (16) and assumption (i) provide that

$$\Big[\sum_{i=1}^{k}\alpha_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))\Big]^T(x-\bar{x})\geq 0,\ \forall x\in C_1. \tag{20}$$

Since $\gamma>0$, from (19) and (20), we get

$$[\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))]^T(x-\bar{x})\geq 0,\ \forall x\in C_1. \tag{21}$$

Picking some $x\in C_1$, then $x+\bar{x}\in C_1$ since C_1 is a closed convex cone. By making $x+\bar{x}$ to play the role of x in (20), we get

$$[\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x)f_i(\bar{x},\bar{y})]x\geq 0,\ \forall\ x\in C_1.$$

Consequently,

$$\sum_{i=1}^{k}\bar{\lambda}_i\big(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x)f_i(\bar{x},\bar{y})\big)\in C_1^*. \tag{22}$$

By considering simultaneously $x=0$ and $x=2\bar{x}$ in (20), we have

$$\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i[(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x)f_i(\bar{x},\bar{y})]=0. \tag{23}$$

Since $\beta=\gamma\bar{y}$ and $\gamma>0$, we get

$$\bar{y}=\frac{\beta}{\gamma}\in C_2. \tag{24}$$

From (22) and (23), it follows that $(\bar{x},\bar{y},\bar{\lambda})\in W^0$. Moreover, if the hypothesis in Theorem 1 hold, then we conclude that $(\bar{x},\bar{y},\bar{\lambda})$ is a weak minimum of (KGMDP), and the two objective values coincide, QED. □

Thanks to the fact that under symmetric duality, the converse duality theorem proof works in the same as for the strong duality theorem, Theorem 1 infers the following result.

Theorem 3 (Converse duality theorem). *Let $(\bar{u}, \bar{v}, \bar{\lambda})$ be a weak efficient solution of (KGMDP); fix $\lambda = \bar{\lambda}$ in (KGMPP) and suppose that*

(i) $\{G'_{f_i}(f_i(\bar{u}, \bar{v}))\nabla_x f_i(\bar{u}, \bar{v})\}_{i=1}^{k}$ *is linearly independent,*
(ii) $R^k_+ \subset K$.

Then $(\bar{u}, \bar{v}, \bar{\lambda}) \in Z^0$ and the objective values of (KGMPP) and (KGMDP) coincide. Moreover, if the assumptions of Theorem 1 are satisfied for every feasible solution of (KGMPP) and (KGMDP), then $(\bar{u}, \bar{v}, \bar{\lambda})$ is a weak efficient solution of (KGMPP).

4. *K*- *N*- *G*-Mond–Weir Type Nondifferentiable Dual Model

Herein, we consider a nondifferentiable multiobjective *K*-*N*-*G*-Mond–Weir primal-dual model over arbitrary cones.

(KGNMPP) *K*-minimize $S(x, y, z) = \Big(G_{f_1}(f_1(x, y)) + s(x|D_1) - y^T z_1,\ G_{f_2}(f_2(x, y)) + s(x|D_2) - y^T z_2, ..., G_{f_k}(f_k(x, y) + s(x|D_k) - y^T z_k)\Big)$

Subject to

$$-\sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(x, y))\nabla_y f_i(x, y) - z_i] \in C_2^*, \tag{25}$$

$$y^T\left(\sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(x, y))\nabla_y f_i(x, y) - z_i]\right) \geq 0, \tag{26}$$

$$\lambda \in \text{int}K^*,\ x \in C_1,\ z_i \in E_i,\ i = 1, 2, ..., k. \tag{27}$$

(KGNMDP) *K*-maximize $T(u, v, w) = \Big(G_{f_1}(f_1(u, v)) - s(v|E_1) + u^T w_1,\ G_{f_2}(f_2(u, v)) - s(v|E_2) + u^T w_2, ..., G_{f_k}(f_k(u, v)) - s(v|E_k) + u^T w_k\Big)$

Subject to

$$\sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u, v))\nabla_x f_i(u, v) + w_i] \in C_1^*, \tag{28}$$

$$u^T\left(\sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u, v))\nabla_x f_i(u, v) + w_i]\right) \leq 0, \tag{29}$$

$$\lambda \in \text{int}K^*,\ v \in C_2,\ w_i \in D_i,\ i = 1, 2, ..., k, \tag{30}$$

where for $i = 1, 2, 3, ..., k$, it holds that:

(I) K^*, C_1^* and C_2^* are the positive polar cones of K, C_1 and C_2, respectively.
(II) Given $f_i : S_1 \times S_2 \to R$, *if* $G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$ has any of its components $G_{f_i} : I_{f_i}(S_1 \times S_2) \to R$ as a strictly increasing function on its domain, G_f is a differentiable function.
(III) D_i and E_i are compact convex sets in R^n and R^m, respectively.
(IV) $s(x|D_i)$ and $s(v|E_i)$ are the support functions of D_i and E_i, respectively.

Remark 4. *In the primal- dual model (K-N-G- Mond-Weir type nondifferentiable dual model), we used support function for a nondifferentiable term.*

Now we are ready to provide three duality theorems for (KGNMPP) and (KGNMDP). Their proofs are easily obtained by mimicking the ones of the three theorems obtained in the previous section.

Let X^0 and Y^0 be the set of feasible solutions of (KGNMPP) and (KGNMDP), respectively.

Theorem 4 (Weak duality theorem)**.** *Let* $(x, y, \lambda, z_1, z_2, ..., z_k) \in X^0$ *and* $(u, v, \lambda, w_1, w_2, ..., w_k) \in Y^0$*. Let*

(i) $\{f_1(., v), f_2(., v), ..., f_k(., v)\}$ *and* $\{(.)^T w_1, (.)^T w_2, ..., (.)^T w_k\})$ *be strongly K-*G_f*-pseudoinvex and strongly* $K - \eta_1$ *pseudoinvex, respectively, at u with respect to* η_1*,*

(ii) $\{f_1(x, .), f_2(x, .), ..., f_k(x, .)\}$ *and* $\{(.)^T z_1, (.)^T z_2, ..., (.)^T z_k\})$*be K-*$G_f$*-pseudoincave and* $K - \eta_2$*-pseudoinvex, respectively, at y with respect to* η_2*,*

(iii) $\eta_1(x, u) + u \in C_1,\ \forall\ x \in C_1,$

(iv) $\eta_2(v, y) + y \in C_2, ,\ \forall\ y \in C_2.$

Then

$$\begin{aligned}(G_{f_1}(f_1(u, v)) - s(v|E_1) + u^T w_1,\ G_{f_2}(f_2(u, v)) - s(v|E_2) + u^T w_2, ..., G_{f_k}(f_k(u, v)) - s(v|E_k)\\ + u^T w_k - \{G_{f_1}(f_1(x, y)) + s(x|D_1) - y^T z_1,\ G_{f_2}(f_2(x, y)) + s(x|D_2) - y^T z_2\\ , ..., G_{f_k}(f_k(x, y)) + s(x|D_k) - y^T z_k\}) \notin int\ K.\end{aligned} \tag{31}$$

Proof. The proof is given by contradiction. Suppose that (31) does not hold. Then,

$$\begin{aligned}(G_{f_1}(f_1(u, v)) - s(v|E_1) + u^T w_1,\ G_{f_2}(f_2(u, v)) - s(v|E_2) + u^T w_2, ..., G_{f_k}(f_k(u, v)) - s(v|E_k)\\ + u^T w_k - \{G_{f_1}(f_1(x, y)) + s(x|D_1) - y^T z_1,\ G_{f_2}(f_2(x, y)) + s(x|D_2) - y^T z_2\\ , ..., G_{f_k}(f_k(x, y)) + s(x|D_k) - y^T z_k\}) \in \text{int}\ K.\end{aligned} \tag{32}$$

For the dual constraint (28) and assumption (iii), we get

$$(\eta_1(x, u) + u)^T \sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u, v)) \nabla_x f_i(u, v) + w_i] \geq 0.$$

Using the dual constraint (29) in the above inequality, we deduce that

$$\eta_1(x, u)^T \sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u, v)) \nabla_x f_i(u, v) + w_i] \geq 0,$$

or equivalently,

$$\sum_{i=1}^{k} \lambda_i [\eta_1(x, u)^T \{G'_{f_i}(f_i(u, v)) \nabla_x f_i(u, v) + w_i\}] \geq 0.$$

Remaining part of proof follows almost similar to the Theorem 1. □

Theorem 5 (Strong duality theorem)**.** *Let* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, ..., \bar{z}_k)$ *be a weak efficient solution of (KGNMPP); fix* $\lambda = \bar{\lambda}$ *in (KGNMDP) and suppose that:*

(i) $\{G'_{f_i}(f_i(\bar{x}, \bar{y})) \nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i\}_{i=1}^{k}$ *is linearly independent,*

(ii) $R^k_+ \subset K.$

Then there exists $\bar{w}_i \in D_i,\ i = 1, 2, ..., k$ *such that* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, ..., \bar{w}_k) \in Y^0$ *and the objective values of (KGNMPP) and (KGNMDP) coincide. Moreover, if the assumptions of Theorem 1 are satisfied for all* *feasible solutions of (KGNMPP) and (KGNMDP), then* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, ..., \bar{w}_k)$ *is a weak efficient solution of (KGNMDP).*

Finally, the following result becomes the sibling result of the last one obtained in the previous section.

Theorem 6 (Converse duality theorem). *Let* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, ..., \bar{w}_k)$ *be a weak efficient solution of (KGNMDP); fix* $\lambda = \bar{\lambda}$ *in (KGNMPP) and suppose that:*

(i) $\left\{G'_{f_i}(f_i(\bar{u}, \bar{v}))\nabla_x f_i(\bar{u}, \bar{v}) - \bar{w}_i\right\}_{i=1}^{k}$ *is linearly independent,*

(ii) $R^k_+ \subset K$.

Then, there exists $\bar{z}_i \in E_i,\ i = 1, 2, ..., k$ *such that* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, ..., \bar{z}_k) \in X^0$ *and the objective values of (KGNMPP) and (KGNMDP) coincide. Moreover, if the assumptions of Theorem 1 are satisfied for all feasible solutions of (KGNMPP) and (KGNMDP), then* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, ..., \bar{z}_k)$ *is a weak efficient solution of (KGNMPP).*

5. Conclusions

By using the notion of K-G_f- pseudo-invex/ strongly $K - G_f$- pseudo-invex functions we have established duality results for (KGMPP) /(KGNMPP)-Mond–Weir dual models applied in multiobjective nondifferentiable symmetric programming problems with objective cone and cone constraints, too. This work may be inspirational for extension to nondifferentiable higher-order symmetric fractional programming.

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