APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 22, no. 1 (2021), 47-65
doi:10.4995/agt.2021.13165

Intermediate rings of complex-valued continuous functions

Amrita Acharyya [a], Sudip Kumar Acharyya [b], Sagarmoy Bag [b] and Joshua Sack [c]

[a] Department of Mathematics and Statistics, University of Toledo, Main Campus, Toledo, OH 43606-3390. (amrita.acharyya@utoledo.edu)
[b] Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata 700019, West Bengal, India (sdpacharyya@gmail.com, sagarmoy.bag01@gmail.com)
[c] Department of Mathematics and Statistics, California State University Long Beach, 1250 Bellflower Blvd, Long Beach, CA 90840, USA (joshua.sack@csulb.edu)

Communicated by A. Tamariz-Mascarúa

Abstract

For a completely regular Hausdorff topological space X, let $C(X,\mathbb{C})$ be the ring of complex-valued continuous functions on X, let $C^*(X,\mathbb{C})$ be its subring of bounded functions, and let $\Sigma(X,\mathbb{C})$ denote the collection of all the rings that lie between $C^*(X,\mathbb{C})$ and $C(X,\mathbb{C})$. We show that there is a natural correlation between the absolutely convex ideals/ prime ideals/maximal ideals/z-ideals/z°-ideals in the rings $P(X,\mathbb{C})$ in $\Sigma(X,\mathbb{C})$ and in their real-valued counterparts $P(X,\mathbb{C})\cap C(X)$. These correlations culminate to the fact that the structure space of any such $P(X,\mathbb{C})$ is βX. For any ideal I in $C(X,\mathbb{C})$, we observe that $C^*(X,\mathbb{C})+I$ is a member of $\Sigma(X,\mathbb{C})$, which is further isomorphic to a ring of the type $C(Y,\mathbb{C})$. Incidentally these are the only C-type intermediate rings in $\Sigma(X,\mathbb{C})$ if and only if X is pseudocompact. We show that for any maximal ideal M in $C(X,\mathbb{C})$, $C(X,\mathbb{C})/M$ is an algebraically closed field, which is furthermore the algebraic closure of $C(X)/M\cap C(X)$. We give a necessary and sufficient condition for the ideal $C_{\mathcal{P}}(X,\mathbb{C})$ of $C(X,\mathbb{C})$, which consists of all those functions whose support lie on an ideal $\mathcal{P}$ of closed sets in X, to be a prime ideal, and we examine a few special cases thereafter. At the end of the article, we find estimates for a few standard parameters concerning the zero-divisor graphs of a $P(X,\mathbb{C})$ in $\Sigma(X,\mathbb{C})$.

2010 MSC: *54C40; 46E25.*

Keywords: *z-ideals; z°-ideals; algebraically closed field; C-type rings; zero divisor graph.*

Received 20 February 2020 – Accepted 06 October 2020

1. Introduction

In what follows, X stands for a completely regular Hausdorff topological space and $C(X,\mathbb{C})$ denotes the ring of all complex-valued continuous functions on X. $C^*(X,\mathbb{C})$ is the subring of $C(X,\mathbb{C})$ containing those functions which are bounded over X. As usual $C(X)$ designates the ring of all real-valued continuous functions on X and $C^*(X)$ consists of those functions in $C(X)$ which are bounded over X. An intermediate ring of real-valued continuous functions on X is a ring that lies between $C^*(X)$ and $C(X)$. Let $\Sigma(X)$ be the aggregate of all such rings. Likewise an intermediate ring of complex-valued continuous functions on X is a ring lying between $C^*(X,\mathbb{C})$ and $C(X,\mathbb{C})$. Let $\Sigma(X,\mathbb{C})$ be the family of all such intermediate rings. It turns out that each member $P(X,\mathbb{C})$ of $\Sigma(X,\mathbb{C})$ is absolutely convex in the sense that $|f| \leq |g|, g \in P(X,\mathbb{C}), f \in C(X,\mathbb{C})$ implies $f \in P(X,\mathbb{C})$. It follows that each such $P(X,\mathbb{C})$ is *conjugate-closed* in the sense that if whenever $f + ig \in P(X,\mathbb{C})$ where $f, g \in C(X)$, then $f - ig \in P(X,\mathbb{C})$. It is realised that there is a natural correlation between the prime ideals/ maximal ideals/ z-ideals/ z°-ideals in the rings $P(X,\mathbb{C})$ and the prime ideals/ maximal ideals/ z-ideals/ z°-ideals in the ring $P(X,\mathbb{C}) \cap C(X)$. In the second and third sections of this article, we examine these correlations in some detail. Incidentally an interconnection between prime ideals in the two rings $C(X,\mathbb{C})$ and $C(X)$ is already observed in Corollary 1.2[7]. As a follow up of our investigations on the ideals in these two rings, we establish that the structure spaces of the two rings $P(X,\mathbb{C})$ and $P(X,\mathbb{C}) \cap C(X)$ are homeomorphic. The structure space of a commutative ring R with unity stands for the set of all maximal ideals of R equipped with the well-known hull-kernel topology. It was established in [21] and [22], independently that the structure space of all the intermediate rings of real-valued continuous functions on X are one and the same viz the Stone-Čech compactification βX of X. It follows therefore that the structure space of each intermediate ring of complex-valued continuous functions on X is also βX. This is one of the main technical results in our article. We like to mention in this context that a special case of this result telling that the structure space of $C(X,\mathbb{C})$ is βX is quite well known, see [19]. We call a ring $P(X,\mathbb{C})$ in the family $\Sigma(X,\mathbb{C})$ a C-type ring if it is isomorphic to a ring of the form $C(Y,\mathbb{C})$ for Tychonoff space Y. We establish that if I is any ideal of $C(X,\mathbb{C})$, then the linear sum $C^*(X,\mathbb{C}) + I$ is a C-type ring. This is the complex analogue of the corresponding result in the intermediate rings of real-valued continuous functions on X as proved in [16]. We further realise that these are the only C-type intermediate rings in the family $\Sigma(X,\mathbb{C})$ when and only when X is pseudocompact i.e. $C(X,\mathbb{C}) = C^*(X,\mathbb{C})$.

It is well-known that if M is a maximal ideal in $C(X)$, then the residue class field $C(X)/M$ is real closed in the sense that every positive element in this field is a square and each odd degree polynomial over this field has a root in the same field [17, Theorem 13.4]. The complex analogue of this result as we realise is that for a maximal ideal M in $C(X,\mathbb{C}), C(X,\mathbb{C})/M$ is an algebraically closed field and furthermore this field is the algebraic closure of $C(X)/M \cap C(X)$.

In section 4 of this article, we deal with a few special problems originating from an ideal $\mathcal{P}$ of closed sets in X and a certain class of ideals in the ring $C(X,\mathbb{C})$. A family $\mathcal{P}$ of closed sets in X is called an ideal of closed sets in X if for any two sets A, B in $\mathcal{P}, A \cup B \in \mathcal{P}$ and for any closed set C contained in A, C is also a member of $\mathcal{P}$. We let $C_{\mathcal{P}}(X,\mathbb{C})$ be the set of all those functions f in $C(X,\mathbb{C})$ whose support $cl_X(X \setminus Z(f))$ is a member of $\mathcal{P}$; here $Z(f) = \{x \in X : f(x) = 0\}$ is the zero set of f in X. We determine a necessary and sufficient condition for $C_{\mathcal{P}}(X,\mathbb{C})$ to become a prime ideal in the ring $C(X,\mathbb{C})$ and examine a few special cases corresponding to some specific choices of the ideal $\mathcal{P}$. The ring $C_\infty(X,\mathbb{C}) = \{f \in C(X,\mathbb{C}) : f$ vanishes at infinity in the sense that for each $n \in \mathbb{N}, \{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact$\}$ is an ideal of $C^*(X,\mathbb{C})$ but not necessarily an ideal of $C(X,\mathbb{C})$. On the assumption that X is locally compact, we determine a necessary and sufficient condition for $C_\infty(X,\mathbb{C})$ to become an ideal of $C(X,\mathbb{C})$.

The fifth section of this article is devoted to finding out the estimates of a few standard parameters concerning zero divisor graphs of a few rings of complex-valued continuous functions on X. Thus for instance we have checked that if $\Gamma(P(X,\mathbb{C}))$ is the zero divisor graph of an intermediate ring $P(X,\mathbb{C})$ belonging to the family $\Sigma(X,\mathbb{C})$, then each cycle of this graph has length 3, 4 or 6 and each edge is an edge of a cycle with length 3 or 4. These are the complex analogues of the corresponding results in the zero divisor graph of $C(X)$ as obtained in [9].

2. Ideals in intermediate rings

Notation: For any subset $A(X)$ of $C(X)$ such that $0 \in A(X)$, we set $[A(X)]_c = \{f + ig : f, g \in A(X)\}$ and call it the *extension* of $A(X)$. Then it is easy to see that $[A(X)]_c \cap C(X) = A(X) = [A(X)]_c \cap A(X)$. From now on, unless otherwise stated, we assume that $A(X)$ is an intermediate ring of real-valued continuous functions on X, i.e. $A(X)$ is a member of the family $\Sigma(X)$. It follows at once that $[A(X)]_c$ is an intermediate ring of complex-valued continuous functions and it is not hard to verify that $[A(X)]_c$ is the smallest intermediate ring in $\Sigma(X,\mathbb{C})$ which contains $A(X)$ and the constant function i. Furthermore $[A(X)]_c$ is conjugate-closed meaning that if $f + ig \in [A(X)]_c$ with $f, g \in A(X)$, then $f - ig \in [A(X)]_c$. The following result tells that intermediate rings in the family $\Sigma(X,\mathbb{C})$ are the extensions of intermediate rings in $\Sigma(X)$.

Theorem 2.1. *Let $P(X,\mathbb{C})$ be an intermediate ring of $C(X,\mathbb{C})$. Then $P(X,\mathbb{C})$ is absolutely convex.*

Proof. Let $|f| \leq |g|$, $f \in C(X,\mathbb{C}), g \in P(X,\mathbb{C})$. Then $f = \frac{f}{1+g^2}(1 + g^2) \in P(X,\mathbb{C})$. Hence $P(X,\mathbb{C})$ is absolutely convex. □

Theorem 2.2. *An intermediate ring $P(X,\mathbb{C})$ of $C(X,\mathbb{C})$ is conjugate closed.*

Proof. Let $f + ig \in P(X, \mathbb{C})$. We have $|f| \leq |f + ig|, |g| \leq |f + ig|$ and $f + ig \in P(X, \mathbb{C})$. Since $P(X, \mathbb{C})$ is absolutely convex, then $f, g \in P(X, \mathbb{C})$. This implies $f, ig \in P(X, \mathbb{C})$ as $i \in P(X, \mathbb{C})$. Thus $f - ig \in P(X, \mathbb{C})$. Hence $P(X, \mathbb{C})$ is conjugate closed. □

Theorem 2.3. *A ring $P(X, \mathbb{C})$ of complex valued continuous functions on X is a member of $\Sigma(X, \mathbb{C})$ if and only if there exists a ring $A(X)$ in the family $\Sigma(X)$ such that $P(X, \mathbb{C}) = [A(X)]_c$.*

Proof. Assume that $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ and let $A(X) = P(X, \mathbb{C}) \cap C(X)$. Then it is clear that $A(X) \in \Sigma(X)$ and $[A(X)]_c \subseteq P(X, \mathbb{C})$.

To prove the reverse containment, let $f + ig \in P(X, \mathbb{C})$. Here $f, g \in C(X)$. Since $P(X, \mathbb{C})$ is conjugate closed, $f - ig \in P(X, \mathbb{C})$, and hence $2f$ and $2ig$ both belong to $P(X, \mathbb{C})$. Since constant functions are bounded and hence in $P(X, \mathbb{C})$, both the constant functions $\frac{1}{2}$ and $\frac{1}{2i}$ are in $P(X, \mathbb{C})$. It follows that both f and g are in $P(X, \mathbb{C}) \cap C(X)$, and hence in $A(X)$. Consequently, $f + ig \in [A(X)]_c$. Thus, $P(X, \mathbb{C}) \subseteq [A(X)]_c$. □

The following facts involving convex sets will be useful. A subset S of $C(X)$ is called *absolutely convex* if whenever $|f| \leq |g|$ with $g \in S$ and $f \in C(X)$, then $f \in S$.

Theorem 2.4. *Let $A(X) \in \Sigma(X)$. Then*

(a) *$A(X)$ is an absolutely convex subring of $C(X)$ (in the sense that if $|f| \leq |g|$ with $g \in A(X)$ and $f \in C(X)$, then $f \in A(X)$)* ([16, Proposition 3.3]).

(b) *A prime ideal P in $A(X)$ is an absolutely convex subset of $A(X)$* ([13, Theorem 2.5]).

The following convenient formula for $[A(X)]_c$ with $A(X) \in \Sigma(X)$ will often be helpful to us.

Theorem 2.5. *For any $A(X) \in \Sigma(X)$, $[A(X)]_c = \{h \in C(X, \mathbb{C}) : |h| \in A(X)\}$.*

Proof. First assume that $h = f + ig \in [A(X)]_c$ with $f, g \in A(X)$. Then $|h| \leq |f| + |g|$. This implies, in view of Theorem 2.4(a), that $h \in A(X)$ and also $|h| \in A(X)$. Conversely, let $h = f + ig \in C(X, \mathbb{C})$ with $f, g \in C(X)$, be such that $|h| \in A(X)$. This means that $(f^2 + g^2)^{\frac{1}{2}} \in A(X)$. Since $|f| \leq (f^2 + g^2)^{\frac{1}{2}}$, this implies in view of Theorem 2.4(a) that $f \in A(X)$. Analogously $g \in A(X)$. Thus $h \in [A(X)]_c$. □

Theorem 2.6. *If I is an ideal in $A(X) \in \Sigma(X)$, then $I_c = \{f + ig : f, g \in I\}$ is the smallest ideal in $[A(X)]_c$ containing I. Furthermore $I_c \cap A(X) = I = I_c \cap C(X)$.*

Proof. It is easy to show that I_c is an ideal in $[A(X)]_c$ containing I. Let K be an ideal of $[A(X)]_c$ containing I. To show $I_c \subseteq K$. Let $f + ig \in K$, where $f, g \in I$. Since $I \subseteq K$, then $f, g \in K$. Now K is an ideal of $[A(X)]_c$, $f, g \in K$

implies $f+ig \in K$. Therefore $I_c \subseteq K$. Hence I_c is the smallest ideal of $[A(X)]_c$ containing I.

Proof of the second part is trivial. $\square$

Theorem 2.7. *If I and J are ideals in $A(X) \in \Sigma(X)$, then $I \subseteq J$ if and only if $I_c \subseteq J_c$. Also $I \subsetneq J$ when and only when $I_c \subsetneq J_c$.*

Proof. If $I \subseteq J$, then clearly $I_c \subseteq J_c$.

Conversely, let $I_c \subseteq J_c$. Let $f \in I$. Since $I \subset I_c$, we have $f \in I_c \subseteq J_c$. Now $f = f + i0$ and $J_c = \{g + ih : g, h \in J\}$. Therefore $f \in J$. Hence $I \subseteq J$.

For the second part we consider $I \subsetneq J$ and $f \in J \setminus I$. Then $f \in J_c \setminus I_c$. Thus $I_c \subsetneq J_c$.

Conversely, let $I_c \subsetneq J_c$ and $f + ig \in J_c \setminus I_c$. Then either f or g is outside I. Let $f \notin I$. Then $f \in J \setminus I$. Hence $I \subsetneq J$. This completes the proof. $\square$

We have the following convenient formula for I_c when I is an absolutely convex ideal of $A(X)$.

Theorem 2.8. *If I is an absolutely convex ideal of $A(X)$ (in particular if I is a prime ideal or a maximal ideal of $A(X)$), then $I_c = \{h \in [A(X)]_c : |h| \in I\}$.*

Proof. Let $h = f + ig \in I_c$. Then $f, g \in I$. Since $|h| \leq |f| + |g|$, the absolute convexity of I implies that $|h| \in I$. Conversely, let $h = f+ig \in [A(X)]_c$ be such that $|h| \in I$. Here $f, g \in A(X)$. Since $|f| \leq (f^2 + g^2)^{\frac{1}{2}} = |h|$, it follows from the absolute convexity of I that $f \in I$. Analogously $g \in I$. Hence $h \in I_c$. $\square$

The above theorem prompts us to define the notion of an absolutely convex ideal in $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ as follows:

Definition 2.9. An ideal J in $P(X, \mathbb{C})$ in $\Sigma(X, \mathbb{C})$ is called absolutely convex if for g, h in $C(X, \mathbb{C})$ with $|g| \leq |h|$ and $h \in J$, it follows that $g \in J$.

The first part of the following proposition is immediate, while the second part follows from Theorem 2.3 and Theorem 2.8.

Theorem 2.10. *Let $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$.*

- (i) *If J is an absolutely convex ideal of $P(X, \mathbb{C})$, then $J \cap C(X)$ is an absolutely convex ideal of the intermediate ring $P(X, \mathbb{C}) \cap C(X) \in \Sigma(X)$.*
- (ii) *An ideal I in $P(X, \mathbb{C}) \cap C(X)$ is absolutely convex in this ring if and only if I_c is an absolutely convex ideal of $P(X, \mathbb{C})$.*
- (iii) *If J is an absolutely convex ideal of $P(X, \mathbb{C})$, then $J = [J \cap C(X)]_c$.*

Proof. (iii) It is trivial that $[J \cap C(X)]_c \subseteq J$. To prove the reverse implication relation let $h = f + ig \in J$, with $f, g \in C(X)$. The absolute convexity of J implies that $|h| \in J$. Consequently $|h| \in J \cap C(X)$. But since $|f| \leq (f^2+g^2)^{\frac{1}{2}} = |h|$, it follows again due to the absolute convexity of $P(X, \mathbb{C})$ as a subring of $C(X, \mathbb{C})$ that $f \in P(X, \mathbb{C})$. We further use absolute convexity of J in $P(X, \mathbb{C})$ to assert that $f \in J$. Analogously $g \in J$. Thus $h = f + ig \in [J \cap C(X)]_c$. Therefore $J \subseteq [J \cap C(X)]_c$. $\square$

Remark 2.11. For any $P(X,\mathbb{C}) \in \Sigma(X,\mathbb{C})$, the assignment $I \mapsto I_c$ provides a one-to-one correspondence between the absolutely convex ideals of $P(X,\mathbb{C}) \cap C(X)$ and those of $P(X,\mathbb{C})$.

The following theorem gives a one-to-one correspondence between the prime ideals of $P(X,\mathbb{C})$ and those of $P(X,\mathbb{C}) \cap C(X)$.

Theorem 2.12. *Let $P(X,\mathbb{C})$ be member of $\Sigma(X,\mathbb{C})$. An ideal J of $P(X,\mathbb{C})$ is prime if and only if there exists a prime ideal Q in $P(X,\mathbb{C}) \cap C(X)$ such that $J = Q_c$.*

Proof. Let J be a prime ideal in $P(X,\mathbb{C})$ and let $Q = J \cap C(X)$ and $A(X) = P(X,\mathbb{C}) \cap C(X)$. Then Q is a prime ideal in the ring $A(X)$. It is easy to see that $Q_c \subseteq J$. To prove the reverse containment, let $h = f + ig \in J$, where $f, g \in P(X,\mathbb{C})$. Note that $P(X,\mathbb{C}) = [A(X)]_c$ by Theorem 2.3. Hence $f, g \in A(X)$ and therefore $f - ig \in P(X,\mathbb{C})$. As J is an ideal of $P(X,\mathbb{C})$, it follows that $(f + ig)(f - ig) \in J$ i.e, $f^2 + g^2 \in J \cap C(X) = Q$. Since Q is a prime ideal in $A(X)$, we can apply Theorem 2.4(b), yielding $f^2 \in Q$ and hence $f \in Q$. Analogously $g \in Q$. Thus $h \in Q_c$. Therefore $J \subseteq Q_c$.

To prove the converse of this theorem, let Q be a prime ideal in $A(X)$. It follows from Theorem 2.8 that $Q_c = \{h \in P(X,\mathbb{C}) : |h| \in Q\}$ and therefore Q_c is a prime ideal in $P(X,\mathbb{C})$. Finally we note that $Q_c \cap C(X) = Q$. □

Remark 2.13. For any $P(X,\mathbb{C}) \in \Sigma(X,\mathbb{C})$, the collection of all prime ideals in $P(X,\mathbb{C})$ is precisely $\{Q_c : Q \text{ is a prime ideal in } P(X,\mathbb{C}) \cap C(X)\}$.

Remark 2.14. The collection of all minimal prime ideals in $P(X,\mathbb{C})$ is precisely $\{Q_c : Q \text{ is a minimal prime ideal in } P(X,\mathbb{C}) \cap C(X)\}$. [This follows from Remark 2.13 and Theorem 2.7].

Theorem 2.15. *For any $P(X,\mathbb{C}) \in \Sigma(X,\mathbb{C})$, the collection of all maximal ideals in $P(X,\mathbb{C})$ is $\{M_c : M$ is a maximal ideal of $P(X,\mathbb{C}) \cap C(X)\}$.*

Proof. Let M be a maximal ideal in $P(X,\mathbb{C}) \cap C(X) = A(X)$. Then by Theorem 2.12, M_c is a prime ideal in $P(X,\mathbb{C})$. Suppose that M_c is not a maximal ideal in $P(X,\mathbb{C})$, then there exists a prime ideal T in $P(X,\mathbb{C})$ such that $M_c \subsetneq T$. By remark 2.11, there exists a prime ideal P in $A(X)$ such that $J = P_c$. So $M_c \subsetneq P_c$. This implies in view of Theorem 2.5 that $M \subsetneq P$, a contradiction to the maximality of M in $A(X)$.

Conversely, let J be a maximal ideal of $P(X,\mathbb{C})$. In particular J is a prime ideal in this ring. By Remark 2.13, $J = Q_c$ for some prime ideal Q in $A(X)$. We claim that Q is a maximal ideal in $A(X)$. Suppose not; then $Q \subsetneq K$ for some proper ideal K in $A(X)$. Then by Theorem 2.7, $Q_c \subsetneq K_c$ and K_c a proper ideal in $P(X,\mathbb{C})$; this contradicts the maximality of $J = Q_c$. □

We next prove analogoues of Remark 2.13 and Theorem 2.15 for two important classes of ideals viz z-ideals and z°-ideals in $P(X,\mathbb{C}) \in \Sigma(X,\mathbb{C})$. These ideals are defined as follows.

Definition 2.16. Let R be a commutative ring with unity. For each $a \in R$, let M_a (respectively P_a) stand for the intersection of all maximal ideals (respectively all minimal prime ideals) which contain a. An ideal I in R is called a z-ideal (respectively z°-ideal) if for each $a \in I, M_a \subseteq I$ (respectively $P_a \subseteq I$).

This notion of z-ideals is consistent with the notion of z-ideal in $C(X)$ (see [17, 4A5]). Since each prime ideal in an intermediate ring $A(X) \in \Sigma(X)$ is absolutely convex (Theorem 2.4(b)), it follows from Theorem 2.10(ii) and Remark 2.13 that each prime ideal in $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ is absolutely convex. In particular each maximal ideal is absolutely convex. Now if I is a z-ideal in $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ and $|f| \leq |g|, g \in I, f \in P(X, \mathbb{C})$, then $M_g \subseteq I$. Let M be a maximal ideal in $P(X, \mathbb{C})$ containing g. It follows due to the absolute convexity of M that $f \in M$. Therefore $f \in M_g \subset I$. Thus each z-ideal in $P(X, \mathbb{C})$ is absolutely convex. Analogously it can be proved that each z°-ideal in $P(X, \mathbb{C})$ is absolutely convex.

The following subsidiary result can be proved using routine arguments.

Lemma 2.17. *For any family $\{I_\alpha : \alpha \in \Lambda\}$ of ideals in an intermediate ring $A(X) \in \Sigma(X)$, $(\bigcap_{\alpha\in\Lambda} I_\alpha)_c = \bigcap_{\alpha\in\Lambda}(I_\alpha)_c$.*

Theorem 2.18. *An ideal J in a ring $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ is a z-ideal in $P(X, \mathbb{C})$ if and only if there exists a z-ideal I in $P(X, \mathbb{C}) \cap C(X)$ such that $J = I_c$.*

Proof. First assume that J is a z-ideal in $P(X, \mathbb{C})$. Let $I = J \cap C(X)$. Since J is absolutely convex, it follows from Theorem 2.10(iii) that $J = I_c$. We show that I is a z-ideal in $P(X, \mathbb{C}) \cap C(X)$. Choose $f \in I$. Suppose $\{M_\alpha : \alpha \in \Lambda\}$ is the set of all maximal ideals in the ring $P(X, \mathbb{C}) \cap C(X)$ which contain f. It follows from Theorem 2.15 that $\{(M_\alpha)_c : \alpha \in \Lambda\}$ is the set of all maximal ideals in $P(X, \mathbb{C})$ containing f. Since $f \in J$ and J is a z-ideal in $P(X, \mathbb{C})$, it follows that $\bigcap_{\alpha\in\Lambda}(M_\alpha)_c \subseteq J$. This implies in the view of Lemma 2.17 that $(\bigcap_{\alpha\in\Lambda} M_\alpha)_c \cap C(X) \subseteq I$, and hence $f \in \bigcap_{\alpha\in\Lambda} M_\alpha \subseteq I$. Thus it is proved that I is a z-ideal in $P(X, \mathbb{C}) \cap C(X)$.

Conversely, let I be a z-ideal in the ring $P(X, \mathbb{C}) \cap C(X)$. We shall prove that I_c is a z-ideal in $P(X, \mathbb{C})$. We recall from Theorem 2.3 that $[P(X, \mathbb{C}) \cap C(X)]_c = P(X, \mathbb{C})$. Choose f from I_c. From Theorem 2.8, it follows that (taking care of the fact that each z-ideal in $P(X, \mathbb{C})$ is absolutely convex) $|f| \in I$. Let $\{N_\beta : \beta \in \Lambda^*\}$ be the set of all maximal ideals in $P(X, \mathbb{C}) \cap C(X)$ which contain the function $|f|$. The hypothesis that I is a z-ideal in $P(X, \mathbb{C}) \cap C(X)$ therefore implies that $\bigcap_{\beta\in\Lambda^*} N_\beta \subseteq I$. This further implies in view of Lemma 2.17 that $\bigcap_{\beta\in\Lambda^*}(N_\beta)_c \subseteq I_c$. Again it follows from Theorem 2.8 that, for any maximal ideal M in $P(X, \mathbb{C}) \cap C(X)$ and any $g \in P(X, \mathbb{C})$, $g \in M_c$ if and only if $|g| \in M$. Thus for any $\beta \in \Lambda^*, |f| \in N_\beta$ if and only if $f \in (N_\beta)_c$. This means that $\{(N_\beta)_c\}_{\beta\in\Lambda^*}$ is the collection of maximal ideals in $P(X, \mathbb{C})$ which contain f, and we have already observed that $f \in \cap_{\beta\in\Lambda^*}(N_\beta)_c \subseteq I_c$. Consequently I_c is a z-ideal in $P(X, \mathbb{C})$. □

If we use the result embodied in Remark 2.14 and take note of the fact that each minimal prime ideal in $P(X,\mathbb{C})$ is absolutely convex and argue as in the proof of Theorem 2.18, we get the following proposition:

Theorem 2.19. *An ideal J in a ring $P(X,\mathbb{C}) \in \Sigma(X,\mathbb{C})$ is a z°-ideal in $P(X,\mathbb{C})$ if and only if there exists a z°-ideal I in $P(X,\mathbb{C}) \cap C(X)$ such that $J = I_c$.*

An ideal J in $P(X,\mathbb{C}) \in \Sigma(X,\mathbb{C})$ is called fixed if $\bigcap_{f\in J} Z(f) \neq \varnothing$. The following proposition is a straightforward consequence of Theorem 2.6.

Theorem 2.20. *An ideal J in a ring $P(X,\mathbb{C}) \in \Sigma(X,\mathbb{C})$ is a fixed ideal in $P(X,\mathbb{C})$ if and only if $J \cap C(X)$ is a fixed ideal in $P(X,\mathbb{C}) \cap C(X)$.*

We recall that a space X is called an *almost P space* if every non-empty G_δ subset of X has non-empty interior. These spaces have been characterized via z-ideals and z°-ideals in the ring $C(X)$ in [8]. We would like to mention that the same class of spaces have witnessed a very recent characterization in terms of fixed maximal ideals in a given intermediate ring $A(X) \in \Sigma(X)$. We reproduce below these two results to make the paper self-contained.

Theorem 2.21 ([8])**.** *X is an almost P space if and only if each maximal ideal in $C(X)$ is a z°-ideal if and only if each z-ideal in $C(X)$ is a z°-ideal.*

Theorem 2.22 ([12])**.** *Let $A(X) \in \Sigma(X)$ be an intermediate ring of real-valued continuous functions on X. Then X is an almost P space if and only if each fixed maximal ideal $M_A^p = \{g \in A(X) : g(p) = 0\}$ of $A(X)$ is a z°-ideal.*

It is further realised in [12] that if X is an almost P space, then the statement of Theorem 2.21 cannot be improved by replacing $C(X)$ by an intermediate ring $A(X)$, different from $C(X)$. Indeed it is shown in [12, Theorem 2.4] that if an intermediate ring $A(X) \neq C(X)$, then there exists a maximal ideal in $A(X)$ (which is incidentally also a z-ideal in $A(X)$), which is not a z°-ideal in $A(X)$.

We record below the complex analogue of the above results.

Theorem 2.23. *X is an almost P space if and only if each maximal ideal of $C(X,\mathbb{C})$ is a z°-ideal if and only if each z-ideal in $C(X,\mathbb{C})$ is a z°-ideal.*

Proof. This follows from combining Theorems 2.15, 2.18, 2.19, and 2.21. □

Theorem 2.24. *Let $P(X,\mathbb{C}) \in \Sigma(X,\mathbb{C})$. Then X is almost P if and only if each fixed maximal ideal $M_P^p = \{g \in P(X,\mathbb{C}) : g(p) = 0\}$ of $P(X,\mathbb{C})$ is a z°-ideal.*

Proof. This follows from combining Theorems 2.15, 2.20, and 2.22. □

Theorem 2.25. *Let X be an almost P space and let $P(X,\mathbb{C})$ be a member of $\Sigma(X,\mathbb{C})$ such that $P(X,\mathbb{C}) \subsetneq C(X,\mathbb{C})$. Then there exists a maximal ideal in $P(X,\mathbb{C})$, which is not a z°-ideal in $P(X,\mathbb{C})$.*

Thus, within the class of almost P-spaces X, $C(X,\mathbb{C})$ is characterized amongst all the intermediate rings $P(X,\mathbb{C})$ of $\Sigma(X,\mathbb{C})$ by the requirement that z-ideals and z°-ideals (equivalently maximal ideals and z°-ideals) in $P(X,\mathbb{C})$ are one and the same.

Proof. This follows from combining Theorems 2.15, 2.18, and 2.19 of this article together with [12, Theorem 2.4]. $\square$

We recall the classical result that X is a P space if and only if $C(X)$ is a von-Neumann regular ring meaning that each prime ideal in $C(X)$ is maximal. Incidentally the following fact was rather recently established:

Theorem 2.26 ([3, 20, 12])**.** *If $A(X) \in \Sigma(X)$ is different from $C(X)$, then $A(X)$ is never a regular ring.*

Theorems 2.12, 2.15, and 2.26 yield in a straight forward manner the following result:

Theorem 2.27. *If $P(X,\mathbb{C}) \in \Sigma(X,\mathbb{C})$ is a proper subring of $C(X,\mathbb{C})$, then $P(X,\mathbb{C})$ is not a von-Neumann regular ring.*

It is well-known that if P is a non maximal prime ideal in $C(X)$ and M is the unique maximal ideal containing P, then the set of all prime ideals in $C(X)$ that lie between P and M makes a Dedekind complete chain containing no fewer than $2^{\aleph_1}$ many members (see [17, Theorem 14.19]). If we use this standard result and combine with Theorems 2.7, 2.12, and 2.15, we obtain the complex-version of this fact:

Theorem 2.28. *Suppose P is a non maximal prime ideal in the ring $C(X,\mathbb{C})$. Then there exists a unique maximal ideal M containing P in this ring. Furthermore, the collection of all prime ideals that are situated between P and M constitutes a Dedekind complete chain containing at least 2^{α_1} many members.*

Thus for all practical purposes (say for example when X is not a P space), $C(X,\mathbb{C})$ is far from being a Noetherian ring. Incidentally we shall decide the Noetherianness condition of $C(X,\mathbb{C})$ by deducing it from a result in Section 4; in particular, we show that $C(X,\mathbb{C})$ is Noetherian if and only if X is a finite set.

3. Structure spaces of intermediate rings

We need to recall a few technicalities associated with the hull-kernel topology on the set of all maximal ideals $\mathcal{M}(A)$ of a commutative ring A with unity. If we set for any element a of A, $\mathcal{M}(A)_a = \{M \in \mathcal{M}(A) : a \in M\}$, then the family $\{\mathcal{M}(A)_a : a \in A\}$ constitutes a base for closed sets of the hull-kernel topology on $\mathcal{M}(A)$. We may write $\mathcal{M}_a$ for $\mathcal{M}(A)_a$ when context is clear. The set $\mathcal{M}(A)$ equipped with this hull-kernel topology is called the *structure space* of the ring A.

For any subset $\mathcal{M}_\circ$ of $\mathcal{M}(A)$, its closure $\overline{\mathcal{M}_\circ}$ in this topology is given by: $\overline{\mathcal{M}_\circ} = \{M \in \mathcal{M}(A) : M \supseteq \bigcap \mathcal{M}_\circ\}$. For further information on this topology, see [17, 7M].

Following the terminology of [14], by a (Hausdorff) compactification of a Tychonoff space X we mean a pair $(\alpha, \alpha X)$, where αX is a compact Hausdorff space and $\alpha : X \to \alpha X$ a topological embedding with $\alpha(X)$ dense in αX. For simplicity, we often designate such a pair by the notation αX. Two compactifications αX and γX of X are called *topologically equivalent* if there exists a homeomorphism $\psi : \alpha X \to \gamma X$ with the property $\psi \circ \alpha = \gamma$. A compactification αX of X is said to possess the *extension property* if given a compact Hausdorff space Y and a continuous map $f : X \to Y$, there exists a continuous map $f^\alpha : \alpha X \to Y$ with the property $f^\alpha \circ \alpha = f$. It is well known that the Stone-Čech compactification βX of X or more formally the pair $(e, \beta X)$, where e is the evaluation map on X induced by $C^*(X)$ defined by the formula: $e(x) = (f(x) : f \in C^*(X))$ such that $e : X \mapsto \mathbb{R}^{C^*(X)}$, enjoys the extension property. Furthermore this extension property characterizes βX amongst all the compactifications of X in the sense that whenever a compactification αX of X has extension property, it is topologically equivalent to βX. For more information on these topic, see [14, Chapter 1].

The structure space $\mathcal{M}(A(X))$ of an arbitrary intermediate ring $A(X) \in \Sigma(X)$ has been proved to be homeomorphic to βX, independently by the authors in [21] and [22]. Nevertheless we offer yet another independent technique to establish a modified version of the same fact by using the above terminology of [14].

Theorem 3.1. *Let $\eta_A : X \to \mathcal{M}(A(X))$ be the map defined by $\eta_A(x) = M_A^x = \{g \in A(X) : g(x) = 0\}$ (a fixed maximal ideal in $A(X)$). Then the pair $(\eta_A, \mathcal{M}(A(X)))$ is a (Hausdorff) compactification of X, which further satisfies the extension property. Hence the pair $(\eta_A, \mathcal{M}(A(X)))$ is topologically equivalent to the Stone-Čech compactification βX of X.*

Proof. Since X is Tychonoff, η_A is one-to-one. Also $cl_{\mathcal{M}(A(X))}\eta_A(X) = \{M \in \mathcal{M}(A(X)) : M \supseteq \bigcap_{x \in X} M_A^x\} = \{M \in \mathcal{M}(A(X)) : M \supseteq \{0\}\} = \mathcal{M}(A(X))$. It follows from a result proved in Theorem 3.3 and Theorem 3.4 [23] that $\mathcal{M}(A(X))$ is a compact Hausdorff space and η_A is an embedding. Thus $(\eta_A, \mathcal{M}(A(X)))$ is a compactification of X. To prove that this compactification of X possesses the extension property we take a compact Hausdorff space Y and a continuous map $f : X \to Y$. It suffices to define a continuous map $f^{\beta_A} : \mathcal{M}(A(X)) \to Y$ with the property that $f^{\beta_A} \circ \eta_A = f$. Let M be any member of $\mathcal{M}(A(X))$ i.e. M is a maximal ideal of the ring $A(X)$. Define $\hat{M} = \{g \in C(Y) : g \circ f \in M\}$. Note that if $g \in C(Y)$ then $g \circ f \in C(X)$. Further note that since Y is compact and $g \in C(Y)$, g is bounded i.e. $g(Y)$ is a bounded subset of $\mathbb{R}$. It follows that $(g \circ f)(X)$ is a bounded subset of $\mathbb{R}$ and hence $g \circ f \in C^*(X)$. Consequently $g \circ f \in A(X)$. Thus the definition of $\hat{M}$ is without any ambiguity. It is easy to see that $\hat{M}$ is an ideal of $C(Y)$. It follows, since M is a maximal ideal and therefore a prime ideal of $A(X)$, that

$\hat{M}$ is a prime ideal of $C(Y)$. Since $C(Y)$ is a Gelfand ring, $\hat{M}$ can be extended to a unique maximal ideal N in $C(Y)$. Since Y is compact, N is fixed (see [17, Theorem 4.11]). Thus we can write: $N = N_y = \{g \in C(Y) : g(y) = 0\}$ for some $y \in Y$. We observe that $y \in \bigcap_{g\in\hat{M}} Z(g)$. Indeed $\bigcap_{g\in\hat{M}} Z(g) = \{y\}$ for if $y_1, y_2 \in \bigcap_{g\in\hat{M}} Z(g)$, for $y_1 \neq y_2$, then $\hat{M} \subseteq N_{y_1}$ and $\hat{M} \subseteq N_{y_2}$ which is impossible as $N_{y_1} \neq N_{y_2}$ and $C(Y)$ is a Gelfand ring. We then set $f^{\beta_A}(M) = y$. Note that $\{f^{\beta_A}(M)\} = \bigcap_{g\in\hat{M}} Z(g)$. Thus $f^{\beta_A} : \mathcal{M}(A(X)) \to Y$ is a well defined map. Now choose $x \in X$ and then $g \in \hat{M}^x_A$; then $g \circ f \in M^x_A$, which implies that $(g \circ f)(x) = 0$. Consequently $f(x) \in Z(g)$ for each $g \in \hat{M}^x_A$. On the other hand $\{f^{\beta_A}(M^x_A)\} = \bigcap_{g\in\hat{M}^x_A} Z(g)$. This implies that $f^{\beta_A}(M^x_A) = f(x)$; in other words $(f^{\beta_A} \circ \eta_A)(x) = f(x)$ and this relation is true for each $x \in X$. Hence $f^{\beta_A} \circ \eta_A = f$.

Now towards the proof of the continuity of the map f^{β_A}, choose $M \in \mathcal{M}(A(X))$ and a neighbourhood W of $f^{\beta_A}(M)$ in the space Y. In a Tychonoff space every neighbourhood of a point x contains a zero set neighbourhood of x, which contains, a co-zero set neighbourhood of x. So there exist some $g_1, g_2 \in C(Y)$, such that $f^{\beta_A}(M) \in Y \setminus Z(g_1) \subseteq Z(g_2) \subseteq W$. It follows that $g_1 g_2 = 0$ as $Z(g_1) \cup Z(g_2) = Y$ which means that $Z(g_1 g_2) = Y$. Furthermore $f^{\beta_A}(M) \notin Z(g_1)$. Since $\{f^{\beta_A}(M)\} = \bigcap_{g\in\hat{M}} Z(g)$, as observed earlier, we then have $g_1 \notin \hat{M}$. This means that $g_1 \circ f \notin M$. In other words $M \in \mathcal{M}(A(X)) \setminus \mathcal{M}_{g_1\circ f}$, which is an open neighbourhood of M in $\mathcal{M}(A(X))$. We shall check that $f^{\beta_A}(\mathcal{M}(A(X)) \setminus \mathcal{M}_{g_1\circ f}) \subseteq W$ and that settles the continuity of f^{β_A} at M. Towards that end, choose a maximal ideal $N \in \mathcal{M}(A(X)) \setminus \mathcal{M}_{g_1\circ f}$. This means that $N \notin \mathcal{M}_{g_1\circ f}$, i.e. $g_1 \circ f \notin N$. Thus $g_1 \notin \hat{N}$. But as $g_1 g_2 = 0$ and $\hat{N}$ is prime ideal in $C(Y)$, it must be that $g_2 \in \hat{N}$. Since $\{f^{\beta_A}(N)\} = \bigcap_{g\in\hat{N}} Z(g)$, it follows that $f^{\beta_A}(N) \in Z(g_2) \subseteq W$. □

To achieve the complex analogue of the above mentioned theorem, we need to prove the following proposition, which is by itself a result of independent interest.

Theorem 3.2. *Let $A(X) \in \Sigma(X)$. Then the map $\psi_A : \mathcal{M}([A(X)]_c) \to \mathcal{M}(A(X))$ mapping $M \to M \cap A(X)$ is a homeomorphism from the structure space of $[A(X)]_c$ onto the structure space of $A(X)$.*

Proof. That the above map ψ_A is a bijection between the structure spaces of $[A(X)]_c$ and $A(X)$ follows from Theorems 2.3, 2.6, 2.7, and 2.15. Recall (same notation as before) that $\mathcal{M}([A(X)]_c)_f$ is the set of maximal ideals in the ring $[A(X)]_c$ containing the function $f \in [A(X)]_c$. A typical basic closed set in the structure space $\mathcal{M}([A(X)]_c)$ is given by $\mathcal{M}([A(X)]_c)_h$ where $h \in [A(X)]_c$. Note that $\mathcal{M}([A(X)]_c)_h = \{J \in \mathcal{M}([A(X)]_c) : h \in J\}$. So for $h \in [A(X)]_c$, $J \in \mathcal{M}([A(X)]_c)_h$ if and only if $h \in J$, and this is true in view of Theorem 2.8 and the absolute convexity of maximal ideals (see Theorem 2.4(b) of the present article) if and only if $|h| \in J \cap A(X)$, and this holds when and only when $J \cap A(X) \in \mathcal{M}(A(X))_{|h|}$, which is a basic closed set in the structure space

$\mathcal{M}(A(X))$ of the ring $A(X)$. Thus

$$\psi_A[\mathcal{M}([A(X)]_c)_h] = \mathcal{M}(A(X))_{|h|} \tag{3.1}$$

Therefore ψ_A carries a basic closed set in the domain space onto a basic closed set in the range space. Now for a maximal ideal N in $A(X)$ and a function $g \in A(X), g$ belongs to N if and only if $|g| \in N$, because of the absolutely convexity of a maximal ideal in an intermediate ring. Consequently $\mathcal{M}(A(X))_g = \mathcal{M}(A(X))_{|g|}$ for any $g \in A(X)$. Hence from relation (3.1), we get: $\psi_A[\mathcal{M}([A(X)]_c)_g] = \mathcal{M}(A(X))_g$ which implies that $\psi_A^{-1}[\mathcal{M}(A(X))_g] = \mathcal{M}([A(X)]_c)_g$. Thus ψ_A^{-1} carries a basic closed set in the structure space $\mathcal{M}(A(X))$ onto a basic closed in the structure space $\mathcal{M}([A(X)]_c)$. Altogether ψ_A becomes a homeomorphism. □

For any $x \in X$ and $A(X) \in \Sigma(X)$, set $M^x_{A[C]} = \{h \in [A(X)]_c : h(x) = 0\}$. It is easy to check by using standard arguments, such as those employed to prove the textbook theorem [17, Theorem 4.1], that $M^x_{A[C]}$ is a fixed maximal in $[A(X)]_c$ and $M^x_{A[C]} \cap A(X) = M^x_A = \{g \in A(X) : g(x) = 0\}$. Let $\zeta_A : X \mapsto \mathcal{M}([A(X)]_c)$ be the map defined by: $\zeta_A(x) = M^x_{A[C]}$. Then we have the following results.

Theorem 3.3. *$(\zeta_A, \mathcal{M}([A(X)]_c))$ is a Hausdorff compactification of X. Furthermore $(\psi_A \circ \zeta_A)(x) = \eta_A(x)$ for all x in X. Hence $(\zeta_A, \mathcal{M}([A(X)]_c))$ is topologically equivalent to the Hausdorff compactification $(\eta_A, \mathcal{M}(A(X)))$ as considered in Theorem 3.1. Consequently $(\zeta_A, \mathcal{M}([A(X)]_c))$ turns out to be topologically equivalent to the Stone-Čech compactification βX of X.*

Proof. Since $\mathcal{M}(A(X))$ is Hausdorff [23], it follows from Theorem 3.2 that $\mathcal{M}([A(X)]_c)$ is a Hausdorff space. Now by following closely the arguments made at the very beginning of the proof of Theorem 3.1, one can easily see that $(\zeta_A, \mathcal{M}([A(X)]_c))$ is a Hausdorff compactification of X. The second part of the theorem is already realised in Theorem 3.2. The third part of the present theorem also follows from Theorem 3.2. □

Definition 3.4. An intermediate ring $A(X) \in \Sigma(X)$ is called C-type in [16], if it is isomorphic to $C(Y)$ for some Tychonoff space Y.

In [16], the authors have shown that if I is an ideal of the ring $C(X)$, then the linear sum $C^*(X)+I$ is a C-type ring and of course $C^*(X)+I \in \Sigma(X)$. Recently the authors in [1] have realised that these are the only C-type intermediate rings of real-valued continuous functions on X if and only if X is pseudocompact. We now show that the complex analogues of all these results are also true. We reproduce the following result established in [15], which will be needed for this purpose.

Theorem 3.5. *A ring $A(X) \in \Sigma(X)$ is C-type if and only if $A(X)$ is isomorphic to the ring $C(\upsilon_A X)$, where $\upsilon_A X = \{p \in \beta X : f^*(p) \in \mathbb{R}$ for each $f \in A(X)\}$ and $f^* : \beta X \mapsto \mathbb{R} \cup \{\infty\}$ is the Stone extension of the function f.*

We extend the notion of C-type ring to rings of complex-valued continuous functions: a ring $P(X,\mathbb{C}) \in \Sigma(X,\mathbb{C})$ is a *C-type ring* if it is isomorphic to a ring $C(Y,\mathbb{C})$ for some Tychonoff space Y.

Theorem 3.6. *Suppose $A(X) \in \Sigma(X)$ is a C-type intermediate ring of real-valued continuous functions on X. Then $[A(X)]_c$ is a C-type intermediate ring of complex-valued continuous functions on X.*

Proof. Since $A(X)$ is a C-type intermediate ring by Theorem 3.5, there exists an isomorphism $\psi : A(X) \mapsto C(\upsilon_A X)$. Let $\hat{\psi} : [A(X)]_c \mapsto C(\upsilon_A X,\mathbb{C})$ be defined as follows: $\hat{\psi}(f+ig) = \psi(f) + i\psi(g)$, where $f, g \in A(X)$. It is not hard to check that $\hat{\psi}$ is an isomorphism from $[A(X)]_c$ onto $C(\upsilon_A X,\mathbb{C})$. □

Theorem 3.7. *Let I be a z-ideal in $C(X,\mathbb{C})$. Then $C^*(X,\mathbb{C}) + I$ is a C-type intermediate ring of complex-valued continuous functions on X. Furthermore these are the only C-type rings lying between $C^*(X,\mathbb{C})$ and $C(X,\mathbb{C})$ if and only if X is pseudocompact.*

Proof. As mentioned above, it is proved in [16] that for any ideal J in $C(X)$, $C^*(X) + J$ is a C-type intermediate ring of real-valued continuous functions on X. In light of this and Theorem 3.6, it is sufficient to prove for the first part of this theorem that $C^*(X,\mathbb{C}) + I = [C^*(X) + I \cap C(X)]_c$. Towards proving that, let $f, g \in C^*(X) + I \cap C(X)$. We can write $g = g_1 + g_2$ where $g_1 \in C^*(X)$ and $g_2 \in I \cap C(X)$. It follows that $ig_1 \in C^*(X,\mathbb{C})$ and $ig_2 \in I$ and this implies that $i(g_1 + g_2) \in C^*(X,\mathbb{C}) + I$. Thus $f + ig \in C^*(X) + I$. Hence $[C^*(X) + I \cap C(X)]_c \subseteq C^*(X,\mathbb{C}) + I$. To prove the reverse inclusion relation, let $h_1 + h_2 \in C^*(X,\mathbb{C}) + I$, where $h_1 \in C^*(X,\mathbb{C})$ and $h_2 \in I$. We can write $h_1 = f_1 + ig_1, h_2 = f_2 + ig_2$, where $f_1, f_2, g_1, g_2 \in C(X)$. Since $h_1 \in C^*(X,\mathbb{C})$, it follows that $f_1, g_1 \in C^*(X)$. Thus $|f_2| \leq |h_2|$ and $h_2 \in I$. This implies, because of the absolute convexity of the z-ideal I in $C(X,\mathbb{C})$, that $f_2 \in I$. Analogously $g_2 \in I$. It is now clear that $f_1 + f_2 \in C^*(X) + I \cap C(X)$ and $g_1 + g_2 \in C^*(X) + I \cap C(X)$. Thus $h_1 + h_2 = (f_1 + f_2) + i(g_1 + g_2) \in [C^*(X) + I \cap C(X)]_c$. Hence $C^*(X,\mathbb{C}) + I \subseteq [C^*(X) + I \cap C(X)]_c$.

To prove the second part of the theorem, we first observe that if X is pseudocompact, then there is practically nothing to prove. Assume therefore that X is not pseudocompact. Hence by [1], there exists an $A(X) \in \Sigma(X)$ such that $A(X)$ is a C-type ring but $A(X) \neq C^*(X) + J$ for any ideal J in $C(X)$. It follows from Theorem 3.6 that $[A(X)]_c$ is a C-type intermediate ring of complex-valued continuous functions belonging to the family $\Sigma(X,\mathbb{C})$. We assert that there does not exist any z-ideal I in $C(X,\mathbb{C})$ with the relation: $C^*(X,\mathbb{C}) + I = [A(X)]_C$ and that finishes the present theorem. Suppose towards a contradiction, there exists a z-ideal I in $C(X,\mathbb{C})$ such that $C^*(X,\mathbb{C}) + I = [A(X)]_C$. Now from the proof of the first part of this theorem, we have already settled that $C^*(X,\mathbb{C}) + I = [C^*(X) + I \cap C(X)]_C$. Consequently $[C^*(X) + I \cap C(X)]_C = [A(X)]_C$ which yields $[C^*(X) + I \cap C(X)]_C \cap C(X) = [A(X)]_C \cap C(X)$, and hence $C^*(X) + I \cap C(X) = A(X)$, a contradiction. □

We shall conclude this section after incorporating a purely algebraic result pertaining to the residue class field of $C(X, \mathbb{C})$ modulo a maximal ideal in the same field.

For each $a = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n$ if $\mathcal{P}_1 a, \mathcal{P}_2 a, \dots, \mathcal{P}_n a$ are the zeroes of the polynomial $P_a(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$, ordered so that $|\mathcal{P}_1 a| \leq |\mathcal{P}_2 a| \leq \cdots \leq |\mathcal{P}_n a|$, then by following closely the arguments of [17, 13.3(a)], the following result can be obtained.

Theorem 3.8. *For each k, the function $\mathcal{P}_k : \mathbb{C}^n \mapsto \mathbb{C}$, described above, is continuous.*

By employing the main argument of [17, Theorem 13.4], we obtain the following proposition as a consequence of Theorem 3.8.

Theorem 3.9. *For any maximal ideal N in $C(X, \mathbb{C})$, the residue class field $C(X, \mathbb{C})/N$ is algebraically closed.*

We recall from Theorem 2.15 that the assignment $M \mapsto M_c$ establishes a one-to-one correspondence between maximal ideals in $C(X)$ and those in $C(X, \mathbb{C})$. Let $\phi : C(X)/M \mapsto C(X, \mathbb{C})/M_c$ be the induced assignment between the corresponding residue class fields, explicitly $\phi(f + M) = f + M_c$ for each $f \in C(X)$. It is easy to check that ϕ is a ring homomorphism and is one-to-one because if $f + M_c = g + M_c$ with $f, g \in C(X)$, then $f - g \in M_c \cap C(X) = M$ and hence $f + M = g + M$. Furthermore, if we choose an element $f + ig + M_c$ from $C(X, \mathbb{C})/M_c$, with $f, g \in C(X)$, then one can verify easily that it is a root of the polynomial $\lambda^2 - 2(f + M_c)\lambda + (f^2 + g^2 + M_c)$ over the field $\phi(C(X)/M)$. Identifying $C(X)/M$ with $\phi(C(X)/M)$, and taking note of Theorem 3.9 we get the following result.

Theorem 3.10. *For any maximal ideal M in $C(X)$, the residue class field $C(X, \mathbb{C})/M_c$ is the algebraic closure of $C(X)/M$.*

4. Ideals of the form $C_{\mathcal{P}}(X, \mathbb{C})$ and $C_\infty^{\mathcal{P}}(X, \mathbb{C})$

Let $\mathcal{P}$ be an ideal of closed sets in X. We set $C_{\mathcal{P}}(X, \mathbb{C}) = \{f \in C(X, \mathbb{C}) : cl_X(X \setminus Z(f)) \in \mathcal{P}\}$ and $C_\infty^{\mathcal{P}}(X, \mathbb{C}) = \{f \in C(X, \mathbb{C}) : \text{for each } \epsilon > 0 \text{ in } \mathbb{R}, \{x \in X : |f(x)| \geq \epsilon\} \in \mathcal{P}\}$. These are the complex analogues of the rings, $C_{\mathcal{P}}(X) = \{f \in C(X) : cl_X(X \setminus Z(f)) \in \mathcal{P}\}$ and $C_\infty^{\mathcal{P}}(X) = \{f \in C(X) : \text{for each } \epsilon > 0, \{x \in X : |f(x)| \geq \epsilon\} \in \mathcal{P}\}$ already introduced in [4] and investigated subsequently in [5], [12]. As in the real case, it is easy to check that $C_{\mathcal{P}}(X, \mathbb{C})$ is a z-ideal in $C(X, \mathbb{C})$ with $C_\infty^{\mathcal{P}}(X, \mathbb{C})$ just a subring of $C(X, \mathbb{C})$. Plainly we have: $C_{\mathcal{P}}(X, \mathbb{C}) \cap C(X) = C_{\mathcal{P}}(X)$ and $C_\infty^{\mathcal{P}}(X, \mathbb{C}) \cap C(X) = C_\infty^{\mathcal{P}}(X)$.

The following results need only routine verifications.

Theorem 4.1. *For any ideal $\mathcal{P}$ of closed sets in X, $[C_{\mathcal{P}}(X)]_c = \{f + ig : f, g \in C_{\mathcal{P}}(X)\} = C_{\mathcal{P}}(X, \mathbb{C})$ and $[C_\infty^{\mathcal{P}}(X)]_c = C_\infty^{\mathcal{P}}(X, \mathbb{C})$.*

Theorem 4.2.

a) *If I is an ideal of the ring $C_{\mathcal{P}}(X)$, then $I_c = \{f + ig : f, g \in I\}$ is an ideal of $C_{\mathcal{P}}(X, \mathbb{C})$ and $I_c \cap C_{\mathcal{P}}(X) = I$.*

b) *If I is an ideal of the ring $C_\infty^{\mathcal{P}}(X)$, then I_c is an ideal of $C_\infty^{\mathcal{P}}(X, \mathbb{C})$ and $I_c \cap C_\infty^{\mathcal{P}}(X) = I$.*

We record below the following consequence of the above theorem.

Theorem 4.3. *If $I_1 \subsetneq I_2 \subsetneq ...$ is a strictly ascending sequence of ideals in $C_{\mathcal{P}}(X)$(respectively $C_\infty^{\mathcal{P}}(X)$), then $I_{1c} \subsetneq I_{2c} \subsetneq \cdots$ becomes a strictly ascending sequence of ideals in $C_{\mathcal{P}}(X, \mathbb{C})$(respectively $C_\infty^{\mathcal{P}}(X, \mathbb{C})$).*

The analogous results for a strictly descending sequence of ideals in both the rings $C_{\mathcal{P}}(X)$ and $C_\infty^{\mathcal{P}}(X)$ are also valid.

Definition 4.4. A space X is called *locally* $\mathcal{P}$ if each point of X has an open neighbourhood W such that $cl_X W \in \mathcal{P}$.

Observe that if $\mathcal{P}$ is the ideal of all compact sets in X, then X is locally $\mathcal{P}$ if and only if X is locally compact.

Towards finding a condition for which $C_{\mathcal{P}}(X, \mathbb{C})$ and $C_\infty^{\mathcal{P}}(X, \mathbb{C})$ are Noetherian ring/Artinian rings, we reproduce a special version of a fact proved in [6]:

Theorem 4.5 (from [6, Theorem 1.1])**.** *Let $\mathcal{P}$ be an ideal of closed sets in X and suppose X is locally $\mathcal{P}$. Then the following statements are equivalent:*

1) *$C_{\mathcal{P}}(X)$ is a Noetherian ring.*
2) *$C_{\mathcal{P}}(X)$ is an Artinian ring.*
3) *$C_\infty^{\mathcal{P}}(X)$ is a Noetherian ring.*
4) *$C_\infty^{\mathcal{P}}(X)$ is an Artinian ring.*
5) *X is finite set.*

We also note the following standard result of Algebra.

Theorem 4.6. *Let $\{R_1, R_2, ..., R_n\}$ be a finite family of commutative rings with identity. The ideals of the direct product $R_1 \times R_2 \times \cdots \times R_n$ are exactly of the form $I_1 \times I_2 \times \cdots \times I_n$, where for $k = 1, 2, \ldots, n$, I_k is an ideal of R_k.*

Now if X is a finite set, with say n elements, then as it is Tychonoff, it is discrete space. Furthermore if X is locally $\mathcal{P}$, then clearly $\mathcal{P}$ is the power set of X. Consequently $C_{\mathcal{P}}(X, \mathbb{C}) = C_\infty^{\mathcal{P}}(X, \mathbb{C}) = C(X, \mathbb{C}) = \mathbb{C}^n$, which is equal to the direct product of $\mathbb{C}$ with itself 'n' times. Since $\mathbb{C}$ is a field, it has just 2 ideals, hence by Theorem 4.6 there are exactly 2^n many ideals in the ring $\mathbb{C}^n$. Hence $C_{\mathcal{P}}(X, \mathbb{C})$ and $C_\infty^{\mathcal{P}}(X, \mathbb{C})$ are both Noetherian rings and Artinian rings. On the other hand if X is an infinite space and is locally $\mathcal{P}$ space then it follows from the Theorem 4.3 and Theorem 4.5 that neither of the two rings $C_{\mathcal{P}}(X, \mathbb{C})$ and $C_\infty^{\mathcal{P}}(X, \mathbb{C})$ is either Noetherian or Artinian. This leads to the following proposition as the complex analogue of Theorem 4.5.

Theorem 4.7. *Let $\mathcal{P}$ be an ideal of closed sets in X and suppose X is locally $\mathcal{P}$. Then the following statements are equivalent:*

1) *$C_{\mathcal{P}}(X, \mathbb{C})$ is a Noetherian ring.*
2) *$C_{\mathcal{P}}(X, \mathbb{C})$ is an Artinian ring.*

3) *$C_\infty^{\mathcal{P}}(X, \mathbb{C})$ is a Noetherian ring.*
4) *$C_\infty^{\mathcal{P}}(X, \mathbb{C})$ is an Artinian ring.*
5) *X is finite set.*

A special case of this theorem, choosing $\mathcal{P}$ to be the ideal of all closed sets in X reads: $C(X, \mathbb{C})$ is a Noetherian ring if and only if X is finite set.

The following gives a necessary and sufficient condition for the ideal $C_{\mathcal{P}}(X, \mathbb{C})$ in $C(X, \mathbb{C})$ to be prime.

Theorem 4.8. *Let $\mathcal{P}$ be an ideal of closed sets in X and suppose X is locally $\mathcal{P}$. Then the following statements are equivalent:*

(1) *$C_{\mathcal{P}}(X, \mathbb{C})$ is a prime ideal in $C(X, \mathbb{C})$.*
(2) *$C_{\mathcal{P}}(X)$ is a prime ideal in $C(X)$.*
(3) *$X \notin \mathcal{P}$ and for any two disjoint co-zero sets in X, one has its closure lying in $\mathcal{P}$.*

Proof. The equivalence of (1) and (2) follows from Theorem 2.12 and Theorem 4.1. Towards the equivalence (2) and (3), assume that $C_{\mathcal{P}}(X)$ is a prime ideal in $C(X)$. If $X \in \mathcal{P}$, then for each $f \in C(X)$, $cl_X(X \setminus Z(f)) \in \mathcal{P}$ meaning that $f \in C_{\mathcal{P}}(X)$ and hence $C_{\mathcal{P}}(X) = C(X)$, a contradiction to the assumption that $C_{\mathcal{P}}(X)$ is a prime ideal and in particular a proper ideal of $C(X)$. Thus $X \notin \mathcal{P}$. Now consider two disjoint co-zero sets $X \setminus Z(f)$ and $X \setminus Z(g)$ in X, with $f, g \in C(X)$. It follows that $Z(f) \cup Z(g) = X$, i.e. $fg = 0$. Since $C_{\mathcal{P}}(X)$ is prime, this implies that $f \in C_{\mathcal{P}}(X)$ or $g \in C_{\mathcal{P}}(X)$, i.e. $cl_X(X \setminus Z(f)) \in \mathcal{P}$ or $cl_X(X \setminus Z(g)) \in \mathcal{P}$.

Conversely let the statement (3) be true. Since a z-ideal I in $C(X)$ is prime if and only if for each $f, g \in C(X)$, $fg = 0$ implies $f \in I$ or $g \in I$ (see [17, Theorem 2.9]) and since $C_{\mathcal{P}}(X)$ is a z-ideal in $C(X)$, it is sufficient to show that for each $f, g \in C(X)$, if $fg = 0$ then $f \in C_{\mathcal{P}}(X)$ or $g \in C_{\mathcal{P}}(X)$. Indeed $fg = 0$ implies that $X \setminus Z(f)$ and $X \setminus Z(g)$ are disjoint co-zero sets in X. Hence by supposition (3), either $cl_X(X \setminus Z(f))\mathcal{P}$ or $cl_X(X \setminus Z(g)) \in \mathcal{P}$ meaning that $f \in C_{\mathcal{P}}(X)$ or $g \in C_{\mathcal{P}}(X)$. ☐

A special case of Theorem 4.8, with $\mathcal{P}$ equal to the ideal of all compact sets in X, is proved in [10]. We examine a second special case of Theorem 4.8.

A subset Y of X is called a *bounded* subset of X if each $f \in C(X)$ is bounded on Y. Let β denote the family of all closed bounded subsets of X. Then β is an ideal of closed sets in X. It is plain that a pseudocompact subset of X is bounded but a bounded subset of X may not be pseudocompact. Here is a counterexample: the open interval $(0, 1)$ in $\mathbb{R}$ is a bounded subset of $\mathbb{R}$ without being a pseudocompact subset of $\mathbb{R}$. However for a certain class of subsets of X, the two notions of boundedness and pseudocompactness coincide. The following well-known proposition substantiates this fact:

Theorem 4.9 (Mandelkar [18])**.** *A support of X, i.e. a subset of X of the form $cl_X(X \setminus Z(f))$ for some $f \in C(X)$, is a bounded subset of X if and only if it is a pseudocompact subset of X.*

It is clear that the conclusion of Theorem 4.9 remains unchanged if we replace $C(X)$ by $C(X, \mathbb{C})$.

Let $C_\psi(X) = \{f \in C(X) : f$ has pseudocompact support$\}$ and recall that $C_\beta(X) = \{f \in C(X) : f$ has bounded support$\}$. We would like to mention here that the closed pseudocompact subsets of a pseudocompact space X might not constitute an ideal of closed sets in X. Indeed a closed subset of a pseudocompact space may not be pseucdocompact. The celebrated example of a Tychonoff plank in [17, 8.20]: $[0, \omega_1] \times [0, \omega] \setminus \{(\omega_1, \omega)\}$, where ω_1 is the 1st uncountable ordinal and ω is the first infinite ordinal, demonstrates this fact. Nevertheless $C_\psi(X)$ is an ideal of the ring $C(X)$. Indeed it follows directly from Theorem 4.9 that $C_\psi(X) = C_\beta(X)$.

A Tychonoff space X is called *locally pseudocompact* if each point on X has an open neighbourhood with its closure pseudocompact. On the other hand, X is called *locally bounded* (or *locally* β) if each point in X has an open neighbourhood with its closure bounded. Since each open neighbourhoad of a point x in a Tychonoff space X contains a co-zero set neighbourhood of x, it follows from Theorem 4.9 that X is locally bounded if and only if X is locally pseudocompact. This combined with Theorem 2.12 leads to the following special case of Theorem 4.8.

Theorem 4.10. *Let X be locally pseudocompact. Then the following statements are equivalent:*

(1) *$C_\psi(X)$ is a prime ideal of $C(X)$.*
(2) *$C_\psi(X, \mathbb{C}) = \{f \in C(X, \mathbb{C}) : f$ has pseudocompact support$\}$ is a prime ideal of $C(X, \mathbb{C})$.*
(3) *X is not pseudocompact and for any two disjoint co-zero sets in X, the closure of one of them is pseudocompact.*

Since for $f \in C(X, \mathbb{C})$, $f \in C_\infty(X, \mathbb{C})$ if and only if $|f| \in C_\infty(X)$, it follows that $C_\infty(X, \mathbb{C})$ is an ideal of $C(X, \mathbb{C})$ if and only if $C_\infty(X)$ is an ideal of $C(X)$. In general however $C_\infty(X)$ need not be an ideal of $C(X)$. If X is assumed to be locally compact, then it is proved in [2] and [11] that $C_\infty(X)$ is an ideal of $C(X)$ when and only when X is pseudocompact. Therefore the following theorem holds.

Theorem 4.11. *Let X be locally compact. Then the following three statements are equivalent:*

1) *$C_\infty(X, \mathbb{C})$ is an ideal of $C(X, \mathbb{C})$.*
2) *$C_\infty(X)$ is an ideal of $C(X)$.*
3) *X is pseudocompact.*

5. Zero divisor graphs of rings in the family $\Sigma(X, \mathbb{C})$

We fix any intermediate ring $P(X, \mathbb{C})$ in the family $\Sigma(X, \mathbb{C})$. Suppose $\mathcal{G} = \mathcal{G}(P(X, \mathbb{C}))$ designates the graph whose vertices are zero divisors of $P(X, \mathbb{C})$ and there is an edge between vertices f and g if and only if $fg = 0$. For any two vertices f, g in $\mathcal{G}$, let $d(f, g)$ be the length of the shortest path between f and

g and $\operatorname{Diam}\mathcal{G} = \sup\{d(f,g) : f,g \in \mathcal{G}\}$. Suppose $\operatorname{Gr}\mathcal{G}$ designates the length of the shortest cycle in $\mathcal{G}$, often called the girth of $\mathcal{G}$. It is easy to check that a vertex f in $\mathcal{G}$ is a divisor of zero in $P(X,\mathbb{C})$ if and only if $Int_X Z(f) \neq \varnothing$. This parallels the statement that a vertex f in the zero-divisor graph $\Gamma C(X)$ of $C(X)$ considered in [9] is a divisor of zero in $C(X)$ if and only if $Int_X Z(f) \neq \varnothing$. We would like to point out in this connection that a close scrutiny into the proof of various results in [9] reveal that several facts related to the nature of the vertices and the length of the cycles related to $\Gamma C(X)$ have been established in [9] by employing skillfully the last mentioned simple characterization of divisors of zero in $C(X)$. It is expected that the anlogous facts pertaining to the various parameters of the graph $\mathcal{G}(P(X,\mathbb{C})) = \mathcal{G}$ should also hold. We therefore just record the following results related to the graph $\mathcal{G}$, without any proof.

Theorem 5.1. *Let f, g be vertices of the graph $\mathcal{G}$. Then $d(f,g) = 1$ if and only if $Z(f) \cup Z(g) = X$; $d(f,g) = 2$ if and only if $Z(f) \cup Z(g) \subsetneq X$ and $Int_X Z(f) \cap Int_X Z(g) \neq \phi$; $d(f,g) = 3$ if and only if $Z(f) \cup Z(g) \subsetneq X$ and $Int_X Z(f) \cap Int_X Z(g) = \varnothing$. Consequently on assuming that X contains at least 3 points,* $\operatorname{Diam}\mathcal{G}$ *and* $\operatorname{Gr}\mathcal{G}$ *are both equal to* 3 (*compare with* [9, Corollary 1.3]).

Theorem 5.2. *Each cycle in $\mathcal{G}$ has length $3, 4$ or 6. Furthermore every edge of $\mathcal{G}$ is an edge of a cycle with length 3 or 4* (*compare with* [9, Corollary 2.3]).

Theorem 5.3. *Suppose X contains at least 2 points. Then*

a) *Each vertex of $\mathcal{G}$ is a 4 cycle vertex.*
b) *$\mathcal{G}$ is a triangulated graph meaning that each vertex of $\mathcal{G}$ is a vertex of a triangle if and only if X is devoid of any isolated point.*
c) *$\mathcal{G}$ is a hypertriangulated graph in the sense that each edge of $\mathcal{G}$ is edge of a triangle if and only if X is a connected middle P space* (*compare with the analogous facts in* [9, Proposition 2.1]).

ACKNOWLEDGEMENTS. *The authors wish to thank the referee for his/her remarks which improved the paper.*

References

[1] S. K. Acharyya, S. Bag, G. Bhunia and P. Rooj, Some new results on functions in C(X) having their support on ideals of closed sets, Quest. Math. 42 (2019), 1017–1090.

[2] S. K. Acharyya and S. K. Ghosh, On spaces X determined by the rings $C_k(X)$ and $C_\infty(X)$, J. Pure Math. 20 (2003), 9–16.

[3] S. K. Acharyya and B. Bose, A correspondence between ideals and z-filters for certain rings of continuous functions-some remarks, Topology Appl. 160 (2013), 1603–1605.

[4] S. K. Acharyya and S. K. Ghosh, Functions in $C(X)$ with support lying on a class of subsets of X, Topology Proc. 35 (2010), 127–148.

[5] S. K. Acharyya and S. K. Ghosh, A note on functions in $C(X)$ with support lying on an ideal of closed subsets of X, Topology Proc. 40 (2012), 297–301

[6] S. K. Acharyya, K. C. Chattopadhyay and P. Rooj, A generalized version of the rings $C_K(X)$ and $C_\infty(X)$-an enquery about when they become Noetheri, Appl. Gen. Topol. 16, no. 1 (2015), 81–87.
[7] N. L. Alling, An application of valuation theory to rings of continuous real and complex-valued functions, Trans. Amer. Math. Soc. 109 (1963), 492–508.
[8] F. Azarpanah, O. A. S. Karamzadeh and A. R. Aliabad, On Z°-ideal in $C(X)$, Fundamenta Mathematicae 160 (1999), 15–25.
[9] F. Azarpanah and M. Motamedi, Zero-divisor graph of $C(X)$, Acta Math. Hungar. 108, no. 1-2 (2005), 25–36.
[10] F. Azarpanah, Algebraic properties of some compact spaces. Real Anal. Exchange 25, no. 1 (1999/00), 317–327.
[11] F. Azarpanah and T. Soundararajan, When the family of functions vanishing at infinity is an ideal of $C(X)$, Rocky Mountain J. Math. 31, no. 4 (2001), 1133–1140.
[12] S. Bag, S. Acharyya and D. Mandal, A class of ideals in intermediate rings of continuous functions, Appl. Gen. Topol. 20, no. 1 (2019), 109–117.
[13] L. H. Byum and S. Watson, Prime and maximal ideals in subrings of $C(X)$, Topology Appl. 40 (1991), 45–62.
[14] R. E. Chandler, Hausdorff Compactifications, New York: M. Dekker, 1976.
[15] D. De and S. K. Acharyya, Characterization of function rings between $C^*(X)$ and $C(X)$, Kyungpook Math. J. 46, no. 4 (2006) , 503–507.
[16] J. M. Domínguez, J. Gómez and M.A. Mulero, Intermediate algebras between $C^*(X)$ and $C(X)$ as rings of fractions of $C^*(X)$, Topology Appl. 77 (1997), 115–130.
[17] L. Gillman and M. Jerison, Rings of Continuous Functions, New York: Van Nostrand Reinhold Co., 1960.
[18] M. Mandelkar, Supports of continuous functions, Trans. Amer. Math. Soc. 156 (1971), 73–83.
[19] W. Wm. McGovern and R. Raphael, Considering semi-clean rings of continuous functions, Topology Appl. 190 (2015), 99–108.
[20] W. Murray, J. Sack and S. Watson, P-space and intermediate rings of continuous functions, Rocky Mountain J. Math. 47 (2017), 2757–2775.
[21] D. Plank, On a class of subalgebras of $C(X)$ with applications to $\beta X \setminus X$, Fund. Math. 64 (1969), 41–54.
[22] L. Redlin and S. Watson, Maximal ideals in subalgebras of $C(X)$, Proc. Amer. Math. Soc. 100, no. 4 (1987), 763–766.
[23] L. Redlin and S. Watson, Structure spaces for rings of continuous functions with applications to real compactifications, Fundamenta Mathematicae 152 (1997), 151–163.