BEYOND THE HYPOTHESIS OF BOUNDEDNESS FOR THE RANDOM COEFFICIENT OF AIRY, HERMITE AND LAGUERRE DIFFERENTIAL EQUATIONS WITH UNCERTAINTIES

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Abstract. In this work, we study the full randomized versions of Airy, Hermite and Laguerre differential equations, which depend on a random variable appearing as an equation coefficient as well as two random initial conditions. In previous contributions, the mean square stochastic solutions to the aforementioned random differential equations were constructed via the Fröbenius method, under the assumption of exponential growth of the absolute moments of the equation coefficient, which is equivalent to its essential boundedness. In this paper we aim at relaxing the boundedness hypothesis to allow more general probability distributions for the equation coefficient. We prove that the equations are solvable in the mean square sense when the equation coefficient has finite moment-generating function in a neighborhood of the origin. A thorough discussion of the new hypotheses is included.

Keywords: random differential equation, second-order linear differential equation, Fröbenius method, mean square calculus, mean fourth calculus.

AMS Classification 2010: 34A30, 34F05, 60H10.

1. Introduction, goals and assumptions

Random differential equations are differential equations with uncertain input coefficients, in the form of random variables or stochastic processes with any type of probability distributions. The references [1, 2] provide good introductions to the theory of random differential equations. These equations must not be confused with stochastic differential equations driven by irregular processes, which are usually studied using a special tool called Itô calculus [3, pp. 96–98].

In the recent years, important efforts have been made in the analysis of random second-order linear differential equations, in order to extend their deterministic counterpart. The main goal has been to construct the rigorous mean square stochastic solution, thus obtaining approximations for its mean and its variance. We refer the reader to [4] for an exposition of mean square and mean fourth calculus. Important random differential equations from Mathematical Physics that have been solved are Airy's equation,

$$\ddot{X}(t) + AtX(t) = 0,\ t \in \mathbb{R}, \quad X(0) = Y_0, \quad \dot{X}(0) = Y_1, \tag{1.1}$$

Hermite's equation,

$$\ddot{X}(t) - 2t\dot{X}(t) + AX(t) = 0,\ t \in \mathbb{R}, \quad X(0) = Y_0, \quad \dot{X}(0) = Y_1, \tag{1.2}$$

and Laguerre's equation,

$$t\ddot{X}(t) + (1-t)\dot{X}(t) + AX(t) = 0,\ t \in \mathbb{R}, \quad X(0) = Y_0. \tag{1.3}$$

It is assumed that the equation coefficient A and the initial conditions Y_0 and Y_1 are random variables on an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space equipped with outcomes $\omega \in \Omega$, $\mathcal{F} \subseteq 2^{\Omega}$ is the σ-algebra of events and $\mathbb{P}$ is the probability measure. For the sake of generality, no statistical independence is assumed between A, Y_0 and Y_1. The term $X(t)$ is a stochastic process, which represents the mean square solution.

Regarding notation, $\mathbb{E}$ denotes the expectation operator and $\|\cdot\|_q$ denotes the q-th Lebesgue norm for random variables, $1 \leq q \leq \infty$. The norm $\|\cdot\|_\infty$ is the essential supremum and $\|\cdot\|_q = (\mathbb{E}[|\cdot|^q])^{1/q}$, $q < \infty$. The cases $q = 2$ and $q = 4$ correspond to mean square and mean fourth calculus, respectively.

In [5–7], the Fröbenius method was utilized to find a random power series solution $X(t) = \sum_{n=0}^{\infty} X_n t^n$, i.e., a mean square analytic solution, to Airy's, Hermite's and Laguerre's equation, respectively. The coefficients X_n are second-order random variables satisfying $\sum_{n=0}^{\infty} \|X_n\|_2 |t|^n < \infty$, for $t \in \mathbb{R}$. Here, we point out that the use of mean square convergence, instead of other strong types of stochastic convergence, is mainly motivated by the fact that it has the following distinctive property:

$$X_n \xrightarrow[n\to\infty]{\|\cdot\|_2} X \Rightarrow \mathbb{E}[X_n] \xrightarrow[n\to\infty]{} \mathbb{E}[X] \text{ and } \mathbb{V}[X_n] \xrightarrow[n\to\infty]{} \mathbb{V}[X].$$

This property plays a key role to construct reliable approximations to the mean and the variance of the solution to random differential equations, whose approximations are obtained after truncating the infinite random series via the Fröbenius method.

A common hypothesis in the works [5–7] was the exponential growth of the absolute moments of A: $\mathbb{E}[|A|^n] \leq \eta \mathcal{H}^n$ for $n \geq 1$, for certain constants $\eta, \mathcal{H} > 0$. This assumption is equivalent to the boundedness of A, $\|A\|_\infty < \infty$, as shown in [8]. Under boundedness of the equation coefficients, in [9, 10] we studied general random second-order linear differential equations, $\ddot{X}(t) + C(t)\dot{X}(t) + D(t)X(t) = 0$, by constructing the mean square convergent power series solution via the Fröbenius method.

The boundedness assumption for A is quite general. Indeed, many standard probability distributions have bounded support: Beta, Triangular, Uniform, Binomial, etc. Moreover, the theorem of existence and uniqueness of mean square solution to general random initial value problems, which is an extension of the classical Picard's theorem with Lipschitz assumption, requires boundedness of the equation coefficients [2, Ch. 5]. By truncating unbounded supports one can use truncated Normal, Gamma and Poisson distributions, for instance (the truncation may be justified by Chebyshev's inequality). However, it would be interesting to prove the existence of mean square solution under a more general assumption than boundedness. We take as main reference here the works [11–13]. Reference [11] studies random first-order linear differential equations, [12] is devoted to a specific random second-order linear differential equation to introduce some random trigonometric functions, and [13] is a doctoral dissertation on the application of the Fröbenius method to solve random differential equations. Following these works, we will assume that Y_0 and Y_1 are fourth-order random variables and that

$$\|A^n\|_4 \leq \eta \mathcal{H}^{n-1}(n-1)!^p \tag{1.4}$$

for $n \geq n_0$, for constants $n_0, \eta, \mathcal{H}, p > 0$. This includes the case of A being bounded, but also the Normal ($p = 1/2$) and the Gamma ($p = 1$) distributions. The assumption (1.4) is a straightforward consequence of polynomial growth for the ratio of moments: $\|A^{n+1}\|_4/\|A^n\|_4 = \mathcal{O}(n^p)$, where the constant corresponding to $\mathcal{O}$ is $\mathcal{H}$, and $\eta = \|A\|_4$.

For Airy's, Hermite's and Laguerre's equation, we will prove that the formal power series solution $X(t) = \sum_{n=0}^{\infty} X_n t^n$ constructed in [5–7] satisfies $\sum_{n=0}^{\infty} \|X_n\|_2 |t|^n < \infty$, for $t \in \mathbb{R}$. This is enough since the random power series can be differentiated termwise in the mean square sense [9, Th. 3.1]. Our reasoning will be more concise and optimized than in such contributions.

In the last part of the paper, we will include a thorough discussion about the new hypotheses that permit extending the results published in [5–7]. We will show the equivalence between condition (1.4) for $p \leq 1$ and $\phi_A(t) < \infty$ in a neighborhood of 0, where $\phi_A(t) = \mathbb{E}[\mathrm{e}^{tA}]$ is the moment-generating function of A. This includes a lot of important probability distributions for A.

2. Random Airy differential equation

The formal solution to (1.1) is given by

$$X(t) = Y_0 X_1(t) + Y_1 X_2(t),$$

$$X_1(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n A^n (3n-2)!!!}{(3n)!} t^{3n}, \quad X_2(t) = t + \sum_{n=1}^{\infty} \frac{(-1)^n A^n (3n-1)!!!}{(3n+1)!} t^{3n+1},$$

see [5]. Essentially, the solution $X(t)$ is a linear combination of the fundamental set $\{X_1(t), X_2(t)\}$, where $X_1(t)$ and $X_2(t)$ solve (1.1) with initial conditions $(X_1(0), \dot{X}_1(0)) = (1, 0)$ and $(X_2(0), \dot{X}_2(0)) = (0, 1)$, respectively.

If Y_0 and Y_1 are fourth-order random variables, we need to prove the mean fourth convergence of $X_1(t)$ and $X_2(t)$, for each $t \in \mathbb{R}$. That is,

$$\sum_{n=1}^{\infty} \frac{\|A^n\|_4 (3n-2)!!!}{(3n)!} |t|^{3n} < \infty, \quad \sum_{n=1}^{\infty} \frac{\|A^n\|_4 (3n-1)!!!}{(3n+1)!} |t|^{3n+1} < \infty.$$

We work with the first series, as the analysis for the second one is analogous. We have, as a direct consequence of (1.4),

$$\sum_{n=1}^{\infty} \frac{\|A^n\|_4 (3n-2)!!!}{(3n)!} |t|^{3n} \leq \eta \sum_{n=1}^{\infty} \frac{\mathcal{H}^{n-1}(n-1)!^p (3n-2)!!!}{(3n)!} |t|^{3n}.$$

We use d'Alembert's ratio test to derive the radius of convergence of the latter power series. We have

$$\lim_{n\to\infty} \frac{\frac{\mathcal{H}^n n!^p (3n+1)!!!}{(3n+3)!} |t|^{3n+3}}{\frac{\mathcal{H}^{n-1}(n-1)!^p (3n-2)!!!}{(3n)!} |t|^{3n}} = \mathcal{H}|t|^3 \lim_{n\to\infty} \frac{n^p (3n+1)}{(3n+3)(3n+2)(3n+1)} = \begin{cases} 0, & 0 \leq p < 2, \\ \frac{\mathcal{H}|t|^3}{9}, & p = 2. \end{cases}$$

Then, for $0 \leq p < 2$, the series converges for all $t \in \mathbb{R}$; while for $p = 2$, it converges for $|t| < \sqrt[3]{9/\mathcal{H}}$.

3. Random Hermite differential equation

The formal solution to (1.2) is given by a linear combination of a fundamental set $\{X_1(t), X_2(t)\}$,

$$X(t) = Y_0 X_1(t) + Y_1 X_2(t), \tag{3.1}$$

$$X_1(t) = 1 + \sum_{n=0}^{\infty} \frac{t^{2n+2}}{(2n+2)!} \prod_{j=0}^{n} (4j - A), \quad X_2(t) = t + \sum_{n=0}^{\infty} \frac{t^{2n+3}}{(2n+3)!} \prod_{j=0}^{n} (4j + 2 - A), \tag{3.2}$$

see [6]. If Y_0 and Y_1 are fourth-order random variables, we prove that

$$\sum_{n=0}^{\infty} \frac{t^{2n+2}}{(2n+2)!} \left\| \prod_{j=0}^{n} (4j - A) \right\|_4 < \infty, \quad \sum_{n=0}^{\infty} \frac{|t|^{2n+3}}{(2n+3)!} \left\| \prod_{j=0}^{n} (4j + 2 - A) \right\|_4 < \infty,$$

for $t \in \mathbb{R}$. We focus on the convergence analysis for the former series, as the analysis for the second one is analogous.

By Hölder's and the triangular inequalities,

$$\left\| \prod_{j=0}^{n} (4j - A) \right\|_4 \leq \prod_{j=0}^{n} \|4j - A\|_{4(n+1)} \leq \prod_{j=0}^{n} \left(4j + \|A\|_{4(n+1)}\right) \leq \left(4n + \|A\|_{4(n+1)}\right)^{n+1}.$$

The last inequality would be tighter if we used the arithmetic-geometric mean inequality, but the final conclusion that we will derive about the radius of convergence is the same. We have, then,

$$\sum_{n=0}^{\infty} \frac{t^{2n+2}}{(2n+2)!} \left\| \prod_{j=0}^{n} (4j - A) \right\|_4 \leq \sum_{n=0}^{\infty} \frac{t^{2n+2}}{(2n+2)!} \left(4n + \|A\|_{4(n+1)}\right)^{n+1}.$$

From (1.4) (just power it to $1/(n+1)$), $\|A\|_{4(n+1)} \leq \eta^{1/(n+1)} \mathcal{H}^{n/(n+1)} n!^{p/(n+1)}$. Then

$$\sum_{n=0}^{\infty} \frac{t^{2n+2}}{(2n+2)!} \left\| \prod_{j=0}^{n} (4j - A) \right\|_4 \leq \sum_{n=0}^{\infty} \frac{t^{2n+2}}{(2n+2)!} \left(4n + \eta^{1/(n+1)} \mathcal{H}^{n/(n+1)} n!^{p/(n+1)}\right)^{n+1}.$$

Now we use the root test:

$$\begin{aligned} &\lim_{n\to\infty} \sqrt[n]{\frac{t^{2n+2}}{(2n+2)!} \left(4n + \eta^{1/(n+1)} \mathcal{H}^{n/(n+1)} n!^{p/(n+1)}\right)^{n+1}} \\ &= \lim_{n\to\infty} \frac{|t|^{2+\frac{2}{n}}}{(2n+2)!^{\frac{1}{n}}} \left(4n + \eta^{1/(n+1)} \mathcal{H}^{n/(n+1)} n!^{p/(n+1)}\right)^{\frac{n+1}{n}} = \begin{cases} 0, \ 0 \leq p < 2, \\ t^2 \frac{\mathcal{H}}{4}, \ p = 2. \end{cases} \end{aligned}$$

As a consequence, for $0 \leq p < 2$ the series converges for all $t \in \mathbb{R}$; for $p = 2$ it converges for $|t| < 2/\sqrt{\mathcal{H}}$.

4. Random Laguerre differential equation

The formal solution to (1.3) is expressed as

$$X(t) = Y_0 \sum_{n=0}^{\infty} t^n \prod_{k=1}^{n} \frac{k-1-A}{k^2}, \tag{4.1}$$

see [7]. In this case, there is only one initial condition because 0 is a regular singular point. If Y_0 is a fourth-order random variable, we prove that

$$\sum_{n=0}^{\infty} |t|^n \left\| \prod_{k=1}^{n} \frac{k-1-A}{k^2} \right\|_4 < \infty.$$

By Hölder's and the triangular inequalities,

$$\begin{aligned} \left\| \prod_{k=1}^{n} \frac{k-1-A}{k^2} \right\|_4 &\leq \prod_{k=1}^{n} \left\| \frac{k-1-A}{k^2} \right\|_{4n} \leq \prod_{k=1}^{n} \frac{k+\|A\|_{4n}}{k^2} \\ &\leq \frac{\left(n+\eta^{1/n}\mathcal{H}^{(n-1)/n}(n-1)!^{p/n}\right)^n}{n!^2}. \end{aligned}$$

At this point, we would like to point out a mistake in the inequality derived for $\| \prod_{k=1}^{n} \frac{k-1-A}{k^2} \|_4$ in [7], as the numerator of the final bound found by the authors should be powered to the n-th. This mistake, however, does not change their conclusions about the radius of convergence.

We use the root test:

$$\begin{aligned} &\lim_{n\to\infty} \sqrt[n]{\frac{\left(n+\eta^{1/n}\mathcal{H}^{(n-1)/n}(n-1)!^{p/n}\right)^n}{n!^2}|t|^n} \\ &= |t| \lim_{n\to\infty} \frac{n+\eta^{1/n}\mathcal{H}^{(n-1)/n}(n-1)!^{p/n}}{n!^{2/n}} = \begin{cases} 0, \ 0 \leq p < 2, \\ |t|\mathcal{H}, \ p = 2. \end{cases} \end{aligned}$$

The series thus converges on $\mathbb{R}$ when $0 \leq p < 2$, and on $(-1/\mathcal{H}, 1/\mathcal{H})$ when $p = 2$.

Remark 4.1. If the random vector (Y_0, Y_1) and A are independent, then we can relax the hypotheses to Y_0 and Y_1 second-order random variables and

$$\|A^n\|_2 \leq \eta \mathcal{H}^{n-1}(n-1)!^p.$$

Indeed, this is because $\|UV\|_2 = \|U\|_2\|V\|_2$ whenever U and V are two independent random variables.

Remark 4.2. In the contributions [5–7], when A is unbounded, its support gets truncated. Suppose that Y_0 and Y_1 are mean fourth integrable and A satisfies (1.4). Consider truncations $A^{(m)} = A\mathbb{1}_{\{|A|\leq a_m\}}$, where $\lim_{m\to\infty} a_m = \infty$. These truncations are bounded and satisfy $\lim_{m\to\infty} \|A^{(m)} - A\|_k = 0$, for all $1 \leq k < \infty$ (by the dominated convergence theorem). Let $X^{(m)}(t)$ be the solution to the equation with coefficient $A^{(m)}$, and let $X(t)$ be the solution when the coefficient is A. These solutions are given by random power series, see the previous sections. We have that $\lim_{m\to\infty} \|X^{(m)}(t) - X(t)\|_2 = 0$, for each $t \in \mathbb{R}$ (this is a consequence of the dominated convergence theorem for series).

5. About the hypotheses

As a consequence of Stirling's formula, $(n-1)! \approx \sqrt{2\pi(n-1)}(\frac{n-1}{\mathrm{e}})^{n-1}$ as $n \to \infty$, hypothesis (1.4) is equivalent to

$$\|A^n\|_4 \leq \gamma \mathcal{C}^{n-1}(n-1)^{p(n-1)} \tag{5.1}$$

for $n \geq n_1$, for constants $n_1, \gamma, \mathcal{C}, p > 0$.

This condition (5.1) might be easier to check in practice. For example, let us prove that the Poisson(λ) distribution for A satisfies (1.4), by taking advantage of the well-known convergence of the Binomial distribution to the Poisson distribution under specific hypotheses on their corresponding parameters. Let $k \geq 1$. Let $V_{s,\alpha} \sim$ Binomial(s, α), where $s \geq k$ is a positive integer and $\alpha \in (0, 1)$. By [15, Prop. 4.7], there is a constant $C > 0$ independent of α, s and k such that

$$\|V_{s,\alpha}\|_k \leq C \frac{k}{\log(\frac{k}{\alpha s})},$$

when $k/(\alpha s) \geq \mathrm{e}$. Let $s \to \infty$, $\alpha \to 0$, $\alpha s \to \lambda$. Then $V_{s,\alpha}$ converges to a Poisson(λ) random variable U, which satisfies

$$\|U\|_k \leq C \frac{k}{\log(\frac{k}{\lambda})}$$

when $k \geq \mathrm{e}\lambda$. In particular, if $A \sim$ Poisson(λ), then

$$\|A^n\|_4 = \|A\|_{4n}^n \leq C^n \frac{(4n)^n}{\log^n(4n/\lambda)} \leq C^n (4n)^n$$

for large n, so (5.1) holds with $p = 1$.

Up to now, we know that (1.4) is fulfilled by the bounded, Normal, Gamma and Poisson distributions, with $p = 0$, $p = 1/2$ and $p = 1$, respectively. We show that condition (1.4) for $p \leq 1$ is equivalent to $\phi_A(t) < \infty$ in a neighborhood of 0, where $\phi_A(t) = \mathbb{E}[\mathrm{e}^{tA}]$ denotes the moment-generating function of A. We use [16, Th. A, p. 5], (a)$\Leftrightarrow$(c): $\phi_A(t) < \infty$ in a neighborhood of 0 if and only if $\mathbb{E}[A^{4k}] \leq C^k(4k)!$ for certain $C > 0$. Now, $C^k(4k)! \sim D^{4k}k^{4k}$, for certain $D > 0$, by Stirling's approximation. Then $\mathbb{E}[A^{4k}] \leq C^k(4k)!$ is equivalent to $\|A^k\|_4 \leq D^k k^k$, which is in turn equivalent to (5.1) with $p = 1$.

Condition (1.4) for $p \leq 2$ and $A \geq 0$ is equivalent to $\phi_{\sqrt{A}}(t) < \infty$ in a neighborhood of 0, by [16, Th. B, p. 6], (a)$\Leftrightarrow$(c).

Example 5.1. Condition (1.4) is not satisfied by the Log-Normal distribution, for any $0 \leq p \leq 2$. If A follows a Log-Normal distribution of parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, then $\mathbb{E}[A^n] = \mathrm{e}^{n\mu + n^2\sigma^2/2}$. In particular, $\|A^n\|_4 = \mathrm{e}^{n\mu + 2n^2\sigma^2}$. This quantity grows faster than $\mathcal{C}^{n-1}(n-1)^{p(n-1)}$ as $n \to \infty$, for any $\mathcal{C}, p > 0$, as the limit of their ratio is infinity. As a conclusion, the random Fröbenius method does not work with the Log-Normal distribution. This issue also occurs with other methods for uncertainty quantification, namely Monte Carlo simulation and polynomial chaos expansions [17]. Due to the large growth of the moments of the Log-Normal distribution and its fat tails, the classical Monte Carlo procedure does not work well with this distribution. On the other hand, polynomial chaos expansions and stochastic Galerkin projections do not converge for the Log-Normal distribution [18].

Example 5.2. Consider Laguerre's equation (1.3) where $A \sim \text{Laplace}(-0.5, 0.5)$ ($\mu = -0.5$ is the location parameter and $b = 0.5$ is the scale parameter) and $Y_0 \sim \text{Uniform}(0.1, 0.2)$. These two random variables are assumed to be independent. The moment-generating function of the Laplace distribution is finite in a neighborhood of zero defined by the interval $(-1/b, 1/b)$. Then, according to the theoretical discussion, there exists a mean square solution $X(t)$ defined by (4.1), $t \in \mathbb{R}$. By taking the N-th partial sum of the random power series, the expectation and the variance of $X(t)$, $\mathbb{E}[X(t)]$ and $\mathbb{V}[X(t)]$, can be approximated as $N \to \infty$. See the formulae from [7, p. 289] (the idea relies on using the linearity of the expectation and the precomputed moments of A). The convergence is exponentially fast with N, but not uniformly in t; as t increases, a larger order N is required. In Tables 1 and 2, we show the approximations of $\mathbb{E}[X(t)]$ and $\mathbb{V}[X(t)]$ for different N's. The results are compared against Monte Carlo simulation with 100,000 realizations. While the Fröbenius method gives correct significant digits very fast, the Monte Carlo procedure shows much slower convergence.

t	$N = 5$	$N = 6$	$N = 7$	$N = 8$	$N = 9$	$N = 10$	MC
0	0.150000	0.150000	0.150000	0.150000	0.150000	0.150000	0.149982
0.2	0.167021	0.167021	0.167021	0.167021	0.167021	0.167021	0.167247
0.4	0.188737	0.188738	0.188738	0.188738	0.188738	0.188738	0.188417
0.6	0.216310	0.216320	0.216321	0.216321	0.216321	0.216321	0.215913
0.8	0.251174	0.251230	0.251238	0.251239	0.251239	0.251239	0.250905
1.0	0.295083	0.295298	0.295333	0.295338	0.295339	0.295339	0.295943
1.2	0.350149	0.350791	0.350917	0.350939	0.350943	0.350943	0.35014
1.4	0.418889	0.420507	0.420879	0.420954	0.420968	0.420970	0.421562

Table 1. Approximations of the expectation of the solution (4.1) to the random Laguerre differential equation (1.3) using the Fröbenius method, for different orders of truncation N and Monte Carlo (MC) simulation. Example 5.2.

t	$N = 5$	$N = 6$	$N = 7$	$N = 8$	$N = 9$	$N = 10$	MC
0	0.000833333	0.000833333	0.000833333	0.000833333	0.000833333	0.000833333	0.000834754
0.2	0.00161217	0.00161217	0.00161217	0.00161217	0.00161217	0.00161217	0.00161509
0.4	0.00428272	0.00428300	0.00428302	0.00428302	0.00428302	0.00428302	0.00427721
0.6	0.0105189	0.0105246	0.0105253	0.0105253	0.0105253	0.0105253	0.0104690
0.8	0.0234742	0.0235277	0.0235358	0.0235368	0.0235370	0.0235370	0.0235577
1.0	0.0489538	0.0492733	0.0493346	0.0493448	0.0493463	0.0493465	0.0486700
1.2	0.0973951	0.0988329	0.0991692	0.0992369	0.0992489	0.0992507	0.0991242
1.4	0.187108	0.192424	0.193900	0.194250	0.194323	0.194000	0.189003

Table 2. Approximations of the variance of the solution (4.1) to the random Laguerre differential equation (1.3) using the Fröbenius method, for different orders of truncation N and Monte Carlo (MC) simulation. Example 5.2.

Example 5.3. Consider Hermite's equation (1.2), with inputs $A \sim \text{Poisson}(2)$ (the parameter 2 is the mean), $Y_0 \sim \text{Uniform}(0.1, 0.2)$ and $Y_1 = -1$, where A and Y_0

are assumed to be independent. The moment-generating function of the Poisson distribution is finite on the whole $\mathbb{R}$, therefore there exists a mean square solution $X(t)$ defined by (3.1)–(3.2), $t \in \mathbb{R}$. By using the formulae from [6, Section 6], based on truncating the power series to the N-th partial sum, the expectation and the variance of $X(t)$, $\mathbb{E}[X(t)]$ and $\mathbb{V}[X(t)]$, can be approximated as $N \to \infty$. In Tables 3 and 4, we show the approximations of $\mathbb{E}[X(t)]$ and $\mathbb{V}[X(t)]$ for different N's, as well as the Monte Carlo estimates with $100,000$ realizations.

t	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=10$	MC
0	0.150000	0.150000	0.150000	0.150000	0.150000	0.150000	0.149969
0.2	-0.0560256	-0.0560256	-0.0560256	-0.0560256	-0.0560256	-0.0560256	-0.0559303
0.4	-0.274510	-0.274510	-0.274510	-0.274510	-0.274510	-0.274510	-0.274442
0.6	-0.507193	-0.507193	-0.507193	-0.507193	-0.507193	-0.507193	-0.507183
0.8	-0.758494	-0.758494	-0.758494	-0.758494	-0.758494	-0.758494	-0.758386
1.0	-1.03813	-1.03814	-1.03814	-1.03814	-1.03814	-1.03814	-1.03799
1.2	-1.36655	-1.3666	-1.36661	-1.36661	-1.36661	-1.36661	-1.36748
1.4	-1.78648	-1.78703	-1.78715	-1.78718	-1.78718	-1.78718	-1.79147

TABLE 3. Approximations of the expectation of the solution (3.1)–(3.2) to the random Hermite differential equation (1.2) using the Fröbenius method, for different orders of truncation N and Monte Carlo (MC) simulation. Example 5.3.

t	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=10$	MC
0	0.000833333	0.000833333	0.000833333	0.000833333	0.000833333	0.000833333	0.000831451
0.2	0.000773883	0.000773883	0.000773883	0.000773883	0.000773883	0.000773883	0.000776030
0.4	0.000596974	0.000596974	0.000596974	0.000596974	0.000596974	0.000596974	0.000597567
0.6	0.000665335	0.000665336	0.000665336	0.000665336	0.000665336	0.000665336	0.000664739
0.8	0.00514877	0.00514884	0.00514884	0.00514884	0.00514884	0.00514884	0.00519014
1.0	0.0343588	, 0.0343611	0.0343612	0.0343613	0.0343613	0.0343613	0.0345285
1.2	0.163114	0.163173	0.163181	0.163182	0.163182	0.163182	0.163717
1.4	0.645706	0.646805	0.647026	0.647070	0.647077	0.647079	0.647087

TABLE 4. Approximations of the variance of the solution (3.1)–(3.2) to the random Hermite differential equation (1.2) using the Fröbenius method, for different orders of truncation N and Monte Carlo (MC) simulation. Example 5.3.

Example 5.4. In this last example, we consider Hermite's equation (1.2), with coefficient $A \sim \text{Weibull}(a, b)$ (a is the scale parameter and b is the shape parameter) and initial conditions $Y_0 \sim \text{Uniform}(0.1, 0.2)$ and $Y_1 = -1$, where A and Y_0 are assumed to be independent. The moments of the Weibull distribution are well-known: $\mathbb{E}[A^m] = a^m\Gamma(1+m/b)$, where Γ is the Gamma function. Using this formula, one can estimate the ratio $\|A^{m+1}\|_2/\|A^m\|_2$ (we use 2-norms instead of 4-norms because of the independence, see Remark 4.1). By Stirling's formula, $\Gamma(x+1) \sim$

$\sqrt{2\pi x}(x/\mathrm{e})^x$ as $x \to \infty$, it is easy to obtain, by direct computations, that

$$\frac{\|A^{m+1}\|_2}{\|A^m\|_2} \sim a \left(\frac{2}{b}\right)^{\frac{1}{b}} m^{\frac{1}{b}}.$$

That is, $p = 1/b$, $\mathcal{H} = a(2/b)^{1/b}$ and $\eta = \|A\|_2 = a\sqrt{\Gamma(1+2/b)}$. We analyze the mean square convergence of the series defined by (3.1)–(3.2). We show the approximations of the variance of $X(t)$, $\mathbb{V}[X(t)]$, for orders of truncation $N = 25, 26, 27$, scale parameter $a = 1$ and shape parameters $b = 2/3, 1/2, 1/3$. Notice that, for $b = 2/3$, we have $p = 3/2 < 2$, so convergence on the whole real line is expected by the theoretical results. For $b = 1/2$, we have that p equals the threshold 2 for convergence: the series defined by (3.1)–(3.2) only converges in a small interval around 0, given by $(-2/\sqrt{\mathcal{H}}, 2/\sqrt{\mathcal{H}}) = (-1/2, 1/2)$. Finally, for $b = 1/3$, we have $p = 3 > 2$, therefore the series given by (3.1)–(3.2) is not expected to converge for any $t \neq 0$. The numerical results are presented in Tables 5, 6 and 7. They agree with our theoretical discussion about the convergence domain. The results of the Monte Carlo simulation using 100,000 realizations are also shown for validation.

t	$N = 25$	$N = 26$	$N = 27$	MC
0	0.000833333	0.000833333	0.000833333	0.000830033
0.2	0.000802005	0.000802005	0.000802005	0.000813572
0.4	0.000687696	0.000687696	0.000687696	0.000669479
0.6	0.00122022	0.00122022	0.00122022	0.0012146
0.8	0.00975692	0.00975692	0.00975692	0.0103762
1.0	0.0559536	0.0559536	0.0559536	0.0541698
1.2	0.233081	0.233081	0.233081	0.233012
1.4	0.830763	0.830763	0.830763	0.827578

Table 5. Approximations of the variance of the solution (3.1)–(3.2) to the random Hermite differential equation (1.2) using the Fröbenius method, for $b = 2/3$, different orders of truncation N and Monte Carlo (MC) simulation. Example 5.4.

t	$N = 25$	$N = 26$	$N = 27$	MC
0	0.000833333	0.000833333	0.000833333	0.000838800
0.2	0.000818187	0.000818187	0.000818187	0.000824624
0.4	0.000691749	0.000691749	0.000691749	0.000690404
0.6	238750	489177	1.00245×10^{6}	0.00361506
0.8	1.05641×10^{18}	6.85769×10^{18}	4.45158×10^{19}	0.0275597
1.0	6.82155×10^{27}	1.08386×10^{29}	1.72173×10^{30}	0.121455

Table 6. Approximations of the variance of the solution (3.1)–(3.2) to the random Hermite differential equation (1.2) using the Fröbenius method, for $b = 1/2$, different orders of truncation N and Monte Carlo (MC) simulation. Example 5.4.

t	$N = 25$	$N = 26$	$N = 27$	MC
0	0.000833333	0.000833333	0.000833333	0.000827802
0.1	3.67222×10^{32}	6.87049×10^{34}	1.38863×10^{37}	0.000927809
0.2	4.95578×10^{62}	1.48136×10^{66}	4.78402×10^{69}	0.00105165
0.3	2.05526×10^{80}	3.11183×10^{84}	5.09011×10^{88}	0.00104901

Table 7. Approximations of the variance of the solution (3.1)–(3.2) to the random Hermite differential equation (1.2) using the Fröbenius method, for $b = 1/3$, different orders of truncation N and Monte Carlo (MC) simulation. Example 5.4.

6. Conclusions and perspectives

In this paper we have studied Airy, Hermite and Laguerre differential equations with random inputs. Using the Fröbenius method, the mean square stochastic solution has been constructed in the form of a random power series. The main goal has been to weaken the usual hypothesis of boundedness for the equation coefficient, so that the class of its probability distributions is enlarged to those having certain growth of the moments. In particular, we include the probability distributions having finite moment-generating function around the origin.

More research on these methods should be carried out in the future. For instance, the solution to the random Legendre differential equation still requires boundedness of the equation coefficient [8, 14]. In the general case of second-order linear differential equations [9, 10], all the equation coefficients are taken as bounded random variables. It would be interesting to generalize the theoretical discussion therein to allow unbounded distributions. This is the aim of our current efforts.

Acknowledgements

This work has been supported by the Spanish Ministerio de Economía y Competitividad grant MTM2017–89664–P. The author Marc Jornet acknowledges the doctorate scholarship granted by Programa de Ayudas de Investigación y Desarrollo (PAID), Universitat Politècnica de València.

Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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