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Additional Information

# Preconditioners for nonsymmetric linear systems with low-rank skew-symmetric part<sup>\*</sup>

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## Abstract

We present a preconditioning technique for solving nonsymmetric linear systems  $Ax = b$ , where the coefficient matrix  $A$  has a skew-symmetric part that can be well approximated with a skew-symmetric low-rank matrix. The method consists of updating a preconditioner obtained from the symmetric part of  $A$ . We present some results concerning to the approximation properties of the preconditioner and the spectral properties of the preconditioning technique. The results of the numerical experiments performed show that our strategy is competitive compared with some specific methods.

*Keywords:* Iterative methods, skew-symmetric matrices, sparse linear systems, preconditioning, low-rank update.

*2000 MSC:* 15B57, 45A05, 65F08, 65F10, 65F50, 65N22

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## 1. Introduction

In this paper we study the iterative solution of nonsingular, nonsymmetric linear systems

$$Ax = b \tag{1}$$

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where the matrix  $A \in \mathbb{R}^{n \times n}$  is large, sparse and its skew-symmetric part has low rank or can be approximated by a skew-symmetric low-rank matrix. Consider  $A = H + K$  where  $H$  and  $K$  are the symmetric and skew-symmetric parts of  $A$ , respectively. It is supposed that the skew-symmetric part can be written as  $K = FCF^T + E$  where  $F \in \mathbb{R}^{n \times s}$  is a full-rank rectangular matrix,  $C \in \mathbb{R}^{s \times s}$  is a nonsingular skew-symmetric matrix with  $s$  even,  $s \ll n$  and  $\|E\| \ll 1$ . Systems like this arise from the discretization of PDEs with certain Neumann boundary conditions, the discretization of integral equations [10] as well as path following methods [1]. In general, any problem whose skew-symmetric part  $K$  has a small number of dominant singular values can be described in this way.

Different strategies have been proposed to solve (1) when the skew-symmetric part  $K$  has exactly rank  $s \ll n$ , i.e.,  $E = O$ . In [1] the authors present a progressive GMRES (PGMRES) method which shows that an orthogonal Krylov subspace basis can be generated with a short recurrence formula. As pointed out in [6], although the method is mathematically equivalent to full GMRES [8], in practice it may suffer from instabilities due to the loss of orthogonality of the generated Krylov subspace basis. In the same paper, the authors propose a Schur complement method (SCM) that also permits the application of short-term formulas. The method obtains an approximate solution by applying the MINRES method  $s + 1$  times. The authors also suggest that it can be applied as a preconditioner for GMRES for the more general case when  $E \neq O$  which is the main problem considered in this paper.

The method proposed is based on the framework presented in [4]. Our approach computes an approximate  $LU$  factorization of the matrix

$$\mathbf{A} = \begin{bmatrix} H + E & F \\ F^T & -C^{-1} \end{bmatrix} \quad (2)$$

which is used as a preconditioner for the linear system (1). This preconditioner can be viewed as a low-rank update of an incomplete LU factorization of the symmetric part  $H$ . Interestingly, the matrix in (2) is similar to the one used in [6] to develop the Schur complement method, but in this work it is used to

update a previously computed preconditioner for the symmetric part  $H$ . Then, the factorization is used as a preconditioner for the (restarted) GMRES and BiCGSTAB [13] methods.

The paper is organized as follows. In Section 2 the proposed preconditioning technique is described. Section 3 is devoted to analyze the approximations properties of the preconditioned matrix. In Section 4 the technique to approximate the skew-symmetric part is described briefly. The results of the numerical experiments for some real and artificial problems are presented in Section 5. Finally, some conclusions are given in Section 6.

## 2. Updated preconditioner method

Our preconditioner  $\mathbf{M}$  is obtained by computing an incomplete LU of the matrix  $\mathbf{A}$  in (2). Assuming that we have calculated an incomplete LU factorization of the symmetric part  $H$ ,  $\hat{H} = L_H D_H L_H^T$ , one has

$$\mathbf{M} = \begin{bmatrix} L_H & 0 \\ F^T L_H^{-T} D_H^{-1} & I \end{bmatrix} \begin{bmatrix} D_H & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} L_H^T & D_H^{-1} L_H^{-1} F \\ 0 & I \end{bmatrix} \quad (3)$$

with  $R = -(C^{-1} + F^T L_H^{-T} D_H^{-1} L_H^{-1} F)$ . The computation of the preconditioner is done in the following steps:

1. Compute incomplete factorization  $L_H D_H L_H^T \approx H$ .
2. Compute block  $T$  by solving  $L_H T = F$ .
3. Compute  $R = -(C^{-1} + T^T D_H^{-1} T)$ .
4. Compute  $L_R U_R = R$ .

Step 2 may involve a sparsification of the matrix  $T$  after its computation to reduce the amount of fill-in introduced. Note that the factorization in step 4 is done exactly when  $s \ll n$ . Otherwise, an incomplete factorization of  $R$  may be necessary to control the amount of fill-in.

The preconditioning step for a Krylov subspace iterative method typically consists of obtaining the preconditioned vector  $\bar{r} = M^{-1}r$ , where  $M^{-1}$  is the preconditioner and  $r$  is the residual vector.  $M^{-1}$  should be a good sparse

approximation of the inverse of the coefficient matrix  $A$ . The preconditioning strategy proposed applies an approximation of the inverse of  $A$  using the relation given by equation (5) below. The approximation of the inverse of  $\mathbf{A}$  is implicitly applied by solving the triangular systems of the LU factorization of  $\mathbf{M}$ , equation (3). Thus, the preconditioning step is done by solving linear systems of the form

$$\mathbf{M} \begin{bmatrix} \bar{r} \\ \bar{r}' \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix},$$

obtaining the preconditioned vector  $\bar{r}$  in three steps:

1. Solve  $L_H D_H r_1 = r$ .
2. Solve  $R \bar{r}' = -F^T L_H^{-T} r_1$ .
3. Solve  $L_H^T \bar{r} = r_1 - D_H^{-1} L_H^{-1} F \bar{r}'$ .

The computation and application of the preconditioner is inexpensive provided that  $s \ll n$ . Note that step 2 implies the solution of a  $s \times s$  linear system which can be done with a direct method. The preconditioner can be viewed as a low-rank update of the incomplete factorization computed for the symmetric part  $H$ . Thus, it will be referred to as updated preconditioner method.

### 3. Approximation properties of the updated preconditioner

In this section we study the approximation properties of the proposed updated preconditioner. We recall that  $A = H + K$  where  $H$  and  $K$  are the symmetric and skew-symmetric parts of  $A$ , respectively, and  $K = FCF^T + E$  where  $F \in \mathbb{R}^{n \times s}$  is a full-rank rectangular matrix,  $C \in \mathbb{R}^{s \times s}$  is a nonsingular skew-symmetric matrix with  $s$  even,  $s \ll n$  and  $\|E\| \ll 1$ . We denote  $H_E = H + E$ .

The proposed preconditioning strategy relies on computing a good approximation of the augmented matrix in equation (2) which is used to accelerate the convergence of a Krylov iterative method. Solving (1) with a preconditioned Krylov method involves the computation of matrix-vector products with  $A$  and

an approximation of its inverse operator  $A^{-1}$  in the preconditioning step. We have the following relations between the linear operators  $A$  and  $\mathbf{A}$ ,

$$A = \begin{bmatrix} I & O \end{bmatrix} \begin{bmatrix} H_E & F \\ F^T & -C^{-1} \end{bmatrix} \begin{bmatrix} I \\ CF^T \end{bmatrix} = \begin{bmatrix} I & O \end{bmatrix} \mathbf{A} \begin{bmatrix} I \\ CF^T \end{bmatrix} \quad (4)$$

and their inverses,

$$A^{-1} = \begin{bmatrix} I & O \end{bmatrix} \begin{bmatrix} H_E & F \\ F^T & -C^{-1} \end{bmatrix}^{-1} \begin{bmatrix} I \\ O \end{bmatrix} = \begin{bmatrix} I & O \end{bmatrix} \mathbf{A}^{-1} \begin{bmatrix} I \\ O \end{bmatrix}, \quad (5)$$

provided that  $H_E$  is nonsingular. Note that if  $H$  is a well conditioned matrix and  $\|E\| \ll 1$ , this condition can be easily satisfied (see Theorem 2.3.4 in [7]).

Next result relates the condition numbers of the matrices  $A$  and  $\mathbf{A}$ .

**Theorem 1.** *Let  $\mathbf{A}$  be the matrix given by equation (2) associated to the linear system (1). Assume that  $FCF^T$  is a reduced unitary diagonalization of the matrix  $K - E$ . Then,*

$$\text{cond}(A) \leq \text{cond}(\mathbf{A}) \sqrt{1 + \sigma_1^2(C)}, \quad (6)$$

where  $\sigma_1(C)$  is the maximum singular value of  $C$ .

*Proof.* Considering the equations (4) and (5), one has

$$\begin{aligned} \text{cond}(A) &= \|A\|_2 \|A^{-1}\|_2 = \left\| \begin{bmatrix} I & O \end{bmatrix} \mathbf{A} \begin{bmatrix} I \\ CF^T \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} I & O \end{bmatrix} \mathbf{A}^{-1} \begin{bmatrix} I \\ O \end{bmatrix} \right\|_2 \\ &\leq \text{cond}(\mathbf{A}) \left\| \begin{bmatrix} I \\ CF^T \end{bmatrix} \right\|_2. \end{aligned}$$

Since  $FCF^T$  is a reduced unitary diagonalization of  $K - E$ , then  $F^T F = I_s$  and  $C \in \mathbb{R}^{s \times s}$  is a block diagonal matrix of the form

$$\begin{bmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{bmatrix}.$$

where  $\lambda_i$  with  $i = 1, \dots, s/2$  are the absolute values of the complex eigenvalues of  $C$ . Under these conditions the nonzero eigenvalues of the matrices  $FC^T CF^T$

and  $C^T C$  are equal and positive since  $C^T C = \text{diag}(\lambda_1^2, \lambda_1^2, \dots, \lambda_{s/2}^2 \lambda_{s/2}^2)$ . Therefore,

$$\left\| \begin{bmatrix} I \\ CF^T \end{bmatrix} \right\|_2^2 = \rho(I + FC^T CF^T) = \rho(I + C^T C) = 1 + \sigma_1^2(C).$$

□

This proposition suggests that one can expect a faster convergence of the iterative method used to solve the linear system (1) if the condition number of the matrix  $\mathbf{A}$  is improved with a proper preconditioner.

To study the quality of the updated preconditioner, first we evaluate the approximation error norm. A comparison with the non-updated preconditioner is also presented. These preconditioners are given by

$$\mathbf{M} = \mathbf{LDU} = \begin{bmatrix} \hat{H} & \hat{F} \\ \hat{F}^T & -C^{-1} \end{bmatrix} \quad \text{and} \quad \mathbf{M}_0 = \begin{bmatrix} \hat{H} & O \\ O & -C^{-1} \end{bmatrix}. \quad (7)$$

The expression for  $\mathbf{M}$  is obtained multiplying the LDU factors in equation (3). Assuming that in step 2 of the computation of the preconditioner a sparsification of the matrix  $T$  has been done, which is denoted by  $\hat{T}$ , one has that the matrix  $F$  is approximated by  $\hat{F} = L_H \hat{T}$ . Moreover, we assume that  $R$  is factorized exactly.

**Theorem 2.** *Let  $\hat{H} = L_H D_H L_H^T$  be an incomplete LDU factorization of  $H$ . Let  $\mathbf{M}$  and  $\mathbf{M}_0$  be the matrices given in (7). Let  $\epsilon = \|\hat{H} - H\|_F^2$ ,  $\delta = \|L_H\|_F^2$ ,  $\gamma = \|E\|_F^2$  and  $c = \|\hat{T} - T\|_F^2$ . Then*

$$\|\mathbf{M} - \mathbf{A}\|_F \leq \sqrt{\epsilon + \gamma + 2\delta c}. \quad (8)$$

Moreover, if  $c \leq \frac{\|F\|_F^2}{\delta}$  then

$$\|\mathbf{M} - \mathbf{A}\|_F \leq \|\mathbf{M}_0 - \mathbf{A}\|_F. \quad (9)$$

*Proof.* From (7) we have

$$\mathbf{M} - \mathbf{A} = \begin{bmatrix} \hat{H} - H_E & \hat{F} - F \\ \hat{F}^T - F^T & O \end{bmatrix} = \begin{bmatrix} \hat{H} - H_E & L_H(\hat{T} - T) \\ (\hat{T} - T)^T L_H^T & O \end{bmatrix}.$$

Then

$$\begin{aligned}
\| \mathbf{M} - \mathbf{A} \|_F^2 &= \| \hat{H} - H_E \|_F^2 + 2 \| L_H(\hat{T} - T) \|_F^2 \\
&\leq \| \hat{H} - H \|_F^2 + \| E \|_F^2 + 2(\| L_H \|_F^2 \| \hat{T} - T \|_F^2) \\
&= \epsilon + \gamma + 2\delta c .
\end{aligned}$$

If  $c \leq \frac{\|F\|_F^2}{\delta}$ , then

$$\| \mathbf{M} - \mathbf{A} \|_F^2 \leq \| \hat{H} - H_E \|_F^2 + 2\delta c \leq \| \hat{H} - H_E \|_F^2 + 2 \| F \|_F^2 = \| \mathbf{M}_0 - \mathbf{A} \|_F^2$$

□

As it could be expected, the above theorem shows that the approximation degree of  $\mathbf{M}$  depends on  $\hat{H}$  and  $\hat{F}$  being a good approximation of  $H$  and  $F$ , respectively, and  $\| E \| \ll 1$ . Moreover, we have proved that if these approximations are good enough, the updated preconditioner  $\mathbf{M}$  is closer to the matrix  $\mathbf{A}$  than the initial one,  $\mathbf{M}_0$ .

**Theorem 3.** *Let the assumptions of Theorem 2 hold, then the preconditioned matrix  $\mathbf{M}^{-1}\mathbf{A}$  can be written as*

$$\mathbf{M}^{-1}\mathbf{A} = \mathbf{I} - \mathbf{M}^{-1}\mathbf{E}_A, \quad (10)$$

where  $\mathbf{E}_A = \mathbf{M} - \mathbf{A}$  satisfies

$$\| \mathbf{M}^{-1}\mathbf{E}_A \|_F \leq \| \mathbf{M}^{-1} \|_F \sqrt{\epsilon + \gamma + 2\delta c} \quad (11)$$

*Proof.* One has

$$\| \mathbf{M}^{-1}\mathbf{E}_A \|_F^2 = \| \mathbf{M}^{-1}(\mathbf{M} - \mathbf{A}) \|_F^2 \leq \| \mathbf{M}^{-1} \|_F^2 \| \mathbf{M} - \mathbf{A} \|_F^2 \leq \| \mathbf{M}^{-1} \|_F^2 (\epsilon + \gamma + 2\delta c)$$

□

**Corollary 4.** *Let the assumptions of Theorem 3 hold. Then, the eigenvalues of the preconditioned matrix  $\mathbf{M}^{-1}\mathbf{A}$  are clustered at 1 in the right half complex plane provided that  $\| \mathbf{M}^{-1} \|_F \sqrt{\epsilon + \gamma + 2\delta c} < 1$ .*



*Proof.* Defining  $\rho = \| \mathbf{M}^{-1} \|_F \sqrt{\epsilon + \gamma + 2\delta c}$ , it immediately follows from the bound (11) and equation (10) that there is a cluster of eigenvalues of  $\mathbf{M}^{-1}\mathbf{A}$  at 1 in the right half complex plane with radius equal to  $\rho < 1$ .  $\square$

Corollary 4 basically means that the quality of the preconditioner depends on the accuracy of the approximations computed for the symmetric and skew-symmetric parts of  $A$ . With a clustered spectrum one can expect a faster convergence of an iterative method although we recall that other aspects may influence the behaviour of Krylov-based iterative methods.

Next, we consider the case in which the symmetric part of  $A$  is indeed positive definite. The following result characterizes the spectrum of  $\mathbf{M}^{-1}\mathbf{A}$ .

**Theorem 5.** *Let  $\mathbf{A}$  and  $\mathbf{M}$  be the matrices given by*

$$\mathbf{A} = \begin{bmatrix} H_E & F \\ F^T & -C^{-1} \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} \hat{H} & \hat{F} \\ \hat{F}^T & -C^{-1} \end{bmatrix}.$$

*Assume that  $H$  is spd,  $F$  and  $\hat{F}$  have full rank  $s$ , and the error matrix  $E_F = F - \hat{F}$  has rank  $p$ ,  $p \leq s$ . Then, the eigenvalues of  $\mathbf{M}^{-1}\mathbf{A}$  are either one or real positive and bounded by*

$$\lambda_{\min}(\hat{H}^{-1}H) \leq \lambda \leq \lambda_{\max}(\hat{H}^{-1}H), \quad (12)$$

*or complex bounded by*

$$|\lambda| \leq 1 + \frac{\|C\|_2 \|E_F\|_2}{\sqrt{1 + \sigma_{\min}^2(\hat{F}C^T)}} \quad (13)$$

*where  $\sigma_{\min}$  represents the smallest singular value.*

*Proof.* The technique to prove the result is standard and similar to the one that can be found in [2]. The eigenvalues and eigenvectors of  $\mathbf{M}^{-1}\mathbf{A}$  are solutions of the following generalized eigenvalue problem  $\mathbf{A}w = \lambda\mathbf{M}w$  written as

$$\begin{bmatrix} H_E & F \\ F^T & -C^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} \hat{H} & F - E_F \\ (F - E_F)^T & -C^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where the eigenvector  $w$  is partitioned according to the block structure of the matrix  $\mathbf{A}$ .

One has equivalently that,

$$\begin{aligned} H_E x + Fy &= \lambda \hat{H}x + \lambda Fy - \lambda E_F y, \\ F^T x &= \lambda F^T x - \lambda E_F^T x + (1 - \lambda)C^{-1}y. \end{aligned} \quad (14)$$

We distinguish the following cases:

1.  $x = 0$ . From the second equation in (14) it follows that  $0 = (1 - \lambda)C^{-1}y$ .

Then  $\lambda = 1$  and therefore  $E_F y = 0$  from the first equation. Since  $y \in \ker E_F$  that has dimension  $s - p$ , we obtain that there are  $s - p$  eigenvectors

$\begin{bmatrix} 0 \\ y \end{bmatrix}$  associated to the unit eigenvalue.

2.  $x \neq 0$ . We consider three cases:

(a)  $F^T x = 0$ . Since  $F$  has rank  $s$  it follows that there are  $n - s$  linearly independent vectors satisfying this condition. From the second equation we have  $\lambda E_F^T x = (1 - \lambda)C^{-1}y$ . Although  $x$  is real, the eigenpair can be complex. Thus, the conjugate transpose is  $\bar{\lambda}x^T E_F = (1 - \bar{\lambda})y^H C^{-T}$ . By multiplying the first equation by  $x^T$  and substituting one has

$$x^T H_E x = \lambda x^T \hat{H}x - \frac{\lambda}{\bar{\lambda}}(1 - \bar{\lambda})y^H C^{-T}y,$$

or equivalently, since  $H_E = H + E$

$$x^T Hx + x^T Ex = \lambda x^T \hat{H}x - \frac{\lambda}{\bar{\lambda}}(1 - \bar{\lambda})y^H C^{-T}y.$$

We recall that  $E$  and  $C$  are skew-symmetric matrices. Therefore, in the equation above the terms  $x^T Ex$  and  $y^H C^{-T}y$  nullify. Then

$$x^T Hx = \lambda x^T \hat{H}x.$$

Since  $H$  and  $\hat{H}$  are spd matrices the eigenpairs are real, and by Courant-Fischer Minimax Theorem (see [7]) it follows that the eigenvalues are bounded by

$$\lambda_{\min}(\hat{H}^{-1}H) \leq \lambda \leq \lambda_{\max}(\hat{H}^{-1}H).$$

- (b)  $F^T x \neq 0$  and  $E_F^T x = 0$ . In this case  $s-p$  linearly independent vectors satisfy these conditions. The second equation reduces to

$$(1 - \lambda)F^T x = (1 - \lambda)C^{-1}y$$

and it is satisfied for eigenvalues equal to 1 or when  $y = CF^T x$ . In this last case, by substituting in the first equation one has

$$H_E x + FCF^T x = \lambda \hat{H} x + \lambda FCF^T x - \lambda E_F CF^T x .$$

Multiplying by  $x^T$  we obtain

$$x^T (H_E + FCF^T) x = \lambda x^T (\hat{H} + FCF^T) x .$$

Since  $FCF^T$  is skew-symmetric, then reasoning similar as in 2.(a) these eigenvalues are bounded by

$$\lambda_{\min}(\hat{H}^{-1}H) \leq \lambda \leq \lambda_{\max}(\hat{H}^{-1}H) .$$

- (c)  $F^T x \neq 0$  and  $E_F^T x \neq 0$ . Multiplying the first equation by  $x^H$  and the second by  $y^H$  one has

$$\begin{aligned} x^H H_E x + x^H F y &= \lambda x^H \hat{H} x + \lambda x^H \hat{F} y , \\ y^H F^T x &= \lambda y^H \hat{F}^T x + (1 - \lambda) y^H C^{-1} y . \end{aligned} \quad (15)$$

Adding both equations we obtain

$$x^H H_E x + 2\text{Re}(x^H F y) - y^H C^{-1} y = \lambda (x^H \hat{H} x + 2\text{Re}(x^H \hat{F} y) - y^H C^{-1} y) .$$

As in case 2.(a), since  $E$  and  $C$  are skew-symmetric matrices, the equation above simplifies to

$$x^H H x + 2\text{Re}(x^H F y) = \lambda (x^H \hat{H} x + 2\text{Re}(x^H \hat{F} y)) . \quad (16)$$

We consider two possibilities in equation (16): if  $x^H \hat{H} x + 2\text{Re}(x^H \hat{F} y) = 0$ , the eigenvalue  $\lambda$  can be complex. In this case from the second equation in (14) one has  $(F^T x - C^{-1}y) = \lambda(\hat{F}^T x - C^{-1}y)$ , equivalent to

$(CF^T x - y) = \lambda(C\hat{F}^T x - y)$ . Note that  $C\hat{F}^T x - y \neq 0$  since we are considering  $E_F^T x \neq 0$ . Then

$$\begin{aligned} |\lambda| &= \frac{\|CF^T x - y\|_2}{\|C\hat{F}^T x - y\|_2} = \frac{\left\| \begin{bmatrix} CF^T & -I \end{bmatrix} w \right\|_2}{\left\| \begin{bmatrix} C\hat{F}^T & -I \end{bmatrix} w \right\|_2} \\ &\leq 1 + \frac{\left\| \begin{bmatrix} CE_F^T & O \end{bmatrix} w \right\|_2}{\left\| \begin{bmatrix} C\hat{F}^T & -I \end{bmatrix} w \right\|_2} \leq 1 + \frac{\|C\|_2 \|E_F\|_2 \|w\|_2}{\left\| \begin{bmatrix} C\hat{F}^T & -I \end{bmatrix} w \right\|_2} \\ &= 1 + \frac{\|C\|_2 \|E_F\|_2}{\left\| \begin{bmatrix} C\hat{F}^T & -I \end{bmatrix} \frac{w}{\|w\|_2} \right\|_2} \leq 1 + \frac{\|C\|_2 \|E_F\|_2}{\sqrt{1 + \sigma_{\min}^2(\hat{F}C^T)}}, \end{aligned}$$

where  $\sigma_{\min}(\hat{F}C^T)$  represents the smallest singular value of a matrix  $\hat{F}C^T$ .

On the other hand, if  $x^H \hat{H}x + 2\text{Re}(x^H \hat{F}y) \neq 0$  then  $\lambda \in \mathbb{R}$ . By subtracting the transpose of the second equation from the first one in (15), we obtain the same equation and the corresponding bound as in 2.(a). Note that  $2p$  is the maximum number of complex eigenvalues. □

To illustrate the bounds deduced in this section we consider the matrix *ADD20* from the University of Florida sparse matrix collection [5]. This matrix has order 2,395 with 13,151 nonzero elements and condition number  $\text{cond}(A) = 1.7637 \times 10^4$ . We approximate its skew-symmetric part with a matrix of rank  $s = 42$ , giving an error matrix with norm  $\|E\|_2 = 9.88 \times 10^{-5}$ . An incomplete Cholesky factorization of  $H$  with dropping parameter equal to  $10^{-4}$  was computed. The matrix  $T$  was also sparsified with a dropping threshold of  $10^{-3}$  with respect to its maximum absolute value. The results were obtained in MATLAB.

First, we studied the bound (6) of Theorem 1. We computed for this matrix  $\text{cond}(\mathbf{A})\sqrt{1 + \sigma_1^2(C)} = 1.0149 \times 10^8$ , that is greater than  $\text{cond}(A)$ , satisfying the bound.

Concerning Theorem 2, the values of the parameters involved in the statement were  $\epsilon = 3.0834 \times 10^{-6}$ ,  $\gamma = 2.1762 \times 10^{-7}$ ,  $\delta = 335.6455$ ,  $c = 2.6701 \times 10^{-9}$

and  $\frac{\|F\|_F^2}{\delta} = 4.8627 \times 10^{-9}$ . The quantities involved in equations (8) and (9) are shown in Table 1 that clearly satisfy the inequalities.

$\  \mathbf{M} - \mathbf{A} \ _F$	$\sqrt{\epsilon + \gamma + 2\delta c}$	$\  \mathbf{M}_0 - \mathbf{A} \ _F$
$1.8 \times 10^{-3}$	$2.3 \times 10^{-3}$	$2.6 \times 10^{-3}$

Table 1: Bounds for Theorem 2

The bound in Theorem 3 is also satisfied since it was obtained 3.0313 and 227.8091 for the left and right side values in inequality (11), respectively.

Finally, with respect Theorem 5, the bounds computed according to the equations (12) and (13) are  $\lambda_{\min}(\hat{H}^{-1}H) = 0.8599$  and  $\lambda_{\max}(\hat{H}^{-1}H) = 1.1311$  for the real eigenvalues, and  $|\lambda| \leq 1.0208$  for the complex ones. These bounds are satisfied since the minimum and maximum real eigenvalues of  $\mathbf{M}^{-1}\mathbf{A}$  are 0.9384 and 1.1309, respectively. Moreover, the norm of the largest complex eigenvalue was 1.0101. Figure 1 illustrates the spectrum of the preconditioned matrix  $\mathbf{M}^{-1}\mathbf{A}$ . It is observed that the eigenvalues are clustered at one in the right half complex plane.

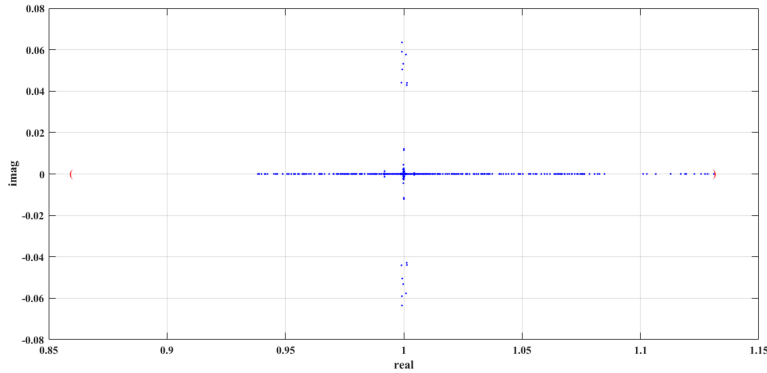


Figure 1: Spectrum of  $\mathbf{M}^{-1}\mathbf{A}$  to illustrate the bounds of Theorem 5. The bounds for the real eigenvalues are indicated with a red parenthesis.

#### 4. Low-rank approximation of the skew-symmetric part

The preconditioner proposed is based on having at disposal a good low-rank approximation of the skew-symmetric part of a matrix. To reach this goal we use the Sparse Column Row approximation (SCR) method presented in [3]. With this method we obtain an approximation of the skew-symmetric part  $K$  of the form  $FCF^T$ , where  $F$  consists of columns of  $K$  and  $C$  is a  $s \times s$  skew-symmetric matrix with  $s$  even. The SCR method is especially suited for computing sparse low-rank approximations. We briefly describe the method.

The SCR method is based on the computation of a SPQR approximation of a given matrix and its transpose. In our case, and since  $K$  is skew-symmetric, it suffices to compute an SPQR approximation of  $K$ . Let

$$KP = \begin{pmatrix} F & F_{\perp} \end{pmatrix} = \begin{pmatrix} Q & Q_{\perp} \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix},$$

where  $P$  is a permutation matrix,  $F$  is the set of  $s$  columns of  $K$  with largest norm and  $R_{11}$  is an  $s \times s$  upper triangular matrix. Then  $KP$  is approximated by  $Q[R_{11} \ R_{12}]$ , with an error  $\epsilon_c = \|R_{22}\|$ . Since  $Q = FR_{11}^{-1}$ , the storage of  $Q$  is not necessary. Defining  $R := R_{11}$ , one gets that the matrix  $C$  which minimizes  $\|K - FCF^T\|$  is  $C = -R^{-1}R^{-T}(F^TKF)R^{-1}R^{-T}$ , see [11], with an approximation error bounded as  $\|K - FCF^T\| \leq \sqrt{2}\epsilon_c$ . Note that  $C$  is skew-symmetric.

#### 5. Numerical Experiments

In this section we compare the updated preconditioner method, referred to as Upd. Prec., with the SCM method used as preconditioner and also an incomplete LU factorization of the symmetric part  $H$ . The iterative methods used are the full GMRES, restarted GMRES(m) and BiCGSTAB. The experiments have been performed with MATLAB. The iterative methods were run until the relative initial residual was reduced to  $10^{-8}$ , allowing a maximum number of 2000 iterations. The incomplete factorization of the symmetric part  $H$  was

computed with MATLAB's function `ilu()` that implements an ILU factorization with threshold [9]. We present the results obtained for different problems that appear in the bibliography and also using some matrices obtained from the University of Florida sparse matrix collection. Concerning the SCM preconditioner, it requires  $s + 1$  applications of MINRES, which could be prohibitive to apply at each iteration of GMRES applied to the preconditioned system. Thus, as the authors suggest in [6], since  $s$  of these applications are needed to solve a linear system with multiple right-hand sides, the solution of this system is computed once and reused at each GMRES iteration.

### 5.1. A class of simple examples

The first example was used in [6] to show the performance of SCM method. Consider the block-diagonal matrix

$$A = \begin{bmatrix} \Lambda_- & & \\ & \Lambda_+ & \\ & & Z \end{bmatrix},$$

where  $\Lambda_- = \text{diag}(\lambda_1, \dots, \lambda_p)$ ,  $\Lambda_+ = \text{diag}(\lambda_{p+1}, \dots, \lambda_{n-s})$  with  $\lambda_1, \dots, \lambda_p$  uniformly spaced in  $[-\beta, -\alpha]$  and  $\lambda_{p+1}, \dots, \lambda_{n-s}$  uniformly spaced in  $[\alpha, \beta]$  for some positive constants  $\alpha < \beta$ ,  $p \ll n$  and  $s$  even such that  $2 \leq s \ll n$ .  $Z = \text{tridiag}(-\gamma, 1, \gamma) \in \mathbb{R}^{s \times s}$  with  $\gamma > 0$ . The matrix  $A$  is indefinite with eigenvalues

- $\lambda_1, \dots, \lambda_p \in [-\beta, -\alpha]$ ,
- $\lambda_{p+1}, \dots, \lambda_{n-s} \in [\alpha, \beta]$ ,
- $s$  complex eigenvalues of  $Z$ .

In this case  $A = H + K$  with

$$H = \begin{bmatrix} \Lambda_- & & \\ & \Lambda_+ & \\ & & I \end{bmatrix}, \quad K = FCF^T = \begin{bmatrix} O & & \\ & O & \\ & & Z - I \end{bmatrix}.$$

For this first problem  $E = O$ , that is, the skew-symmetric part is not approximated.

We study how to solve the system (1) with  $b$  equal to  $1/\sqrt{n}$  in all its components,  $n = 10^5$ ,  $\alpha = 1/8$ ,  $\beta = 1$ ,  $\gamma = 1$ . Figure 2 compares the CPU time of the different methods tested.

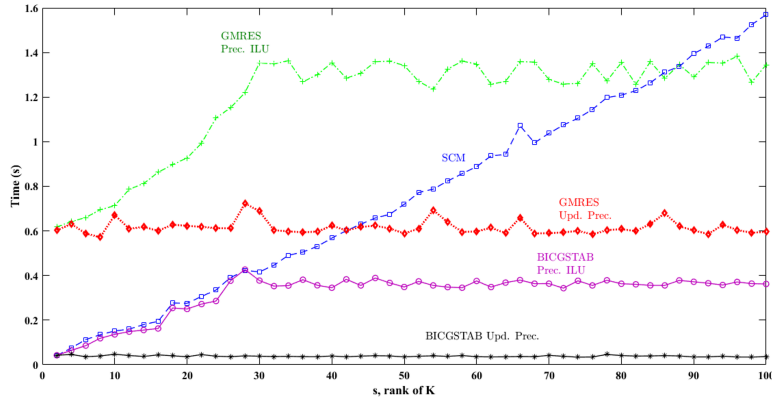


Figure 2: CPU solution time for the first example with the different methods tested for different values of the rank of the skew-symmetric part of  $A$ ,  $s$ .

For all the values of the rank  $s$  it can be observed that using BiCGSTAB preconditioned with the updated preconditioned method performs the best. In the case of full GMRES, it starts to be competitive compared with SCM for values of  $s$  greater than 40. Note that the solution time of the SCM increases linearly with the rank of the skew-symmetric part, while it remains almost constant for the other methods.

In the next example we modify the previous one in order to obtain a class of problems for which the skew-symmetric part of the coefficient matrix is approximated by a low-rank matrix, that is,  $A = H + FCF^T + E$  with  $E \neq O$  and  $\|E\| \ll 1$ . The problem is defined with the following matrices,

$$A = \begin{bmatrix} \Psi & & \\ & \Gamma & \\ & & \Omega \end{bmatrix}, FCF^T = \begin{bmatrix} O & & \\ & O & \\ & & \frac{1}{2}(\Omega - \Omega^T) \end{bmatrix}, E = \begin{bmatrix} O & & \\ & \frac{1}{2}(\Gamma - \Gamma^T) & \\ & & O \end{bmatrix}$$



where  $\Psi$  is of size  $n/2$  from the discretization of the 2D Poisson operator,  $\Gamma = \text{tridiag}(-\gamma, -4, \gamma)$  and  $\Omega = \text{tridiag}(-\omega, -4, \omega)$  are tridiagonal matrices of dimension  $n/2 - s$  and  $s \ll n$ , respectively. We consider  $n = 250000$ ,  $\gamma = 0.01$ ,  $\omega = 10$  and  $s$  an even number with values from 10 to 40 representing the rank of the matrix  $FCF^T$ . For these matrices the error matrix has 2-norm equal to 0.02. Under these conditions the skew-symmetric part of  $A$  has rank equal to  $n^2/2$  and it is approximated by a matrix of rank  $s$ . The matrix  $A$  is indefinite with eigenvalues lying in the intervals  $(0, 8]$  and  $[-4 - 20i, -4 + 20i]$ , which follows from Gerschgorin's theorem [12]. Figure 3 shows the eigenvalue distribution for a matrix generated with  $n = 50$  and  $s = 20$ .

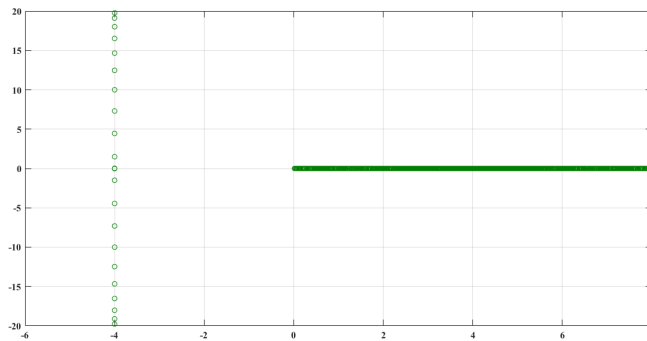


Figure 3: Eigenvalues for the matrix with  $n=50$  and  $s=20$

In Tables 2 and 3, respectively, we present the number of iterations and time needed to solve the system  $Ax = b$  with  $b$  a random vector. The SCM, restarted GMRES(90), BiCGSTAB and GMRES methods were preconditioned with an ILU factorization computed for  $H$  with drop tolerance  $10^{-2}$ , and with the proposed preconditioner. It is observed that Upd. Prec. performs considerably better than the other ones in number of iterations and CPU time.

Iterations				
s	10	20	30	40
GMRES(90) Prec. ILU	228	233	274	398
GMRES(90) Upd. Prec.	99	99	99	99
GMRES(90) SCM	206	206	206	206
BiCGSTAB Prec. ILU	260	664	993	†
BiCGSTAB Upd. Prec.	114	125	113	125

Table 2: Number of iterations for the second problem with different values of  $s$ . A † means no convergence in 2000 iterations.

Time (s)				
s	10	20	30	40
GMRES(90) Prec. ILU	75.3	79.8	101.7	140.0
GMRES(90) Upd. Prec.	36.0	35.8	36.3	36.2
GMRES(90) SCM	73.6	74.2	76.3	77.6
BiCGSTAB Prec. ILU	14.6	37.3	56.4	†
BiCGSTAB Upd. Prec.	6.7	7.3	6.7	7.4

Table 3: CPU time for the second problem with different values of  $s$ . A † means no convergence in 2000 iterations.

## 5.2. The Bratu problem

The next example corresponds to the 2-dimensional Bratu problem. It consists of finding the solution  $u(x, y)$  of the nonlinear boundary problem

$$-\Delta u - \lambda \exp(u) = 0 \text{ in } \Omega, \quad \text{with } u = 0 \text{ on } \partial\Omega \quad (17)$$

depending on the parameter  $\lambda$ ,  $\Delta$  is the Laplacian,  $\Omega$  the unit square and  $\partial\Omega$  its boundary. We discretize this problem using the five-point finite differences as in [1, 6], in a grid of  $500 \times 500$  points. After this, we obtain a system with coefficient matrix of order  $n = 2.5 \times 10^5$  with skew-symmetric part of exactly rank equal

to 2. Table 4 shows the results for the tested methods. The non-preconditioned BiCGSTAB and restarted GMRES(m) methods were also tested.

Method	Time (s)	Iter
GMRES(100)	†	
BiCGSTAB	26.6	827
GMRES(100) Prec. ILU	45.1	123
GMRES(100) Upd. Prec.	46.3	131
BiCGSTAB Prec. ILU	13.1	194
BiCGSTAB Upd. Prec.	11.3	156
SCM	38.2	255

Table 4: CPU solution time and iterations for the Bratu problem

It can be observed that BiGSTAB preconditioning with our technique has the edge over the SCM method and also works better than the ILU preconditioner computed for  $H$ . Compared with the preconditioned GMRES(100), both preconditioners performed similarly.

### 5.3. Problems from the University of Florida sparse matrix collection

Table 5 shows the matrices used in this subsection. These matrices arise from different applications. In this table  $n$  and  $nnz$  indicate the size and number of nonzeros of the matrices, respectively. The rank of the matrix  $FCF^T$  that approximates the skew-symmetric part is indicated with  $s$ , and the norm of the error matrix  $E$  is indicated in the last column.  $\|E\|_2 = 0$  means that the skew-symmetric part has low rank. The full and restarted GMRES methods were used. Tables 6 and 7 show the results of the experiments.

We can observe that for these matrices the Upd. Prec. technique obtains the better results in terms of CPU time. The number of iterations is comparable to the SCM preconditioner, but this method spends more CPU time to obtain the solution because the preconditioner application is more expensive. We remark that, compared with the incomplete LU factorization of  $H$ , when the skew-symmetric part  $K$  is not exactly approximated, as it happens with the matrices

Matrix name	Application	$n$	$nnz$	$s$	$\ E\ _2$
IPROB	Linear programming	3001	9000	4	0
PESA	Directed weighted graph	11738	79566	2	0
BIG	Directed weighted graph	13209	91465	2	0
ASIC_100K	Circuit simulation	99340	940,621	6	0.7091
HCIRCUIT	Circuit simulation	105676	513072	58	0.0472
SCIRCUIT	Circuit simulation	170998	958936	126	0.0026

Table 5: Set of tested matrices from the University of Florida sparse matrix collection

	IPROB	PESA	BIG
	m = 10	m = 200	m = 200
	$\rho$ /Iter/time(s)	$\rho$ /Iter/time(s)	$\rho$ /Iter/time(s)
GMRES Prec. ILU	2.15/22/0.2	1.16/295/3.8	1.15/295/3.8
GMRES Upd. Prec.	2.15/21/0.2	1.16/244/2.8	1.16/270/3.2
GMRES SCM	2.15/27/0.7	1.16/263/3.4	1.15/265/5.4
GMRES(m) Prec. ILU	2.15/30/0.2	1.16/698/5.5	1.15/1928/17.3
GMRES(m) Upd. Prec.	2.15/30/0.1	1.16/398/3.4	1.16/832/7.5
GMRES(m) SCM	2.15/28/0.8	1.16/378/3.8	1.15/761/9.1

Table 6: Results for the matrices IPROB, PESA, BIG

in Table 7, the technique proposed improves considerably the convergence of the restarted GMRES method.

## 6. Conclusions

We have presented a method for preconditioning nonsymmetric matrices whose skew-symmetric can be well approximated by a low-rank matrix. The method can be viewed as an update of a preconditioner computed for the symmetric part of the system matrix. Some approximation properties of the preconditioner and the eigenvalue distribution of the preconditioned matrix have

	ASIC_100K	HCIRCUIT	SCIRCUIT
	m = 20	m = 50	m=200
	$\rho$ /Iter/time(s)	$\rho$ /Iter/time(s)	$\rho$ /Iter/time(s)
GMRES(m) Prec. ILU	0.87/120/2.1	0.86/155/8.4	1.09/1171/148.2
GMRES(m) Upd. Prec.	0.87/38/0.8	0.89/80/1.6	1.12/568/61.4
GMRES(m) SCM	0.87/36/10.1	0.88/76/21.3	1.09/569/149.7

Table 7: Results for the matrices ASIC\_100K, HCIRCUIT and SCIRCUIT

been presented. The method has been compared with others that appear in the literature for this kind of matrices. From the numerical results conducted it has been observed that the proposed preconditioner is competitive in terms of solution time and number of iterations spent.

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