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Digital homotopic distance between digital functions

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Abstract

In this paper, we define digital homotopic distance and give its relation with LS category of a digital function and of a digital image. Moreover, we introduce some properties of digital homotopic distance such as being digitally homotopy invariance.

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1. Introduction

Macias-Virgos and Mosquera-Lois introduced homotopic distance between maps in [16]. One of the benefits of homotopic distance is to cover the concepts of Lusternik-Schnirelmann category (denoted by cat, see [8]) and topological complexity (denoted by TC, see [9]). If one computes the homotopic distance between some specific maps, then they end up with cat or TC of the domain of these maps. So there is a well-defined relation between homotopic distance, TC, cat and even the sectional category (secat) of some specific fibrations.

We investigate an analog of this relationship in the digital setting. After constructing the digital homotopic distance between digital functions, one of our aims is to show the relation between digital homotopic distance and digital

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LS category of a digital image as defined in [2] and digital LS cat of a digital function as defined in [18].

Our another aim is to investigate how the adjacency relation effects the digital homotopic distance; see Theorem 3.5 and Theorem 3.6.

2. Background

In this section, we recall some definitions and theorems from digital topology.

A digital image X is a subset of $\mathbb{Z}^n$ with an adjacency relation which is defined as follows.

Definition 2.1 ([6])**.** Let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ in $\mathbb{Z}^n$. Then for $1 \leq \ell \leq n$, p and q are said to be c_ℓ-adjacent if

(i) there are at most ℓ indices i which satisfies $|p_i - q_i| = 1$ We would like to call attention to that digital LS category is also introduced with a different point of view using subdivisions by Lupton, Oprea, and Scoville in [14] and [15].

(ii) $p_j = q_j$ for all other indices j satisfying $|p_i - q_i| \neq 1$

Here c_ℓ indicates the number of adjacent points in $\mathbb{Z}^n$. For example, $c_1 = 2$ in $\mathbb{Z}$; $c_1 = 4$ and $c_2 = 8$ in $\mathbb{Z}^2$. Also notice that adjacency relations are often denoted by Greek letters.

Definition 2.2 ([4])**.** A digital interval which is a subset of $\mathbb{Z}$ can be defined as follows

$$[a, b]_{\mathbb{Z}} = \{n \in \mathbb{Z} | a \leq n \leq b\}$$

where 2-adjacency is assumed.

Definition 2.3 ([5])**.** Let (X, κ) and (Y, λ) be digital images. Given a function $f : X \to Y$, if $f(x)$ and $f(y)$ are λ-adjacent or $f(x) = f(y)$ in Y whenever x and y are κ-adjacent in X, then f is called (κ, λ)-*continuous*.

Definition 2.4 ([5, 13])**.** Let $f, g : X \to Y$ be (κ, λ)-*continuous* functions. If there exist $m \in \mathbb{Z}^+$ and a function

$$F : X \times [0, m]_{\mathbb{Z}} \to Y$$

with the following conditions, then F is called a (κ, λ)-*homotopy*, and f and g are called (κ, λ)-*homotopic* in Y (denoted by $f \simeq_{\kappa,\lambda} g$).

(i) For all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$.

(ii) For all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$, $F_x(t) = F(x, t)$ is $(2, \lambda)$-continuous.

(iii) For all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$, $F_t(x) = F(x, t)$ is (κ, λ)-continuous.

Proposition 2.5 ([5])**.** *If $f : X \to Y$ and $g : Y \to Z$ are (κ, λ)-continuous and (λ, γ)-continuous functions, respectively, then $g \circ f : X \to Z$ is (κ, γ)-continuous.*

Definition 2.6 ([7])**.** If κ and λ are two adjacency relations on X, then we say κ dominates λ (denoted by $\kappa \geq_d \lambda$) if for $x, y \in X$ and if x, y are κ-adjacent imply x, y are λ-adjacent.

Proposition 2.7 ([7])**.** *Let $\kappa, \kappa_1, \kappa_2$ be adjacency relations on X and $\lambda, \lambda_1, \lambda_2$ be adjacency relations on Y.*

(a) *If $f : X \to Y$ is (κ, λ_1)-continuous and $\lambda_1 \geq_d \lambda_2$, then f is (κ, λ_2)-continuous.*
(b) *If $f : X \to Y$ is (κ_1, λ)-continuous and $\kappa_2 \geq_d \kappa_1$, then f is (κ_2, λ)-continuous.*

Definition 2.8 ([1, 17])**.** Let (X, κ) and (Y, λ) be digital images. Two elements $(x_1, y_1), (x_2, y_2) \in X \times Y$ are called $NP(\kappa, \lambda)$-adjacent if either

(i) $x_1 = x_2$ and y_1, y_2 are λ-adjacent or
(ii) $y_1 = y_2$ and x_1, x_2 are κ-adjacent or
(iii) x_1, x_2 are κ-adjacent and y_1, y_2 are λ-adjacent.

Definition 2.9. If a (κ, λ)-continuous function $f : X \to Y$ is (κ, λ)-homotopic to a constant map $c : X \to Y$, $c(x) = y_0$, then f is said to be (κ, λ)-nullhomotopic.

If $f : X \to X$ is (κ, κ)-nullhomotopic, then we omit one of the adjacency relations and simply write "κ-nullhomotopic".

3. Digital Homotopic Distance

Recall that a covering of a space X is a collection of subsets of X whose union is X.

Definition 3.1. Let $f, g : X \to Y$ be (κ, λ)-continuous functions. The (κ, λ) *homotopic distance* (so-called *digital homotopic distance*) between f and g is the least non-negative integer n such that there exists a covering $U_0, U_1, \ldots, U_n$ of the digital image X with the property $f|_{U_i} \simeq_{\kappa,\lambda} g|_{U_i}$ for each i. It is denoted by $\mathrm{D}_{\kappa,\lambda}(f, g)$.

If there is no such covering, we define $\mathrm{D}_{\kappa,\lambda}(f, g) = \infty$.

Proposition 3.2. *Let $f : (X, \kappa) \to (Y, \lambda)$ be continuous. If X is finite and κ-connected, then $\mathrm{D}_{\kappa,\lambda}(f, g) < \infty$.*

Proof. Let $U_x = \{x\}$. Then $\{U_x \,|\, x \in X\}$ is a finite covering of X. Since X is κ-connected, for each $x \in X$ there is a (c_1, κ)-continuous $f_x : [0, m_x]_{\mathbb{Z}} \to X$ such that $f_x(0) = x$ and $f_x(m_x) \in f^{-1}(g(x))$. Then the function $H : U_x \times [0, m_x]_{\mathbb{Z}} \to Y$ defined by $H(x, t) = f(f_x(t))$ is a homotopy between $f|_{U_x}$ and $g|_{U_x}$. It follows that $D_{\kappa,\lambda}(f, g) < \infty$. □

Proposition 3.3. *Let $f, g : X \to Y$ be (κ, λ)-continuous. The following properties hold.*

(a) $\mathrm{D}_{\kappa,\lambda}(f, g) = \mathrm{D}_{\kappa,\lambda}(g, f)$.
(b) $\mathrm{D}_{\kappa,\lambda}(f, g) = 0$ *iff* $f \simeq_{\kappa,\lambda} g$.

Proposition 3.4. *Let (X,κ) and (Y,λ) be digital images. Let $f,f',g,g' : X \to Y$ be (κ,λ)-continuous functions. If $f \simeq_{\kappa,\lambda} f'$ and $g \simeq_{\kappa,\lambda} g'$ then $\mathrm{D}_{\kappa,\lambda}(f,g) = \mathrm{D}_{\kappa,\lambda}(f',g')$.*

Proof. Suppose $\mathrm{D}_{\kappa,\lambda}(f',g') = n$. Then there exist subsets $U_0, U_1, \ldots, U_n$ covering (X,κ) such that $f'|_{U_i} \simeq_{\kappa,\lambda} g'|_{U_i}$ for all i.

From the assumption $f \simeq_{\kappa,\lambda} f'$ and $g \simeq_{\kappa,\lambda} g'$, we have $f|_{U_i} \simeq_{\kappa,\lambda} f'|_{U_i}$ and $g|_{U_i} \simeq_{\kappa,\lambda} g'|_{U_i}$ for all i.

Therefore, for all i, we have

$$f|_{U_i} \simeq_{\kappa,\lambda} f'|_{U_i} \simeq_{\kappa,\lambda} g'|_{U_i} \simeq_{\kappa,\lambda} g|_{U_i}.$$

Hence $\mathrm{D}_{\kappa,\lambda}(f,g) \leq n$.

The other way around can be proved similarly. Thus we conclude that $\mathrm{D}_{\kappa,\lambda}(f',g') = \mathrm{D}_{\kappa,\lambda}(f,g)$. □

The following theorems state how the adjacency relations in the domain and in the image affect the digital homotopic distance.

Theorem 3.5. *Let (X,κ), (Y,λ) and (Y,λ') be digital images. Let $f,f' : X \to Y$ be (κ,λ)-continuous and $g,g' : X \to Y$ (κ,λ')-continuous functions. If $f \simeq_{\kappa,\lambda} f'$, $g \simeq_{\kappa,\lambda'} g'$ and $\lambda' \geq_d \lambda$, then $\mathrm{D}_{\kappa,\lambda}(f,g) \leq \mathrm{D}_{\kappa,\lambda'}(f',g')$.*

Proof. Suppose $\mathrm{D}_{\kappa,\lambda'}(f',g') = n$. Then there exist subsets $U_0, U_1, \ldots, U_n$ covering (X,κ) such that $f'|_{U_i} \simeq_{\kappa,\lambda'} g'|_{U_i}$ for all i.

From the assumption $f \simeq_{\kappa,\lambda} f'$ and $g \simeq_{\kappa,\lambda'} g'$, we have $f|_{U_i} \simeq_{\kappa,\lambda} f'|_{U_i}$ and $g|_{U_i} \simeq_{\kappa,\lambda'} g'|_{U_i}$ for all i.

On the other hand, since $\lambda' \geq_d \lambda$ and g,g' are (κ,λ')-continuous, by Proposition 2.7(a) g,g' are (κ,λ)-continuous. Moreover, (κ,λ')-homotopies are also (κ,λ)-homotopies. Hence, since $g|_{U_i} \simeq_{\kappa,\lambda'} g'|_{U_i}$ and $f'|_{U_i} \simeq_{\kappa,\lambda'} g'|_{U_i}$, thus $g|_{U_i} \simeq_{\kappa,\lambda} g'|_{U_i}$ and $f'|_{U_i} \simeq_{\kappa,\lambda} g'|_{U_i}$ for all i.

Therefore, for all i, we have

$$f|_{U_i} \simeq_{\kappa,\lambda} f'|_{U_i} \simeq_{\kappa,\lambda} g'|_{U_i} \simeq_{\kappa,\lambda} g|_{U_i}.$$

It follows that $f|_{U_i} \simeq_{\kappa,\lambda} g|_{U_i}$ for all i. So we have $\mathrm{D}_{\kappa,\lambda}(f,g) \leq n$. Thus we conclude that $\mathrm{D}_{\kappa,\lambda}(f,g) \leq \mathrm{D}_{\kappa,\lambda'}(f',g')$. □

Theorem 3.6. *Let (X,κ), (X,κ') and (Y,λ) be digital images. Let $f,f' : X \to Y$ be (κ,λ)-continuous and $g,g' : X \to Y$ (κ',λ)-continuous functions. If $f \simeq_{\kappa,\lambda} f'$, $g \simeq_{\kappa',\lambda} g'$ and $\kappa' \geq_d \kappa$, then $\mathrm{D}_{\kappa',\lambda}(f',g') \leq \mathrm{D}_{\kappa,\lambda}(f,g)$.*

Proof. Suppose $\mathrm{D}_{\kappa,\lambda}(f,g) = n$. Then there exist subsets $U_0, U_1, \ldots, U_n$ covering (X,κ) such that $f|_{U_i} \simeq_{\kappa,\lambda} g|_{U_i}$ for all i.

From the assumption $f \simeq_{\kappa,\lambda} f'$ and $g \simeq_{\kappa',\lambda} g'$, we have $f|_{U_i} \simeq_{\kappa,\lambda} f'|_{U_i}$ and $g|_{U_i} \simeq_{\kappa',\lambda} g'|_{U_i}$ for all i.

On the other hand, since $\kappa' \geq_d \kappa$ and f,f' are (κ,λ)-continuous, by Proposition 2.7(b) f,f' are (κ',λ)-continuous. Moreover, (κ,λ)-homotopies are also (κ',λ)-homotopies. Hence, since $f|_{U_i} \simeq_{\kappa,\lambda} f'|_{U_i}$ and $f|_{U_i} \simeq_{\kappa,\lambda} g|_{U_i}$, it follows that $f|_{U_i} \simeq_{\kappa',\lambda} f'|_{U_i}$ and $f|_{U_i} \simeq_{\kappa',\lambda} g|_{U_i}$ for all i.

Therefore, for all i, we have

$$f'|_{U_i} \simeq_{\kappa',\lambda} f|_{U_i} \simeq_{\kappa',\lambda} g|_{U_i} \simeq_{\kappa',\lambda} g'|_{U_i}.$$

It follows that $f'|_{U_i} \simeq_{\kappa,\lambda} g'|_{U_i}$ for all i. So we have $\mathrm{D}_{\kappa',\lambda}(f',g') \leq n$. Notice that U_i's also admit κ'-adjacency. Thus we conclude that $\mathrm{D}_{\kappa',\lambda}(f',g') \leq \mathrm{D}_{\kappa,\lambda}(f,g)$. □

Proposition 3.7. *If $f, g : X \to Y$ are (κ,λ)-continuous maps and if $\{U_0, U_1, \ldots, U_n\}$ is a finite covering of X, then we have*

$$\mathrm{D}_{\kappa,\lambda}(f,g) \leq \sum_{i=0}^{n} \mathrm{D}_{\kappa,\lambda}(f|_{U_i}, g|_{U_i}) + n.$$

Proof. Suppose $\mathrm{D}_{\kappa,\lambda}(f|_{U_i}, g|_{U_i}) = m_i$ for each $i = 0, 1, \ldots, n$. So there exist $U_i^0, U_i^1, \ldots, U_i^{m_i}$ covering (U_i, κ) such that $f|_{U_i^j} \simeq_{\kappa,\lambda} g|_{U_i^j}$ for all i, j.

Consider the collection $\mathcal{U} = \{U_0^0, U_0^1, \ldots, U_0^{m_0}, U_1^0, U_1^1, \ldots, U_1^{m_1}, \ldots, U_n^0, U_n^1, \ldots, U_n^{m_n}\}$. Notice that $\bigcup_{V \in \mathcal{U}} V = X$. Moreover, $f|_V \simeq_{\kappa,\lambda} g|_V$ for all $V \in \mathcal{U}$. Hence

$$\begin{aligned} \mathrm{D}_{\kappa,\lambda}(f,g) &\leq (m_0+1) + (m_1+1) + \ldots + (m_n+1) \\ &= (m_0 + m_1 + \ldots + m_n) + n \\ &= \sum_{i=0}^{n} \mathrm{D}_{\kappa,\lambda}(f|_{U_i}, g|_{U_i}) + n. \end{aligned}$$

□

4. Relations Between the Digital Analogs of Homotopic Distance and cat

In this section we introduce the relation between digital homotopic distance and digital LS category both of a digital image and a digital function. Let us first recall the definition of digital LS categories.

Definition 4.1 ([2])**.** The digital LS category of a digital image (X,κ) is the least non-negative integer k such that there is a covering $U_0, U_1, \cdots, U_k$ of X such that inclusion map $\iota_i : U_i \hookrightarrow X$ is digitally κ-nullhomotopic in X for each $i = 0, 1, \ldots, k$. It is denoted by $\mathrm{cat}_\kappa(X) = k$.

Definition 4.2 ([18])**.** Let $f : X \to Y$ be a (κ,λ)-continuous function. The digital LS category of f is the least non-negative integer k such that there is a covering $\{U_0, \ldots, U_k\}$ of X such that $f|_{U_j}$ is (κ,λ)-nullhomotopic for each $j = 0, 1, \ldots, k$. It is denoted by $\mathrm{cat}_{\kappa,\lambda}(f)$.

By Definition 4.2, if $f : X \to Y$ is (κ,λ)-continuous and $c : X \to Y$ is a digital constant function, then $\mathrm{cat}_{\kappa,\lambda}(f) = \mathrm{D}_{\kappa,\lambda}(f,c)$ provided X and Y are κ- and λ-connected, respectively.

By Definition 4.1, $\mathrm{cat}_\kappa(X) = \mathrm{D}_{\kappa,\kappa}(\mathrm{Id}, c)$ where $c : X \to X$ is a digital constant map and X is κ-connected.

Example 4.3. Consider the digital image $(\{0,2\}, c_1)$. $\mathrm{cat}_{c_1}(\{0,2\}) = 1$ but $\mathrm{D}_{c_1,c_1}(\mathrm{Id}, c) = \infty$.

Digital Lusternik-Schnirelmann category can be written in terms of digital homotopic distance as follows.

Theorem 4.4. *For a fixed $x_0 \in X$, let $i_1 : X \to X \times X$, $i_1(x) = (x, x_0)$ and $i_2 : X \to X \times X$, $i_2(x) = (x_0, x)$. Then $\mathrm{cat}_\kappa(X) = \mathrm{D}_{\kappa,NP(\kappa,\kappa)}(i_1, i_2)$.*

Proof. Let us first show that $\mathrm{D}_{\kappa,NP(\kappa,\kappa)}(i_1, i_2) \leq \mathrm{cat}_\kappa(X)$. Let $\mathrm{cat}_\kappa(X) = k$. Then there are $U_0, U_1, \ldots, U_k$ covering X such that $j_i : U_i \hookrightarrow X$ is κ-nullhomotopic for each $i = 0, 1, \ldots, k$.

Let $F^i : U_i \times [0, m_i]_{\mathbb{Z}} \to X$ be (κ, κ)-homotopy such that

(a1) $F^i(x, 0) = j_i(x) = x$ and $F^i(x, m_i) = c(x) = x_0$ where $c : X \to X$, $c(x) = x_0$ is a constant map.
(a2) For all $x \in U_i$, the induced function $F^i_x : [0, m_i]_{\mathbb{Z}} \to X$, $F^i_x(t) = F^i(x, t)$ is $(2, \kappa)$-continuous.
(a3) For all $t \in [0, m_i]_{\mathbb{Z}}$, the induced function $F^i_t : U_i \to X$, $F^i_t(x) = F^i(x, t)$ is (κ, κ)-continuous.

Define $H^i : U_i \times [0, 2m_i]_{\mathbb{Z}} \to X \times X$ as follows.

$$H^i(x, t) = \begin{cases} (F^i(x, t), x_0), & t \in [0, m_i]_{\mathbb{Z}} \\ (x_0, F^i(2m_i - t)), & t \in [m_i, 2m_i]_{\mathbb{Z}} \end{cases}$$

Then we have

(b1) $H^i(x, 0) = i_1(x)$ and $H^i(x, 2m_i) = i_2(x)$.
(b2) For all $x \in U_i$, the induced function $H^i_x : [0, m_i]_{\mathbb{Z}} \to X \times X$, $H^i_x(t) = H^i(x, t)$ is $(2, NP(\kappa, \kappa))$-continuous:

Suppose t_1, t_2 are 2-adjacent.

Case I: If $t_1, t_2 \in [0, m_i]_{\mathbb{Z}}$, then $H^i_x(t_1) = (F^i(x, t_1), x_0)$ and $H^i_x(t_2) = (F^i(x, t_2), x_0)$ are $NP(\kappa, \kappa)$-continuous since the first components are κ-adjacent or equal from (a2) and the second components are equal.

Case II: If $t_1, t_2 \in [m_i, 2m_i]_{\mathbb{Z}}$, then $H^i_x(t_1) = (x_0, F^i(x, 2m_i - t_1))$ and $H^i_x(t_2) = (x_0, F^i(x, 2m_i - t_2))$ are $NP(\kappa, \kappa)$-continuous since the first components are equal and the second components are κ-adjacent or equal from (a2).

Case III: If $m_i \in \{t_1, t_2\}$ then either Case I or Case II applies.

(b3) For all $t \in [0, m_i]_{\mathbb{Z}}$, the induced function $H^i_t : U_i \to X$, $H^i_t(x) = H^i(x, t)$ is $(\kappa, NP(\kappa, \kappa))$-continuous:

Suppose $x, y \in U_i$ are κ-adjacent.

Case I: If $t \in [0, m_i]_{\mathbb{Z}}$, then $H^i_t(x) = H^i(x, t) = (F(x, t), x_0)$ and $H^i_t(y) = H^i(y, t) = (F(y, t), x_0)$. So the first components are κ-adjacent or equal due to (a3) and the second components are equal.

Case II: If $t \in [m_i, 2m_i]_{\mathbb{Z}}$, then $H^i_t(x) = H^i(x, t) = (x_0, F(x, 2m_i - t))$ and $H^i_t(y) = H^i(y, t) = (x_0, F(y, 2m_i - t))$. So the first components are equal and the second components are κ-adjacent or equal due to (a3).

Hence $i_1|_{U_i} \simeq_{\kappa,NP(\kappa,\kappa)} i_2|_{U_i}$. This establishes the desired inequality.

Now let us show that $\mathrm{cat}_\kappa(X) \leq \mathrm{D}_{\kappa,NP(\kappa,\kappa)}(i_1,i_2)$. Let $\mathrm{D}_{\kappa,NP(\kappa,\kappa)}(i_1,i_2) = k$. Then there are $U_0, U_1, \ldots, U_k$ covering $X \times X$ such that $i_1|_{U_j} \simeq_{\kappa,NP(\kappa,\kappa)} i_2|_{U_j}$ for each $j = 0, 1, \ldots, k$.

Let $G^j : U_j \times [0,m_j]_{\mathbb{Z}} \to X \times X$ be $(\kappa, NP(\kappa,\kappa))$-homotopy such that

(c1) $G^j(x,0) = i_1(x)$ and $G^j(x,m_j) = i_2(x)$.
(c2) For all $x \in U_j$, the induced function $G^j_x : [0,m_j]_{\mathbb{Z}} \to X \times X$, $G^j_x(t) = G^j(x,t)$ is $(2, NP(\kappa,\kappa))$-continuous.
(c3) For all $t \in [0,m_j]_{\mathbb{Z}}$, the induced function $G^j_t : U_j \to X \times X$, $G^j_t(x) = G^j(x,t)$ is $(\kappa, NP(kappa,\kappa))$-continuous.

Let $\iota_j : U_j \hookrightarrow X$ be the inclusion function and $c : U_j \to X$, $c(x) = x_0$ be a constant function.

Define $K^j : U_j \times [0,m_j]_{\mathbb{Z}} \to X$ by $K^j(x,t) = \mathrm{pr}_1 \circ G^j(x,t)$ where $\mathrm{pr}_1 : X \times X \to X$ is the projection to the first factor. Notice that pr_1 is $(NP(\kappa,\kappa),\kappa)$-continuous, [10]. Then we have

(d1) $K^j(x,0) = x = \iota_j(x)$ and $K^j(x,m_j) = x_0 = c(x)$.
(d2) For all $x \in U_j$, the induced function $K^j_x : [0,m_j]_{\mathbb{Z}} \to X$, $K^j(x,t) =: K^j_x(t) = \mathrm{pr}_1 \circ G^j_x(t)$ is $(2, NP(\kappa,\kappa))$-continuous due to Proposition 2.5, the $(NP(\kappa,\kappa),\kappa)$-continuity of pr_1 and the $(2, NP(\kappa,\kappa))$-continuity of G^j_x.
(d3) For all $t \in [0,m_j]_{\mathbb{Z}}$, the induced function $K^j_t : U_j \to X \times X$, $K^j(x,t) =: K^j_t(x) = \mathrm{pr}_1 \circ G^j_t(x)$ is (κ,κ)-continuous due to due to Proposition 2.5, the $(NP(\kappa,\kappa),\kappa)$-continuity of pr_1 and the $(2, NP(\kappa,\kappa))$-continuity of G^j_x.

Thus we have $\iota_j|_{U_j} \simeq_{\kappa,\kappa} c$ for each j. ☐

Theorem 4.5 ([18]). $\mathrm{cat}_{\kappa,NP(\kappa,\kappa)}(\Delta_X) = \mathrm{cat}_\kappa(X)$ *where* $\Delta_X : X \to X \times X$, $\Delta_X(x) = (x,x)$ *is* $(\kappa, NP(\kappa,\kappa))$*-continuous.*

Corollary 4.6. *Let* i_1, i_2 *be inclusions as defined in Theorem 4.4 and* $\Delta_X : X \to X \times X$ *be* $(\kappa, NP(\kappa,\kappa))$*-continuous digital diagonal function. Then*

$$\mathrm{D}_{\kappa,NP(\kappa,\kappa)}(\Delta_X, c) = \mathrm{D}_\kappa(i_1,i_2).$$

Proof. This follows from Theorems 4.4 and 4.5.

☐

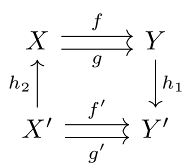
5. Digitally Homotopy Invariance of Digital Homotopic Distance

Theorem 5.1 which states that the digital homotopic distance is homotopy invariant, is the main theorem of this section. Before we mention the theorem, let us recall right and left digital homotopy equivalences.

A (κ,λ)-continuous function $f : X \to Y$ is left digital homotopy inverse if there exists a (λ,κ)-continuous function $g : Y \to X$ such that $g \circ f \simeq_{\kappa,\kappa} \mathrm{Id}_X$.

A (κ, λ)-continuous function $f : X \to Y$ is right digital homotopy inverse if there exists a (λ, κ)-continuous function $h : Y \to X$ such that $f \circ g \simeq_{\lambda,\lambda} \mathrm{Id}_Y$.

Theorem 5.1. *Let $f, g : X \to Y$ be (κ, λ)-continuous and $f', g' : X' \to Y'$ be (κ', λ')-continuous functions. If $h_2 : X' \to X$ and $h_1 : Y \to Y'$ have left and right digital homotopy equivalences respectively such that the following diagram is commutative both with respect to f and g in the following sense: $h_1 \circ f \circ h_2 \simeq_{\kappa',\lambda'} f'$ and $h_1 \circ g \circ h_2 \simeq_{\kappa',\lambda'} g'$. Then $\mathrm{D}_{\kappa,\lambda}(f, g) = \mathrm{D}_{\kappa',\lambda'}(f', g')$.*



An immediate consequence of Theorem 5.1 is the following.

Corollary 5.2 ([2]). *Digital LS category is digitally homotopy invariant.*

For a proof of Theorem 5.1 we need the following lemmas.

Lemma 5.3. *Let $f, g : X \to Y$ be (κ, λ)-continuous and $h : Y \to Z$ be (λ, γ)-continuous functions. Then $\mathrm{D}_{\kappa,\gamma}(h \circ f, h \circ g) \leq \mathrm{D}_{\kappa,\lambda}(f, g)$.*

Proof. Let $\mathrm{D}_{\kappa,\lambda}(f, g) = k$. Then there exists a covering $U_0, \ldots, U_k$ of X such that $f|_{U_i} \simeq_{\kappa,\lambda} g|_{U_i}$ for each $i = 0, 1, \ldots, k$. Then for each i, we have

$$(h \circ f)|_{U_i} = h \circ f|_{U_i} \simeq_{\kappa,\gamma} h \circ g|_{U_i} = (h \circ g)|_{U_i}$$

where the (κ, γ)-homotopy follows from Proposition 2.5. Hence $\mathrm{D}_{\kappa,\gamma}(h \circ f, h \circ g) \leq \mathrm{D}_{\kappa,\lambda}(f, g)$. □

Lemma 5.4. *Let $f, g : X \to Y$ be (κ, λ)-continuous and $h : Z \to X$ be (γ, κ)-continuous functions. Then $\mathrm{D}_{\gamma,\lambda}(f \circ h, g \circ h) \leq \mathrm{D}_{\kappa,\lambda}(f, g)$.*

Proof. Let $\mathrm{D}_{\kappa,\lambda}(f, g) = k$. Then there exists a covering $U_0, \ldots, U_k$ of X such that $f|_{U_i} \simeq_{\kappa,\lambda} g|_{U_i}$ for each $i = 0, 1, \ldots, k$.

Consider $h^{-1}(U_j) \subseteq Z$. Notice that $\{h^{-1}(U_j)\}_{j=0}^{k}$ is a covering of X and the restriction map $h_j : h^{-1}(U_j) \to Z$ can be written in terms of h as $h_j : h^{-1}(U_j) \xrightarrow{h} U_j \xhookrightarrow{\iota} Z$, $h_j = \iota \circ h$. Then we have

$$\begin{aligned}(f \circ h)|_{h^{-1}(U_j)} &= f_{h^{-1}(U_j)} \circ h_j \simeq_{\gamma,\lambda} g|_{h^{-1}(U_j)} \circ h_j = g|_{h^{-1}(U_j)} \circ (\iota \circ h)|_{h^{-1}(U_j)} \\ &= g \circ (\iota \circ h)|_{h^{-1}(U_j)} = (g \circ h)|_{h^{-1}(U_j)}\end{aligned}$$

So $(f \circ h)|_{h^{-1}(U_j)} \simeq_{\gamma,\lambda} (g \circ h)|_{h^{-1}(U_j)}$ for each i. Notice that the (γ, λ)-homotopy on above line follows from Proposition 2.5. Hence $\mathrm{D}_{\gamma,\lambda}(f \circ h, g \circ h) \leq k$. □

By using these lemmas we prove the following propositions.

Proposition 5.5. *Let $f, g : X \to Y$ be (κ, λ)-continuous and $h_1 : Y \to Y'$ be (λ, λ')-continuous function with a left digital homotopy inverse. Then* $\mathrm{D}_{\kappa,\lambda'}(h_1 \circ f, h_1 \circ g) = \mathrm{D}_{\kappa,\lambda}(f, g)$.

Proof. By Proposition 3.4 and Lemma 5.3, it follows that $\mathrm{D}_{\kappa,\lambda}(f, g) = \mathrm{D}_{\kappa,\lambda}(h \circ h_1 \circ f, h \circ h_1 \circ g) \leq \mathrm{D}_{\kappa,\lambda'}(h_1 \circ f, h_1 \circ g) \leq \mathrm{D}_{\kappa,\lambda}(f, g)$. □

Proposition 5.6. *Let $f, g : X \to Y$ be (κ, λ)-continuous and $h_2 : X' \to X$ be (κ', κ)-continuous function with a right digital homotopy inverse. Then* $\mathrm{D}_{\kappa',\lambda}(f \circ h_2, g \circ h_2) = \mathrm{D}_{\kappa,\lambda}(f, g)$.

Proof. By Proposition 3.4 and Lemma 5.4, it follows that $\mathrm{D}_{\kappa,\lambda}(f, g) = \mathrm{D}_{\kappa,\lambda}(f \circ h_2 \circ h, g \circ h_2 \circ h) \leq \mathrm{D}_{\kappa',\lambda}(f \circ h_2, g \circ h_2) \leq \mathrm{D}_{\kappa,\lambda}(f, g)$.

□

Proof of Theorem 5.1. By Proposition 5.5 and Proposition 5.6, we have

$$\mathrm{D}(f', g') = \mathrm{D}(h_1 \circ f \circ h_2, h_1 \circ g \circ h_2) = \mathrm{D}(f \circ h_2, g \circ h_2) = \mathrm{D}_{\kappa,\lambda}(f, g).$$

6. Future Work

There is a relation between usual homotopic distance and TC. A similar relation can be found between digital homotopic distance and digital TC (as defined in [12]).

Higher digital topological complexity is studied by Is and Karaca in [11]. Motivated from the fact that if digital homotopic distance is a generalization of digital TC, it can be predicted that a higher analog of homotopic distance (defined in a similar way as in [16]; see also [3]) can be realized as a generalization of higher digital TC.

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