POWER-AGGREGATION OF PSEUDOMETRICS AND THE MCSHANE-WHITNEY EXTENSION THEOREM FOR LIPSCHITZ p-CONCAVE MAPS

J. RODRÍGUEZ-LÓPEZ* AND E. A. SÁNCHEZ PÉREZ**

Abstract. Given a countable set of families $\{\mathcal{D}_k : k \in \mathbb{N}\}$ of pseudometrics over the same set D, we study the power-aggregations of this class, that are defined as convex combinations of integral averages of powers of the elements of $\cup_k \mathcal{D}_k$. We prove that a Lipschitz function f is dominated by such a power-aggregation if and only if a certain property of super-additivity involving the powers of the elements of $\cup_k \mathcal{D}_k$ is fulfilled by f. In particular, we show that a pseudo-metric is equivalent to a power-aggregation of other pseudometrics if this kind of domination holds. When the super-additivity property involves a p-power domination, we say that the elements of $\mathcal{D}_k$ are p-concave. As an application of our results, we prove under this requirement a new extension result of McShane-Whitney type for Lipschitz p-concave real valued maps.

1. Introduction

One of the main tools for multi-objective optimization is the aggregation of real valued (objective) functions for obtaining a new real valued objective map. The aggregation function usually represents the way the decision maker wants to combine the different objectives of the multi-valued optimization problem for getting a meaningful solution. In the basic cases, these functions are weighted linear combinations as well as p-sums of the set of original functions. In case the objective functions are metrics, we have a classical optimization problem. Particular cases in which the aggregation function is given by a weighted p-norm —if the elements of the metric space constitute a subset of a linear space—, have already become classical tools in multi-objective optimization (see for example [2], and the notion of Lipschitz p-stability in [3, 1.3.5.]). It is also on the basis of more sophisticated methods, as the so called OWA (Ordered Weighted Averaging) (see [3, 1.4.4]).

Motivated in part by this generalized use of averages and p-norms, we are interested in showing a link among this applied context and some classical tools coming from pure topology and functional analysis, that would contribute with new ideas for the foundations of the multi-objective optimization and to widen the set of theoretical tools in this

2010 *Mathematics Subject Classification.* Primary 46N10; Secondary 54A10, 26A16, 54E45.

Key words and phrases. Aggregation; Pseudometric; Optimization; Lipschitz function; extension; p-average.

* Support of the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigaciones and FEDER under research grant MTM2015-64373-P (MINECO/FEDER, UE) (Spain) is gratefully acknowledged.

** Support of the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigaciones and FEDER under research grant MTM2016-77054-C2-1-P (Spain) is gratefully acknowledged.

field. Concerning multi-objective optimization, the Lipschitz properties of the aggregation functions are, in a way, decisive for the stability of the chosen method ([3, 1.3.5.]). Moreover, extension of Lipschitz inequalities from a particular test set to the whole metric space in which the optimization is being carried on is also a fundamental tool for the design of the technique to be used; this is the way the stability of the method can be preserved when an extension of the aggregation function to a bigger set is required. For this reason, we are also interested in providing a new extension result for real-valued Lipschitz p-concave functions from the metric subspaces to the full metric spaces preserving the p-concavity for the extended function.

On the other hand, in modern functional analysis classical summability in Banach spaces is rewritten in terms of domination of linear maps, what gives a characterization of the so called p-summing operators (see for example [10]). This domination is provided by integral p-averages, and the generalization to the case of Lipschitz maps has been developed in the last ten years (see [6, 12, 20]). The analytical tools used there can be applied to the characterization of power-aggregations. Consequently, the aim of this paper is to obtain some representation theorems for this classical family of aggregation functions, that is often used, but for which —as far as we know— there is no general supporting theoretical framework yet, and to prove some new extension theorems for real valued maps satisfying a certain concavity-type Lipschitz inequality.

Let us introduce the main ideas. Let D be a set and let $\mathcal{D}$ be a uniformly bounded family of pseudometrics over D. A usual way of considering a uniform treatment of the family of pseudometrics is given by means of the definition of the supremum pseudometric

$$d_{\mathcal{D},\infty}(a,b) := \sup_{d\in\mathcal{D}} d(a,b), \quad a,b\in D,$$

that can also be considered as an aggregation formula for the family $\mathcal{D}$ in the context of an optimization problem. However, for most applications, this formula is not very useful. For example, the supremum metric is not compatible, in general, with the product topology of a countable family of pseudometric spaces [11, Theorem 7.2]. Furthermore, it also has a bad behaviour with respect to topological and uniform properties as the next examples show.

Example 1.1. Given $n\in\mathbb{N}$, let us consider the metric d_n on $\mathbb{R}$ given by

$$d_n(x,y) = \min\{1, n|x-y|\}$$

for all $x,y\in\mathbb{R}$. Then $\mathcal{D} = \{d_n : n\in\mathbb{N}\}$ is a uniformly bounded family of metrics on $\mathbb{R}$ and all of these metrics generate the Euclidean topology on $\mathbb{R}$. Nevertheless, it is obvious that $d_{\mathcal{D},\infty}$ is the discrete metric generating the discrete topology on $\mathbb{R}$ different from the Euclidean topology. Hence, the supremum metric is not topologically well-behaved.

Example 1.2. Let us consider the set $X = \{\frac{1}{n} : n\in\mathbb{N}\}$ and for each $k\in\mathbb{N}$ define $d_k : X\times X\to\mathbb{R}^+\cup\{0\}$ by

$$d_k\left(\frac{1}{n},\frac{1}{m}\right) = \begin{cases} \left|\frac{1}{n}-\frac{1}{m}\right| & \text{if } n,m>k \\ 1 & \text{if } k\geq\min\{n,m\} \\ 0 & \text{if } n=m \end{cases}.$$

It is clear that $\mathcal{D} = \{d_k : k\in\mathbb{N}\}$ is a uniformly bounded family of metrics on X. Moreover, $(\frac{1}{n})_n$ is a Cauchy sequence on (X,d_k) for all $k\in\mathbb{N}$ but it is not Cauchy with respect to the supremum metric $d_{\mathcal{D},\infty}$ which is obviously the discrete metric on X.

This is the reason we introduce a new family of aggregations for a family $\mathcal{D}$. In fact, we will define and prove the main results for general aggregations defined as convex combinations of compositions of continuous functions with pseudometrics belonging to countably many families that are pointwise compact. This will define our general framework, that will be fixed in several steps—finite families of pseudometrics and power functions—, for finishing with some applications of the easiest case: p-aggregations. These are generalization of weighted p-averages of a finite family of pseudometrics. Since this is the simplest case, let us illustrate such notion now. Take a finite set of pseudometrics $d_1, ..., d_n$ and weights $\alpha_1, ..., \alpha_n$, that are positive numbers that add up to 1. Then the associated p-average —a p-aggregation in our abstract formalism— is given by the formula

$$d_{\{d_1,\ldots,d_n\},p}(a,b) = \Big(\sum_{i=1}^{n} \alpha_i d_i(a,b)^p \Big)^{1/p}, \quad a,b \in D.$$

In the abstract case, the same notion can be written as follows. Suppose that $(\mathcal{D}, \mathcal{O})$ is a topological space. Let $1 \leq p \leq \infty$, and suppose that, for every fixed couple $(a,b) \in D \times D$, the function $\mathcal{F}_{(a,b)} : \mathcal{D} \to \mathbb{R}^+ \cup \{0\}$ given by $\mathcal{F}_{(a,b)}(d) := d(a,b)$ is Lebesgue p-integrable with respect to a measure μ that is defined over the σ-algebra of all the Borel sets of $\mathcal{D}$, i. e. $\mathcal{F}_{(a,b)} \in L^p(\mathcal{D}, \mu)$. In case everything is well-defined, we will say that μ defines a *p-aggregation of $\mathcal{D}$* by the formula

$$d_{\mathcal{D},p,\mu}(a,b) = \Big\| \mathcal{F}_{(a,b)} \Big\|_{L^p(\mathcal{D},\mu)} = \Big(\int_{\mathcal{D}} d(a,b)^p \, d\mu(d) \Big)^{1/p}, \quad a,b \in D.$$

The paper is organized in five sections. After this introduction, we will prove our main separation result—Lemma 2.4— in Section 2, together with some remarks concerning the general setting. In Section 3 we introduce the notion of power-concave function, and some main results and examples are shown.

After this, we will present such p-aggregations —what will be done in Section 4—, and we will show a concrete result concerning families of pseudometrics for functions satisfying a Lipschitz type p-concavity inequality (see Definition 4.3). Besides proving a complete description of the involved property, we will use it for giving a characterization of when the supremum metric of a family of uniformly bounded pseudometrics is in fact a p-aggregation (see Corollary 4.10). Finally, Section 5 is devoted to prove the announced extension result, as a consequence of the characterization theorem (Theorem 4.6): a McShane-Whitney extension theorem for real valued Lipschitz maps preserving p-concavity (Corollary 5.1).

2. Aggregations of pseudometrics and domination

Let us introduce here some definitions and concepts from General Topology. Let $\mathbb{R}^+$ be the positive real numbers. Let D be a nonempty set. A pseudometric on D is a function $d : D \times D \to \mathbb{R}^+ \cup \{0\}$ such that for every $a, b, c \in D$, the following hold:

(1) $d(a,a) = 0$.
(2) $d(a,b) = d(b,a)$.
(3) $d(a,b) \leq d(a,c) + d(c,b)$.

If it happens that $d(a,b) = 0$ implies $a = b$ for every $a, b \in D$, then d is called a metric.

The family of open balls $\{B_\varepsilon(a) : a \in D, \varepsilon > 0\}$ associated to a pseudometric allows us to define a topology $\tau(d)$ on D that clearly has a countable basis.

We will say that two pseudometrics d and ρ over the same set D are equivalent if there are constants $0 < K_1, K_2$ such that

$$\rho(a,b) \leq K_1 d(a,b) \leq K_2 \rho(a,b) \quad \text{for all } \ a,b \in D.$$

This obviously implies that $\tau(\rho) = \tau(d)$.

A Lipschitz function (or K-Lipschitz function) between metric spaces (D,d) and (E,ρ) is a function $f : D \to E$ that satisfies that

$$\rho(f(a), f(b)) \leq K' d(a,b), \quad a,b \in D,$$

for a certain constant K'. The Lipschitz constant K of f is defined to be the infimum of all the constants K' above.

The McShane-Whitney Theorem states that for a subspace S of a metric space (D,d), and a Lipschitz function $f : S \to \mathbb{R}$ with Lipschitz constant K, there exists an extension of f to D which is Lipschitz with the same constant as f.

Such an extension can be given, for example, by the formula

$$f^M(b) := \sup_{a \in S} \{f(a) - K\, d(b,a)\}, \quad b \in D,$$

that is the so called McShane extension of f, or by the Whitney formula, that is

$$f^W(b) := \inf_{a \in S} \{f(a) + K\, d(b,a)\}, \quad b \in D.$$

A look to these formulas and some simple calculations show that the results remain true if we change metrics by pseudometrics in the definition of Lipschitz function.

From now on, $\mathcal{D}$ will denote a family of uniformly bounded pseudometrics over a fixed nonempty set D.

It is well-known that the pointwise bound of all such pseudometrics, defined as

$$d_{\mathcal{D},\infty}(a,b) = \sup_{d \in \mathcal{D}} d(a,b), \quad a,b \in D,$$

is a pseudometric on D.

From the point of view of the classical p-norms of Lebesgue Banach spaces of integrable functions, this is an ∞-norm. Let us define in what follows the "p-version" of this ∞-norm.

Let $1 \leq p \leq \infty$. For every fixed couple $(a,b) \in D \times D$, let us consider the function $\mathcal{F}_{(a,b)} : \mathcal{D} \to \mathbb{R}^+ \cup \{0\}$ given by $\mathcal{F}_{(a,b)}(d) := d(a,b)$ for all $d \in \mathcal{D}$. Set $\mathcal{F} = \{\mathcal{F}_{(a,b)} : (a,b) \in D \times D\}$. Then we want to obtain a classical Banach space from $\mathcal{F}$ by considering an appropriate compact Hausdorff topology on $\mathcal{D}$ which makes all the functions in $\mathcal{F}$ continuous. In this way $\mathcal{F}$ would be a subspace of the Banach space of continuous functions over a compact Hausdorff space with the max norm. In order to achieve this, we need a topology $\mathcal{O}$ over $\mathcal{D}$ making all the functions $\mathcal{F}_{(a,b)}$ continuous. Of course, the smallest topology verifying this property is the weak topology on $\mathcal{D}$ induced by $\mathcal{F}$ [19]. In this topology a net $(d_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{D}$ is convergent to d if and only if $(d_\lambda(a,b))_{\lambda \in \Lambda}$ is convergent to $d(a,b)$ for all $a,b \in D$ so it is the induced topology on $\mathcal{D}$ by the pointwise topology τ_p on $(\mathbb{R}^+ \cup \{0\})^{D \times D}$, i. e. the product topology on the cartesian product $\Pi_{(a,b) \in D \times D} \mathbb{R}^+ \cup \{0\}$. Hence, we have to consider only topologies finer than the pointwise topology. Notice that $((\mathbb{R}^+ \cup \{0\})^{D \times D}, \tau_p)$ is Hausdorff.

It must be said that compactness of $(\mathcal{D}, \mathcal{O})$ will be a necessary requirement in the paper, since we will need to identify the functionals acting in a space of continuous functions with countably additive measures. That is, we will use the Riesz Theorem that identifies the dual of a space of continuous functions $C(\mathcal{D})$ over a compact Hausdorff space $\mathcal{D}$ with the space of countably additive regular Borel measures $\mathcal{M}(\mathcal{D})$.

Moreover, if we want $\mathcal{O}$ to be compact the only possibility is to work with the pointwise topology. In fact, since the identity function $i_{\mathcal{D}} : (\mathcal{D}, \mathcal{O}) \to (\mathcal{D}, \tau_p)$ is continuous and $(\mathcal{D}, \tau_p)$ is Hausdorff we have that $i_{\mathcal{D}}$ must be a homeomorphism. Therefore, the only candidates for $\mathcal{D}$ are the families of pseudometrics over D which are compact subspaces of $(\mathbb{R}^+ \cup \{0\})^{D\times D}$ with the product topology. In the search for this kind of families the following result is fundamental:

Theorem 2.1 ([19])**.** *Let X be a nonempty set and (Y, τ) a Hausdorff topological space. A subset $\mathcal{F} \subseteq Y^X$ is compact relative to the topology of pointwise convergence if and only if:*

(i) *$\mathcal{F}$ is closed in Y^X;*
(ii) *for each $x \in X$, the set $\{f(x) : f \in \mathcal{F}\}$ has compact closure.*

We provide some simple examples of compact topologies over a family of pseudometrics.

Example 2.2.

(a) An obvious example is when $\mathcal{D}$ is a finite family of pseudometrics defined on a nonempty set D and $\mathcal{O}$ is a Hausdorff topology on $\mathcal{D}$. Then $(\mathcal{D}, \mathcal{O})$ is a compact Hausdorff space and the functions $\mathcal{F}_{a,b}$ are obviously continuous.

This example could be even the most interesting one for applications. Indeed, the domination by a p-average that will be proved later on makes sense in this case and is non-trivial, specially if we are interested in the control of the constant appearing in this domination. We will see that this is a consequence of a p-concavity type domination by a family of pseudometrics, that could allow us to control the parameters appearing when an optimization method is applied.

(b) Now, let $D = [0, 1]$ and consider the family of pseudometrics $\mathcal{D} := \Big\{d_n : n \in \mathbb{N} \cup \{0\}\Big\}$ where

$$d_n(a, b) = \begin{cases} \frac{1}{n}|b - a| & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$

for every $a, b \in D$.

Let $\mathcal{O}$ be the restriction of the pointwise topology on $[0, 1]^{[0,1]\times[0,1]}$ to $\mathcal{D}$. It is easy to check that $\mathcal{D}$ is closed in $[0, 1]^{[0,1]\times[0,1]}$. Moreover, $\{d_n(a, b) : n \in \mathbb{N} \cup \{0\}\} = \{\frac{1}{n}|a-b| : n \in \mathbb{N}\} \cup \{0\}$ is obviously compact for every $(a, b) \in [0, 1]\times[0, 1]$ so, by the above theorem, $(\mathcal{D}, \mathcal{O})$ is compact.

We will need the next boundedness properties.

Definition 2.3.

(i) A family $\mathcal{D}$ of pseudometrics on a set D is said to be pointwise uniformly bounded if for every $a, b \in D$, $\sup_{d\in\mathcal{D}} d(a, b) < \infty$.
(ii) A family $\mathcal{F}$ of continuous functions $\mathcal{F} = \{\phi |\, \phi : \mathbb{R} \to \mathbb{R}\}$ is said to be locally uniformly bounded if for every $c \in \mathbb{R}^+$,

$$\sup_{\phi\in\mathcal{F},\, t\in[-c,c]} |\phi(t)| < \infty.$$

In order to simplify the formulas, we use the duality between ℓ^1 and ℓ^∞ in the notation of the following lemma.

Lemma 2.4. *Let D be a nonempty set and fix a countable set $\{\mathcal{D}_k : k \in \mathbb{N}\}$ of families of pseudometrics on D such that the topology of pointwise convergence is compact on any $\mathcal{D}_k$, $k \in \mathbb{N}$, and assume that the set $\cup_{k\in\mathbb{N}}\mathcal{D}_k$ is pointwise uniformly bounded.*

Let (E,ρ) be a pseudometric space, $(\beta_k)_{k\in\mathbb{N}}$ a sequence in ℓ^1 and $\mathcal{F} = \{\phi_k : k \in \mathbb{N}\cup\{0\}\}$ a locally uniformly bounded family of continuous self maps on $\mathbb{R}^+\cup\{0\}$.

The following statements are equivalent for a map $f : D \to E$.

(1) *For every $n \in \mathbb{N}$ and for every set of elements $a_1, ..., a_n, b_1, ..., b_n \in D$, we have that*

$$\sum_{i=1}^{n} \phi_0 \circ \rho(f(a_i), f(b_i))$$
$$\leq \Big\langle \Big(\beta_k\Big)_{k\in\mathbb{N}}, \Big(\sup_{d\in\mathcal{D}_k} \big(\sum_{i=1}^{n} \phi_k \circ d(a_i, b_i)\big)\Big)_{k\in\mathbb{N}} \Big\rangle.$$

(2) *There is a sequence $(\mu_k)_{k\in\mathbb{N}}$ of regular Borel probability measures over the corresponding spaces $\mathcal{D}_k$ such that*

$$\phi_0 \circ \rho(f(a), f(b)) \leq \Big\langle \Big(\beta_k\Big)_{k\in\mathbb{N}}, \Big(\int_{\mathcal{D}_k} \phi_k \circ d(a,b)\, d\mu_k(d)\Big)_{k\in\mathbb{N}} \Big\rangle$$

for every $a, b \in D$.

Proof. The proof of (2) $\Rightarrow$ (1) is given by a direct calculation, just taking into account that the μ_k's are probability measures and for every $a, b \in D$ the series

$$\sum_{k=1}^{\infty} \beta_k \int_{\mathcal{D}_k} \phi_k \circ d(a,b)\, d\mu_k(d_{k,\tau})$$

appearing in (2), converges.

Thus, the proof of (1) $\Rightarrow$ (2) is our main concern. It is given by a classical separation argument; a standard application of the geometric version of the Hahn-Banach Theorem to subsets of a $C(K)$-space would give the result. Instead we use a minimax theorem that is often used in optimization and convex analysis, known as Ky Fan's Lemma.

First, recall that all the spaces $\mathcal{D}_k$ are by assumption compact Hausdorff spaces when endowed with the topology of pointwise convergence. Therefore, for each k we can consider the Banach space $(C(\mathcal{D}_k), \|\cdot\|_\infty)$ of all continuous real functions acting in $\mathcal{D}_k$. By the Riesz-Markov-Kakutani representation theorem, the Banach dual spaces $C(\mathcal{D}_k)^*$, $k \in \mathbb{N}$, can be identified with the spaces $\mathcal{M}(\mathcal{D}_k)$ of regular Borel measures over $\mathcal{D}_k$.

For every $k \in \mathbb{N}$, the set of all the probability measures $\mathcal{P}(\mathcal{D}_k)$, $k \in \mathbb{N}$, is a closed compact convex set in $\mathcal{M}(\mathcal{D}_k)$, with respect to the corresponding weak* topologies. Note that, by the uniform boundedness requirements for both the set $\cup_{k\in\mathbb{N}}\mathcal{D}_k$ (it is pointwise uniformly bounded) and the set $\mathcal{F}$ (it is locally uniformly bounded), we have that for $a_1, b_1, ..., a_n, b_n \in D$ and $0 < \alpha_1, ..., \alpha_n$ such that $\sum_{i=1}^{n} \alpha_i = 1$, the sequence

$$s(\alpha_1, a_1, b_1, ..., \alpha_n, a_n, b_n) = \Big(\sup_{d\in\mathcal{D}_k} \big(\sum_{i=1}^{n} \alpha_i\, \phi_k \circ d(a_i, b_i)\big)\Big)_{k\in\mathbb{N}}$$

is bounded, and so is $\langle s(\alpha_1, a_1, b_1, ..., \alpha_n, a_n, b_n), (\beta_k)_{k\in\mathbb{N}}\rangle$. Recall that by Tychonoff's Theorem, the product Π of the topological spaces $\mathcal{P}(\mathcal{D}_k)$, $k \in \mathbb{N}$, is compact. We can define the functions $F_{\alpha_1,a_1,b_1,...,\alpha_n,a_n,b_n} : \Pi \to \mathbb{R}^+$ by

$$F_{\alpha_1,a_1,b_1,...,\alpha_n,a_n,b_n}((\mu_k)_{k\in\mathbb{N}})$$

$$= \Big\langle \Big(\beta_k\Big)_{k\in\mathbb{N}}, \Big(\sum_{i=1}^n \alpha_i \int_{\mathcal{D}_k} \phi_k \circ d(a_i,b_i) d\mu_k(d)\Big)_{k\in\mathbb{N}} \Big\rangle - \sum_{i=1}^n \alpha_i\, \phi_0 \circ \rho(f(a_i), f(b_i)).$$

$$= \sum_{i=1}^n \alpha_i \Big(\sum_{k=1}^\infty \beta_k \int_{\mathcal{D}_k} \phi_k \circ d(a_i,b_i) d\mu_k(d) \Big) - \sum_{i=1}^n \alpha_i\, \phi_0 \circ \rho(f(a_i), f(b_i)).$$

Taking into account the convergence of the series of the first term of the above formula, and the fact that for each k and i the function $d \mapsto \phi_k \circ d(a_i,b_i)$ belongs to $C(\mathcal{D}_k)$, we get that $F_{\alpha_1,a_1,b_1,\ldots,\alpha_n,a_n,b_n}$ is continuous with respect to the product topology of Π. It is also convex, and the set of all such functions is closed under convex combinations.

Finally, we use a trick explained in [12, S.2] (the observation seems to be due to M. Mendel and G. Schechtman; in this connection, Professor Joe Diestel is said to have pointed out that this trick is possibly older, related to the solution of the functional equations). Note that using the inequality in (1), repeating the elements a_i, b_i as many times as necessary in the sums, and dividing by the appropriate natural number both parts of the inequality, we get that

$$\sum_{i=1}^n \alpha_i\, \phi_0 \circ \rho(f(a_i), f(b_i)) \le \sum_{k=1}^\infty \beta_k \sup_{d\in\mathcal{D}_k} \big(\sum_{i=1}^n \alpha_i\, \phi_k \circ d(a_i,b_i) \big)$$

holds for every rational numbers $\alpha_1, ..., \alpha_n$, and then by continuity for all non-negative real numbers such that $\sum_{k=1}^n \alpha_k = 1$.

Conditions are given to apply Ky Fan's Lemma (see for example [10, 9.10]) to the set of all the functions as $F_{\alpha_1,a_1,b_1,\ldots,\alpha_n,a_n,b_n}$. It gives that there is a unique element $(\mu_k)_{k\in\mathbb{N}} \in \Pi$ such that $F_{\alpha_1,a_1,b_1,\ldots,\alpha_n,a_n,b_n}((\mu_k)_{k\in\mathbb{N}}) \le 0$ for every such a function. In particular, when only a pair $a, b \in D$ is chosen we get

$$\phi_0 \circ \rho(f(a), f(b)) - \sum_{k=1}^\infty \beta_k \int_{\mathcal{D}_k} \phi_k \circ d(a_i,b_i) d\mu_k(d) \le 0.$$

This gives the result.

□

Remark 2.5. A extreme case of Lemma 2.4 is given when the infimum of all the suprema of the families of pseudometrics is considered. A direct application of the lemma gives— in the context of the lemma and under the same hypothesis—, the equivalence among the following facts.

(1) For every $n \in \mathbb{N}$ and for every set of elements $a_1, ..., a_n, b_1, ..., b_n \in D$, we have that

$$\sum_{i=1}^n \phi_0 \circ \rho(f(a_i), f(b_i)) \le \inf_{k\in\mathbb{N}} \Big(\sup_{d\in\mathcal{D}_k} \big(\sum_{i=1}^n \phi_k \circ d(a_i,b_i) \big) \Big).$$

(2) There is a sequence $(\mu_k)_{k\in\mathbb{N}}$ of regular Borel probability measures over the corresponding spaces $\mathcal{D}_k$ such that

$$\phi_0 \circ \rho(f(a), f(b)) \le \inf_{k\in\mathbb{N}} \Big(\int_{\mathcal{D}_k} \phi_k \circ d(a,b)\, d\mu_k(d) \Big)$$

for every $a, b \in D$.

To see this, it is enough to apply the lemma taking as $(\beta_k)_{k\in\mathbb{N}}$ all the vectors of the canonical basis e_k separately, and compute the infimum.

3. Power-type aggregations and power-concavity of pseudometrics

Following the plan that we explained in the Introduction, we are interested in applying the general separation result provided in the previous section to the case of functions $\phi_k(\cdot) = |\cdot|^{p_k}$, where $0 < p_k < \infty$ is a real number for every $k \in \mathbb{N}$, and ϕ_0 is the identity.

Definition 3.1. Fix a countable set $\{\mathcal{D}_k : k \in \mathbb{N}\}$ of families of pseudometrics on a set D such that the topology of pointwise convergence is compact on any $\mathcal{D}_k$, $k \in \mathbb{N}$, and such that $\cup_{k\in\mathbb{N}}\mathcal{D}_k$ is pointwise uniformly bounded. Let $(\beta_k)_{k\in\mathbb{N}}$ be a sequence of weights in ℓ^1. Let $0 < r < \infty$ and take a strictly positive sequence of real numbers $(p_k)_{k\in\mathbb{N}} \in \ell^\infty$.

(a) Consider a sequence $(\mu_k)_{k\in\mathbb{N}}$ of probability measures over the sets $\mathcal{D}_k$. We say that the function

$$q(a,b) := \left(\sum_{k=1}^{\infty} \beta_k \int_{\mathcal{D}_k} d(a,b)^{p_k}\, d\mu_k(d)\right)^{1/r}, \quad a,b \in D,$$

is a power-aggregation with respect to the measures $\mu_1, ..., \mu_k, ...$ with indices $r, p_1, ..., p_k, ...$ and weights $\beta_1, ..., \beta_k, ...$

(b) Let (E, ρ) be a pseudometric space. We say that a function $f : E \to D$ is power-concave with indices $r, p_1, ..., p_k, ...$ and with respect to the families of pseudometrics $\{\mathcal{D}_k : k \in \mathbb{N}\}$ for the weights $(\beta_k)_{k\in\mathbb{N}}$ if for every finite set $a_1, b_1, ..., a_n, b_n$ of elements of D, the following inequality holds:

$$\sum_{i=1}^{n} \rho(f(a_i), f(b_i))^r \leq \sum_{k=1}^{\infty} \beta_k \sup_{d\in\mathcal{D}_k} \Big(\sum_{i=1}^{n} d(a_i, b_i)^{p_k}\Big).$$

In particular, we will say that a pseudometric d over D is power-concave with indices $r, p_1, ..., p_k, ...$ and with respect to the families of pseudometrics above for the weights $(\beta_k)_{k\in\mathbb{N}}$ if the identity map $Id : (D, d) \to (D, d)$ is power-concave with indices $r, p_1, ..., p_k, ...$.

Recall that—as explained before—for a pointwise uniformly bounded set of pseudometrics $\mathcal{D}$ over D, we can define a pseudometric $d_{\mathcal{D},\infty}(a, b)$ as $\sup_{d\in\mathcal{D}} d(a, b)$, $a, b \in D$.

As usual, we say that a function $f : (E, \rho) \to (D, d)$ between metric spaces is *Hölder continuous* if for every $a, b \in D$, $d(f(a), f(b)) \leq K\rho(a, b)^\alpha$ for a certain $0 < \alpha$ and a constant K. Moreover, if we have a family of pseudometrics $\{d_\tau : \tau \in \mathcal{T}\}$ over D, we say that the function f is *uniformly Hölder continuous* with indices $\{\alpha_\tau : \tau \in \mathcal{T}\}$ if for every pair $a, b \in D$ there is a constant K such that $d_\tau(f(a), f(b)) \leq K\rho(a, b)^{\alpha_\tau}$ for every $\tau \in \mathcal{T}$.

Let us give now our main result on power-aggregations of pseudometrics. Recall that we are assuming that all the sets $\mathcal{D}_k$, $k \in \mathbb{N}$, are pointwise uniformly bounded, and so finite unions of these sets are too.

Theorem 3.2. *Consider a pseudometric ρ in D and consider families of pseudometrics and weights $(\beta_k)_{k\in\mathbb{N}}$ as in Definition 3.1. Suppose that the pseudometric ρ satisfies:*

(a) *It is power-concave with indices $r, p_1, ..., p_k, ...$ and with respect to the families of pseudometrics $\{\mathcal{D}_k : k \in \mathbb{N}\}$ for the weights $(\beta_k)_{k\in\mathbb{N}}$.*

(b) *The identity map $i : (D, \rho) \to (D, d_{\mathcal{D}_k,\infty})$, $k \in \mathbb{N}$, is uniformly Hölder continuous with indices $\{r/p_k : k \in \mathbb{N}\}$.*

Then there is a sequence $(\mu_k)_{k\in\mathbb{N}}$ of regular Borel probability measures over the corresponding spaces $\mathcal{D}_k$ such that ρ is equivalent to a power-aggregation q with respect to the *measures $\mu_1, ..., \mu_k, ...$ with indices $r, p_1, ..., p_k, ...$ and weights $\beta_1, ..., \beta_k, ...$. Moreover, under these requirements ρ is also equivalent to*

$$t(a,b) = \Big(\sum_{k=1}^{\infty} \beta_k \, d_{\mathcal{D}_k,\infty}(a,b)^{p_k} \Big)^{1/r}, \quad a,b \in D.$$

Conversely, if the sequence $\{\frac{1}{\beta_k}\}_{k\in\mathbb{N}}$ is bounded and positive and there are constants Q_1, Q_2 and Q_3 such that $\rho \le Q_1 \cdot q \le Q_2 \cdot t \le Q_3 \cdot \rho$ for q and t as in Definition 3.1 (a) and above, respectively, then (a) and (b) holds for ρ.

Proof. Let us show that (a) and (b) imply the desired inequalities for ρ. Indeed, if $a, b \in D$, Lemma 2.4 together with the assumption (b) give

$$\rho(a,b) \le \Big(\sum_{k=1}^{\infty} \beta_k \sup_{d\in\mathcal{D}_k} d(a,b)^{p_k} \Big)^{1/r}$$

$$= \Big(\sum_{k=1}^{\infty} \beta_k \, d_{\mathcal{D}_k,\infty}(a,b)^{p_k} \Big)^{1/r} \le \Big(\sum_{k=1}^{\infty} \beta_k \, K^{p_k} \rho(a,b)^r \Big)^{1/r} \le \Big(\sum_{k=1}^{\infty} \beta_k \Big) \sup_{k\in\mathbb{N}} K^{p_k} \, \rho(a,b),$$

what is bounded since $(p_k)_{k\in\mathbb{N}}$ is a bounded sequence.

For the converse, Lemma 2.4 and the inequality $\rho \le Q_1 \cdot q$ gives that ρ is power-concave with indices $r, p_1, ..., p_k, ...$ and with respect to the families of pseudometrics $\{\mathcal{D}_k : k \in \mathbb{N}\}$ for the weights $\{\beta_k Q_1^r\}_{k\in\mathbb{N}}$.

Furthermore, the inequality $Q_2 \cdot t \le Q_3 \cdot \rho$ gives (b). Indeed, for $a, b \in D$ and each $j \in \mathbb{N}$ we have that

$$\begin{aligned} d_{\mathcal{D}_j,\infty}(a,b)^{p_j/r} &\le \frac{1}{\beta_j} \Big(\sum_{k=1}^{\infty} \beta_k \, d_{\mathcal{D}_k,\infty}(a,b)^{p_k} \Big)^{1/r} = \frac{1}{\beta_j} t(a,b) \le \frac{1}{\beta_j} \frac{Q_3}{Q_2} \rho(a,b) \beta_j \\ &\le \sup_{k\in\mathbb{N}} \frac{1}{\beta_k} \frac{Q_3}{Q_2} \, \rho(a,b). \end{aligned}$$

Thus, we get that

$$d_{\mathcal{D}_j,\infty}(a,b) \le \sup_{n\in\mathbb{N}} \left\{ \left(\sup_{k\in\mathbb{N}} \left(\frac{Q_3}{\beta_k} \right) \right)^{r/p_n} \right\} \rho(a,b)^{r/p_j}$$

for every $a, b \in D$ and for every $j \in \mathbb{N}$. Hence (b) holds. □

Remark 3.3. Although we have proved the requirements for a pseudometric ρ to be equivalent to a power-aggregation, in general a power-aggregation is not a metric. However, some easy requirements gives that it is a metric. For example, for $r = 1$ it is enough to assure that all the elements d of $\mathcal{D}_k$ satisfy that d^{p_k} is a metric. We will see other easy situation like this in the next section.

In the search of examples and applications, let us show more concrete conditions under which a domination by a power-aggregation can be obtained. For the case of a finite set of families of pseudometrics, the inequality in the following result provides a sufficient condition for having a special power-concave domination for a metric. It is closely related with a great class of domination properties for multilinear operators that have been studied in recent years. Just as an example of such kind of results, the reader can find information about in [5, 16] and the references therein. In particular, the argument that proves it for $m = 2$ is similar to the one that proves Kwapien's Theorem for (p,q)-dominated operators (see the proof for example in [9, 19.2]); a fruitful general version of this result also for the bilinear case can be found in [8, Th.1].

Corollary 3.4. *Let D be a nonempty set and fix a finite set $\{\mathcal{D}_1,...,\mathcal{D}_m\}$ of families of pseudometrics on D such that the topology of pointwise convergence is compact on any $\mathcal{D}_k$, $k=1,...,m$. Let (E,ρ) be a pseudometric space and let $f:(D,d_{\mathcal{D},\infty})\to(E,\rho)$ be a Lipschitz map. Set $1\le r,p_1,...,p_m<\infty$ such that $1/r=\sum_{k=1}^m 1/p_k$.*

Suppose that for every $n\in\mathbb{N}$ and for every set of elements $a_1,...,a_n,b_1,...,b_n\in D$, we have that

$$\Big(\sum_{i=1}^n \rho(f(a_i),f(b_i))^r\Big)^{\frac{1}{r}} \le \sup_{d\in\mathcal{D}_1}\Big(\sum_{i=1}^n d(a_i,b_i)^{p_1}\Big)^{\frac{1}{p_1}}\cdot\ldots\cdot\sup_{d\in\mathcal{D}_m}\Big(\sum_{i=1}^n d(a_i,b_i)^{p_m}\Big)^{\frac{1}{p_m}}.$$

Then there are regular Borel probability measures μ_k over $\mathcal{D}_k$, $k=1,...,m$, such that

$$\rho(f(a),f(b))^r \le \sum_{k=1}^m \frac{r}{p_k}\int_{\mathcal{D}_k} d(a,b)^{p_k}\,d\mu_k(d)$$

for every $a,b\in D$.

Proof. Note that, by Young's inequality and taking into account that $\sum_{k=1}^m r/p_k=1$, we have that for $a_1,b_1,...,a_n,b_n\in D$,

$$\sup_{d\in\mathcal{D}_1}\Big(\sum_{i=1}^n d(a_i,b_i)^{p_1}\Big)^{\frac{r}{p_1}}\cdot\ldots\cdot\sup_{d\in\mathcal{D}_m}\Big(\sum_{i=1}^n d(a_i,b_i)^{p_m}\Big)^{\frac{r}{p_m}}$$
$$\le \sum_{k=1}^m \frac{r}{p_k}\sup_{d\in\mathcal{D}_k}\Big(\sum_{i=1}^n d(a_i,b_i)^{p_k}\Big).$$

Thus, the result follows directly from Lemma 2.4. □

Example 3.5 (A $(2,2)$-dominated operator acting in a Hilbert grid.)**.** Let us provide an example of a function from a metric space satisfying a domination as in Corollary 3.4. We use the framework of the operator ideals theory, in particular the properties of the so called (p,q)-dominated operators, mainly a characterization of this class due to Kwapien; the reader can find all the information that is needed in [9, 19.2]. Consider a self-adjoint positive linear operator $T:\ell^2\to\ell^2$ that is $(2,2)$-dominated. This means that for every $x_1,...,x_n\in\ell^2$ and $y_1',...,y_n'\in(\ell^2)^*=\ell^2$,

$$\sum_{i=1}^n |\langle T(x_i),y_i'\rangle| \le K\sup_{x'\in B_{\ell^2}}\Big(\sum_{i=1}^n|\langle x_i,x'\rangle|^2\Big)^{\frac{1}{2}}\sup_{y\in B_{\ell^2}}\Big(\sum_{i=1}^n|\langle y_i',y\rangle|^2\Big)^{\frac{1}{2}}.$$

By Kwapien's Theorem, we have that the operator is in particular 2-summing (see Remark 4.5), and so it is also Hilbert-Schmidt and compact (see [9, Prop. 11.6]). Since it is self-adjoint, we use the Spectral Theorem to find a representation of T as $T(\cdot)=\sum_{i=1}^\infty \lambda_i e_i\langle e_i,\cdot\rangle$, and we also know that $\lambda_i\ge 0$ and $\sum_{i=1}^\infty\lambda_i^2=1$. Now take $s\in\mathbb{N}$. We fix the set $D=\Big\{k(e_i,e_i)\in\ell^2\times\ell^2 : i\in\mathbb{N}, -s\le k\le s, k\in\mathbb{Z}\Big\}$, and the function $f:D\to\mathbb{R}$ given by $f(k(e_i,e_i)):=\langle T(ke_i),ke_i\rangle=\lambda_i k^2$. Let us show that this map satisfies the inequality

$$\big|f(k(e_i,e_i))-f(k'(e_j,e_j))\big| \le M\big|\langle T(ke_i-k'e_j),ke_i-k'e_j\rangle\big|, \tag{1}$$

for all $k(e_i,e_i)$, $k'(e_j,e_j)\in D$ and a certain constant M. Indeed, we have that if $i\ne j$,

$$\big|\langle T(ke_i-k'e_j),ke_i-k'e_j\rangle\big| = \lambda_i k^2+\lambda_j k'^2,$$

and if $i = j$, $\big|\langle T(ke_i - k'e_i), ke_i - k'e_i\rangle\big| = \big|\langle \lambda_i(ke_i - k'e_i), ke_i - k'e_i\rangle\big| = \lambda_i(k-k')^2.$

Note now that $\big|f(k(e_i,e_i)) - f(k'(e_j,e_j))\big| = \big|\lambda_i k^2 - \lambda_j k'^2\big|$. Thus, if $i \neq j$,

$$\big|f(k(e_i,e_i)) - f(k'(e_j,e_j))\big| = \big|\lambda_i k^2 - \lambda_j k'^2\big| \leq \lambda_i k^2 + \lambda_j k'^2,$$

and if $i = j$, $\big|f(k(e_i,e_i)) - f(k'(e_i,e_i))\big| = |\lambda_i|\,\big|k^2 - k'^2\big| = \lambda_i\,\big|k-k'\big|\,\big|k+k'\big|$

$$\leq \lambda_i\,\big|k-k'\big|\,2s \leq 2s\,\lambda_i(k-k')^2.$$

Therefore, the inequality (1) holds for $M = 2s$. Now we consider the set(s) of pseudometric(s)

$$\mathcal{D}_t = \{d_{t,x'}(k(e_i,e_i), k'(e_j,e_j)) = |\langle ke_i - k'e_j, x'\rangle| : x' \in B_{\ell^2}\}, \quad t = 1, 2,$$

appearing in the right had side of the $(2,2)$-summing inequality of T. Therefore, for every $a_1, b_1, ..., a_n, b_n \in D$,

$$\sum_{i=1}^n \big|f(a_i)) - f(b_i)\big|$$

$$\leq 2s \sum_{i=1}^n |\langle T(x_i), y_i'\rangle| \leq 2s\,K \sup_{x' \in B_{\ell^2}} \Big(\sum_{i=1}^n |\langle x_i, x'\rangle|^2\Big)^{\frac{1}{2}} \sup_{y \in B_{\ell^2}} \Big(\sum_{i=1}^n |\langle y_i', y\rangle|^2\Big)^{\frac{1}{2}}.$$

Corollary 3.6. *Under the same conditions as in Corollary 3.4, suppose that the sets of pseudometrics $\{\mathcal{D}_1, ..., \mathcal{D}_m\}$ satisfy that given $k \in \{1, \dots, n\}$ and $d \in \mathcal{D}_k$ there is $q_d \in \mathcal{D}_k$, such that for every $(d_1, \dots, d_m) \in \mathcal{D}_1 \times \dots \times \mathcal{D}_m$ we have that*

$$\max_{k=1,\dots,m} \{d_k(a,b)^{p_k}\} \leq K \prod_{k=1}^m q_{d_k}(a,b)^r$$

for every $a, b \in D$, where K is the Lipschitz constant of f. Then the following statements are equivalent.

(i) *For every $n \in \mathbb{N}$ and for every set of elements $a_1, ..., a_n, b_1, ..., b_n \in D$, there exists a constant $K_1 > 0$ such that*

$$\Big(\sum_{i=1}^n \rho(f(a_i), f(b_i))^r\Big)^{\frac{1}{r}} \leq K_1 \sup_{d \in \mathcal{D}_1} \Big(\sum_{i=1}^n d_{1,\tau}(a_i,b_i)^{p_1}\Big)^{\frac{1}{p_1}} \dots \sup_{d \in \mathcal{D}_m} \Big(\sum_{i=1}^n d_{m,\tau}(a_i,b_i)^{p_m}\Big)^{\frac{1}{p_m}}.$$

(ii) *For $n \in \mathbb{N}$ and $a_1, ..., a_n, b_1, ..., b_n \in D$, there exists a constant $K_2 > 0$ such that*

$$\sum_{i=1}^n \rho(f(a_i), f(b_i))^r \leq K_2 \sum_{i=1}^n \max_{k=1,\dots,m} \sup_{d \in \mathcal{D}_k} \{d(a_i,b_i)^{p_k}\}.$$

(iii) *There are regular Borel probability measures μ_k over $\mathcal{D}_k$, $k = 1, ..., m$, and a constant $K_3 > 0$ such that*

$$\rho(f(a), f(b))^r \leq K_3 \sum_{k=1}^m \frac{r}{p_k} \int_{\mathcal{D}_k} d(a,b)^{p_k}\, d\mu_k(d)$$

for every $a, b \in D$.

Proof. (i) implies (iii) is given by Corollary 3.4, changing ρ by ρ/K_1^r.

For (iii) implies (ii). We can assume w.l.o.g. that $K_3 = 1$. Consider the following computations. Since the sets $\mathcal{D}_k$ are compact, given $k \in \{1,\dots,m\}$ and $i \in \{1,\dots,n\}$ there exists $d_{k,i} \in \mathcal{D}_k$ such that $\sup_{d\in\mathcal{D}_k} d(a_i,b_i) = d_{k,i}(a_i,b_i)$. Then

$$\sum_{i=1}^n \rho(f(a_i),f(b_i))^r \leq \sum_{i=1}^n \sum_{k=1}^m \frac{r}{p_k} \int_{\mathcal{D}_k} \frac{1}{\gamma^{p_k}}\, d(a_i,b_i)^{p_k}\, d\mu_k(d)$$

$$= \sum_{k=1}^m \frac{r}{p_k} \int_{\mathcal{D}_k} \sum_{i=1}^n d(a_i,b_i)^{p_k}\, d\mu_k(d) \leq \sum_{k=1}^m \frac{r}{p_k} \sup_{d\in\mathcal{D}_k} \big(\sum_{i=1}^n d(a_i,b_i)^{p_k}\big)$$

$$\leq \sum_{k=1}^m \frac{r}{p_k} \big(\sum_{i=1}^n d_{k,i}(a_i,b_i)^{p_k}\big) = \sum_{i=1}^n \Big(\sum_{k=1}^m \frac{r}{p_k} d_{k,i}(a_i,b_i)^{p_k}\Big)$$

$$\leq \sum_{i=1}^n \max_{k=1,\dots,m} d_{k,i}(a_i,b_i)^{p_k} = \sum_{i=1}^n \max_{k=1,\dots,m} \sup_{d\in\mathcal{D}_k} \{d(a_i,b_i)^{p_k}\}.$$

Finally, for (ii) implies (i) we use the specific requirement for the families $\mathcal{D}_k$ appearing in the statement of the corollary. We can assume again that $K_2 = 1$. Note that given $k \in \{1,\dots,m\}$ and $i \in \{1,\dots,n\}$ by compactness of $\mathcal{D}_k$, there exist elements $d_{k,i} \in \mathcal{D}_k$ such that

$$\sum_{i=1}^n \rho(f(a_i),f(b_i))^r$$

$$\leq \sum_{i=1}^n \max_{k=1,\dots,m} \sup_{d\in\mathcal{D}_k} \{d(a_i,b_i)^{p_k}\} = \sum_{i=1}^n \max_{k=1,\dots,m} d_{k,i}(a_i,b_i)^{p_k}.$$

By assumption, and using Hölder inequality, we have that for every $k \in \{1,\dots,n\}$ and every $i \in \{1,\dots,m\}$ we can find $q_{d_{k,i}} \in \mathcal{D}_k$ such that

$$\sum_{i=1}^n \max_{k=1,\dots,m} d_{k,i}(a_i,b_i)^{p_k} \leq K \sum_{i=1}^n \prod_{k=1}^m q_{d_{k,i}}(a_i,b_i)^r$$

$$\leq K \Big(\sum_{i=1}^n q_{d_{1,i}}(a_i,b_i)^{p_1}\Big)^{\frac{r}{p_1}} \dots \Big(\sum_{i=1}^n q_{d_{m,i}}(a_i,b_i)^{p_m}\Big)^{\frac{r}{p_m}}$$

$$\leq K \sup_{d\in\mathcal{D}_1} \Big(\sum_{i=1}^n d(a_i,b_i)^{p_1}\Big)^{\frac{r}{p_1}} \dots \sup_{d\in\mathcal{D}_m} \Big(\sum_{i=1}^n d(a_i,b_i)^{p_m}\Big)^{\frac{r}{p_m}}.$$

This finishes the proof. □

4. p-concavity for Lipschitz maps and p-aggregations of pseudometrics

In this section we center our attention in the case of a unique family of pseudometrics $\mathcal{D}$ and a unique parameter $1 \leq p < \infty$. In particular, this provides the classical definition of p-aggregation of a set of metrics, as can be found in the mathematical literature.

Definition 4.1. Let $1 \leq p < \infty$, D be a nonempty set and $\mathcal{D}$ a uniformly pointwise bounded family of pseudometrics on D such that the topology of pointwise convergence on $\mathcal{D}$ is compact. Fix a regular Borel probability measure μ over $\mathcal{D}$. The *p-aggregation of $\mathcal{D}$ with respect to μ* is defined as the function $d_{\mathcal{D},p,\mu} : D \times D \to \mathbb{R}^+ \cup \{0\}$ given by

$$d_{\mathcal{D},p,\mu}(a,b) := \|\mathcal{F}(a,b)\|_{L^p(\mathcal{D},\mu)} = \Big(\int_{\mathcal{D}} d(a,b)^p \, d\mu(d) \Big)^{1/p}, \quad a,b \in D.$$

If μ is a convex combination of a finite set of Dirac's deltas $\delta_{d_1}, ..., \delta_{d_n}$ of elements of $\mathcal{D}$ for coefficients $0 < \alpha_1, ..., \alpha_n < 1$, $\sum_{i=1}^n \alpha_i = 1$, we get

$$d_{\mathcal{D},p,\mu}(a,b) = \Big(\int_{\mathcal{D}} d(a,b)^p \, d(\sum_{i=1}^n \alpha_i \delta_{d_i})(d) \Big)^{1/p} = \Big(\sum_{i=1}^n \alpha_i \, d_i(a,b)^p \Big)^{1/p},$$

$a, b \in D$. The last formula provides the typical example of p-aggregation used in optimization theory, that is clearly again a metric. In fact, this is true even when the general integral formula is used, as we show below.

Lemma 4.2. *Let D be a nonempty set and $\mathcal{D}$ be a family of pseudometrics on D such that the topology of pointwise convergence on $\mathcal{D}$ is compact. Fix a regular Borel probability measure μ over $\mathcal{D}$. Then the associated p-aggregation of $\mathcal{D}$ with respect to μ is a pseudometric.*

Proof. The triangle inequality that holds for each of the elements of $\mathcal{D}$, together with the fact that $\| \cdot \|_{L^p(\mathcal{D},\mu)}$ is a norm, gives the triangle inequality for the p-aggregation, and so it gives the result. □

We next introduce the following *p-concavity type property,* which is the key of the characterization of when the supremum pseudometric of a family of uniformly bounded pseudometrics is a p-aggregation.

Definition 4.3. Let $\mathcal{D}$ be a family of uniformly pointwise bounded pseudometrics on a nonempty set D. Let (E, ρ) be another pseudometric space and $f : (D, d_{\mathcal{D},\infty}) \to (E, \rho)$ be a Lipschitz map. We say that f is *Lipschitz-p-concave* if there exists $K > 0$ such that for every $n \in \mathbb{N}$, and for every set of elements $a_1, ..., a_n, b_1, ..., b_n \in D$, we have that

$$\Big(\sum_{i=1}^n \rho(f(a_i), f(b_i))^p \Big)^{1/p} \leq K \sup_{d \in \mathcal{D}} \Big(\sum_{i=1}^n d(a_i, b_i)^p \Big)^{1/p}.$$

Remark 4.4. With this name we intend to suggest the relationship between the new defined notion and the concept of p-concavity coming from the theory of Banach lattices. Let $1 \leq p, p' < \infty$ such $1/p + 1/p' = 1$. A Banach space valued linear operator T acting in a Banach lattice X is said to be p-concave if there is a constant $K > 0$ such that for every $x_1, ..., x_n \in X$,

$$\Big(\sum_{i=1}^n \|T(x_i)\|^p \Big)^{1/p} \leq K \Big\| \big(\sum_{i=1}^n |x_i|^p \big)^{1/p} \Big\|_X.$$

Note that we are treating the pseudometrics $d_{k,\tau}$ as elements of a space of continuous functions. When X is a space of continuous functions $C(K)$, it is well-known that the norm can be written as

$$\Big\| \big(\sum_{i=1}^n |x_i|^p \big)^{1/p} \Big\|_{C(K)} = \sup_{w \in K} \sup_{(\lambda_i) \in B_{\ell^{p'}}} \Big| \sum_{i=1}^n \lambda_i x_i(w) \Big|$$

$$= \sup_{\mu \in B_{(C(K))^*}} \sup_{(\lambda_i) \in B_{\ell^{p'}}} \Big| \sum_{i=1}^n \lambda_i \langle x_i, \mu \rangle \Big| = \sup_{\mu \in B_{(C(K))^*}} \Big(\sum_{i=1}^n |\langle x_i, \mu \rangle|^p \Big)^{1/p}.$$

Now consider vectors as $x_i^1 - x_i^2$ instead of isolated elements x_i in this expression. Taking into account that in this case $\|T(x_1) - T(x_2)\| = \rho(T(x_1), T(x_2))$, if we change T by a Lipschitz map and the norm by a general distance, we get Definition 4.3.

Related notions and even the same name has been used for a similar inequality in the context of Lipschitz operators between Banach spaces. The interested reader can find more information about this in [1, 7]. In [12], Farmer and Johnson gave a definition of p-summing operator for the case of Lipschitz maps between metric spaces. Our definition of Lipschitz p-concavity is, in a sense, a generalization of the one given there. Further developments have been made in this direction in recent years; we refer to the papers [6, 20] and the references therein for more information about the topic.

Remark 4.5. Let us show that p-summing operators are indeed in the category of Lipschitz-p-concave operators. Given $1 \leq p < \infty$, a linear operator $T : (E, |\cdot|) \to (F, \|\cdot\|)$ between Banach spaces is called p-summing (see, for example, [10, p.31]) if there is a constant $K > 0$ such that for every finite set of vectors $x_1, ..., x_n \in E$,

$$\Big(\sum_{i=1}^{n} \|T(x_i)\|^p\Big)^{1/p} \leq K \sup_{x' \in B_{E^*}} \Big(\sum_{i=1}^{n} |\langle x_i, x'\rangle|^p\Big)^{1/p}.$$

where B_{E^*} is the closed unit ball of the dual space E^* of E.

Let us consider the family of pseudometrics $\mathcal{D} = \{d_{x'} : x' \in B_{E^*}\}$ where $d_{x'}(y, z) = |\langle y - z, x'\rangle|$ for every $y, z \in E$. Clearly, this set is uniformly bounded and $d_{\mathcal{D},\infty}(y, z) = \sup_{x' \in B_{E^*}} d_{x'}(y, z) = |y - z|$. Then it is straightforward to check that $T : (E, d_{\mathcal{D},\infty}) \to (F, d_{\|\cdot\|})$ is Lipschitz-p-concave if and only if $T : E \to F$ is p-summing. In fact, if T is Lipschitz-p-concave we can find $K > 0$ such that for every finite set $\{x_1, \ldots, x_n\} \subseteq E$ we have that

$$\begin{aligned}\Big(\sum_{i=1}^{n} \|T(x_i)\|^p\Big)^{1/p} = \Big(\sum_{i=1}^{n} \|T(x_i)\|^p\Big)^{1/p} &= \Big(\sum_{i=1}^{n} d_{\|\cdot\|}(T(x_i), T(0_E))^p\Big)^{1/p} \\ &\leq K \sup_{x' \in B_{E^*}} \Big(\sum_{i=1}^{n} d_{x'}(x_i, 0_E)^p\Big)^{1/p} \\ &= K \sup_{x' \in B_{E^*}} \Big(\sum_{i=1}^{n} |\langle x_i, x'\rangle|^p\Big)^{1/p}.\end{aligned}$$

Conversely, if T is p-summing, given $n \in \mathbb{N}$ and $a_1, ..., a_n, b_1, ..., b_n \in E$ we have

$$\begin{aligned}\Big(\sum_{i=1}^{n} d_{\|\cdot\|}(T(a_i), T(b_i))^p\Big)^{1/p} &= \Big(\sum_{i=1}^{n} \|T(a_i - b_i)\|^p\Big)^{1/p} \\ &\leq K \sup_{x' \in B_{E^*}} \Big(\sum_{i=1}^{n} |\langle a_i - b_i, x'\rangle|^p\Big)^{1/p} \\ &= K \sup_{x' \in B_{E^*}} \Big(\sum_{i=1}^{n} d_{x'}(a_i, b_i)^p\Big)^{1/p}.\end{aligned}$$

Theorem 4.6. *Let D be a nonempty set and $\mathcal{D}$ be a pointwise uniformly bounded family of pseudometrics on D such that the topology of pointwise convergence on $\mathcal{D}$ is compact. Let (E, ρ) be a pseudometric space and $f : (D, d_{\mathcal{D},\infty}) \to (E, \rho)$ be a Lipschitz map. The following statements are equivalent.*

(1) *The function f is Lipschitz-p-concave with constant $K > 0$.*

(2) *There is a regular Borel probability measure μ over $\mathcal{D}$ such that*

$$\rho(f(a), f(b)) \leq K\Big(\int_{\mathcal{D}} d(a,b)^p \, d\mu(d)\Big)^{1/p}$$

for every $a, b \in D$. That is, f is K-Lipschitz for a p-aggregation $d_{\mathcal{D},p,\mu}$ of $\mathcal{D}$.

The result is a concrete consequence of Corollary 3.6 for the easiest case $m = 1$ and $r = p$. However, this result can be proved using other results that can be found in the references. All of them use a Hahn-Banach separation argument, or—equivalently in this case—Ky Fan's Lemma (see for example [10, 9.10]). The original proof of the Pietsch's Domination Theorem for p-summing linear operators (see [10, 2.12]) can be adapted to get a proof. It is also a direct corollary of the unified Pietsch domination theorem provided in [4, Th. 2.2]. Therefore, the first consequence of the above theorem is the classical Pietsch's Theorem.

Remark 4.7 (Pietsch's Domination Theorem [10, p.44])**.** A linear operator $T : E \to F$ between Banach spaces is p-summing if and only if there is a regular Borel measure μ on B_{E^*}, when it is endowed with the weak* topology, such that for all $x \in E$,

$$\|T(x)\| \leq K\Big(\int_{B_{E^*}} |\langle x, x'\rangle|^p d\mu(x')\Big)^{1/p}.$$

This is a direct consequence of Remark 4.5, the linearity of T and how the metric is defined in a Banach space

Let us show two easy examples with families of metrics satisfying the requirements for the application of Theorem 4.6.

Example 4.8. Consider the Banach spaces of sequences ℓ_1 and ℓ_2, and recall that $\ell_1 \subset \ell_2$. Take $D = B_{\ell_1}$ and define the family $\mathcal{D} = \{d_\tau : \tau \in [0,1]\}$ of pseudometrics on D where

$$d_\tau(a,b) = \tau\|b-a\|_{\ell_1} + (1-\tau)\|b-a\|_{\ell_2}$$

for every $a, b \in D$ and every $\tau \in [0,1]$. We next check that $\mathcal{D}$ endowed with the pointwise topology $\mathcal{O}$ is compact. In fact, the function $h : [0,1] \to \mathcal{D}$ given by $h(\tau) = d_\tau$ for all $\tau \in [0,1]$ is a homeomorphism. We only have to take into account that the following equation holds:

$$|d_\alpha(a,b) - d_\lambda(a,b)| = (\|b-a\|_{\ell_1} + \|b-a\|_{\ell_2})|\lambda - \alpha|$$

for every $\alpha, \lambda \in [0,1]$ and every $a, b \in D$. This gives that a net $(\tau_\lambda)_\lambda$ in $[0,1]$ converges to τ if and only if $(d_{\tau_\lambda})_\lambda$ is pointwise convergent to d_τ.

Let $(E, \|\cdot\|_E)$ be a Banach space and take a Lipschitz operator $T : D \to E$ of Lipschitz constant equal to one. Then

$$\|T(a) - T(b)\|_E \leq \|b-a\|_{\ell_2} \leq \tau\|b-a\|_{\ell_1} + (1-\tau)\|b-a\|_{\ell_2} = d_\tau(a,b)$$

for all $\tau \in [0,1]$. Thus, it can be considered as a Lipschitz map from $(D, d_{\mathcal{D},\infty})$ to $(E, \|\cdot\|)$. For $0 < p < \infty$, and using this inequality for computing the next inequality term-wise, we get for every set of elements $a_1, ..., a_n, b_1, ..., b_n \in B_{\ell_1}$,

$$\sum_{i=1}^n \|T(a_i) - T(b_i)\|_E^p \leq \sum_{i=1}^n d_\tau(a_i, b_i)^p \leq \sup_{\tau \in [0,1]} \Big(\sum_{i=1}^n d_\tau(a_i, b_i)^p\Big).$$

Thus, T is Lipschitz-p-concave with constant $K \leq 1$.

Example 4.9. Consider a bounded set D of $\mathbb{R}^2$, and the family of pseudometrics $\mathcal{D}$ on D defined as

$$d_\tau((x_1,x_2),(x_1',x_2')) = \tau|x_1 - x_1'| + (1-\tau)|x_2 - x_2'|$$

for all $\tau \in [0,1]$, $(x_1,x_2),(x_1',x_2') \in D$. Take

$$M := \sup_{(x_1,x_2)\in D} 2|x_1| + \sup_{(x_1,x_2)\in D} 2|x_2|,$$

and consider the function $f : D \to \mathbb{R}$ given by $f(x_1,x_2) = x_1^2 + x_2^2$, $(x_1,x_2) \in D$. Given a finite family $\{(x_{1,i},x_{2,i}),(x_{1,i}',x_{2,i}') : i \in \{1,\dots,n\}\}$ of elements in D, then

$$\sum_{i=1}^n \big|f(x_{1,i},x_{2,i}) - f(x_{1,i}',x_{2,i}')\big| = \sum_{i=1}^n \big|(x_{1,i}^2 - x_{1,i}'^2) + (x_{2,i}^2 - x_{2,i}'^2)\big|$$

$$= \sum_{i=1}^n \big|(x_{1,i}+x_{1,i}')(x_{1,i}-x_{1,i}') + (x_{2,i}+x_{2,i}')(x_{2,i}-x_{2,i}')\big|$$

$$\le \max_{i=1,\dots,n} \big|x_{1,i}+x_{1,i}'\big|\Big(\sum_{i=1}^n \big|x_{1,i}-x_{1,i}'\big|\Big) + \max_{i=1,\dots,n} \big|x_{2,i}+x_{2,i}'\big|\Big(\sum_{i=1}^n \big|x_{2,i}-x_{2,i}'\big|\Big).$$

Take $\tau_0 = \frac{\max_{i=1,\dots,n} \left|x_{1,i}+x_{1,i}'\right|}{\left(\max_{i=1,\dots,n}\left|x_{1,i}+x_{1,i}'\right| + \max_{i=1,\dots,n}\left|x_{2,i}+x_{2,i}'\right|\right)}$. Then we have that

$$\sum_{i=1}^n \big|f(x_{1,i},x_{2,i}) - f(x_{1,i}',x_{2,i}')\big|$$

$$\le M\left(\tau_0\Big(\sum_{i=1}^n \big|x_{1,i}-x_{1,i}'\big|\Big) + (1-\tau_0)\Big(\sum_{i=1}^n \big|x_{2,i}-x_{2,i}'\big|\Big)\right)$$

$$= M\sum_{i=1}^n d_{\tau_0}((x_{1,i},x_{2,i}),(x_{1,i}',x_{2,i}')) \le M \sup_{d_\tau\in\mathcal{D}} \sum_{i=1}^n d_\tau((x_{1,i},x_{2,i}),(x_{1,i}',x_{2,i}')).$$

Therefore, the function f is Lipschitz 1-concave with constant less than or equal to M.

For the particular case $\rho = d_{\mathcal{D},\infty}$ and f the identity map, we get the following direct result.

Corollary 4.10. *Let D be a nonempty set and $\mathcal{D}$ be a family of pseudometrics on D such that the topology of pointwise convergence on $\mathcal{D}$ is compact. Let (E,ρ) be a pseudometric space and $f : (D, d_{\mathcal{D},\infty}) \to (E,\rho)$ be a Lipschitz map. Then the following statements are equivalent.*

(1) *The identity map $Id : (D, d_{\mathcal{D},\infty}) \to (D, d_{\mathcal{D},\infty})$ is Lipschitz-p-concave with constant $K > 0$.*
(2) *The metric $d_{\mathcal{D},\infty}$ is equivalent to a p-aggregation $d_{\mathcal{D},p,\mu}$ of $\mathcal{D}$.*

Proof. Assume (1). By Theorem 4.6, the identity map is Lipschitz-p-concave, if and only if there is a constant $K > 0$ and a probability measure μ over $\mathcal{D}$ such that

$$d_{\mathcal{D},\infty}(a,b) \le K\left(\int_{\mathcal{D}} d(a,b)^p d\mu(d)\right)^{1/p} \quad a,b \in D.$$

On the other hand, it is obvious that

$$\left(\int_{\mathcal{D}} d(a,b)^p d\mu(d)\right)^{1/p} \le \sup_{d\in\mathcal{D}} d(a,b), \quad a,b \in D,$$

and so we get (2). The converse is a direct application of the implication (2) $\Rightarrow$ (1) of Theorem 4.6. □

For instance, the family presented in Example 4.8(i) satisfies the requirements in Corollary 4.10, and so $d_{\mathcal{D},\infty}$ is equivalent to a p-aggregation.

5. Application: McShane-Whitney uniform extensions of Lipschitz-p-concave functions

Consider a uniformly bounded family $\mathcal{D}$ of pseudometrics on a nonempty set D. Let S be a subset of D, that is considered as a pseudometric subspace of all the spaces (D, d), where $d \in \mathcal{D}$. Suppose that the function $f : (S, d) \to \mathbb{R}$ is a K_d-Lipschitz map, such that the set of all the Lipschitz constants $\{K_d : d \in \mathcal{D}\}$ is uniformly bounded, and $\inf_d K_d = K$. Since the set $\mathcal{D}$ is uniformly bounded, we have that $f : (S, d_{\mathcal{D},\infty}) \to \mathbb{R}$ is K-Lipschitz, and so the McShane-Whitney extension theorem for Lipschitz real-valued maps gives an extension to D.

We will show in what follows that this result can also be obtained for extensions of Lipschitz-p-concave functions. We prove in this way that the McShane-Whitney extension theorem for Lipschitz real maps also works for real operators that are Lipschitz-p-concave, preserving this property. So, using the characterization of the domination of a Lipschitz map by a p-aggregation of a family of pseudometrics provided by Theorem 4.6, we get the following result.

Corollary 5.1. *Let D be a nonempty set and $\mathcal{D}$ be a family of pseudometrics on D such that the topology of pointwise convergence on $\mathcal{D}$ is compact. Let $S \subseteq D$. Let $f : (S, d_{\mathcal{D},\infty}) \to (\mathbb{R}, d_{|\cdot|})$ be a Lipschitz map that is Lipschitz-p-concave with constant $K > 0$.*

Then there is a Lipschitz extension of f to D that is also Lipschitz-p-concave with the same constant $K > 0$.

Proof. The proof of the result is direct after Theorem 4.6. Under the given assumptions, there is a Borel regular measure μ over $\mathcal{D}$ such that

$$\big|f(a) - f(b)\big| \leq K \left(\int_{d\in\mathcal{D}} d(a,b)^p \, d\mu(d) \right)^{1/p} = K d_{\mathcal{D},p,\mu}(a,b)$$

for all $a, b \in S$. Then f is Lipschitz in the metric space $(S, d_{\mathcal{D},p,\mu})$. Note that $d_{\mathcal{D},p,\mu}$ is a pseudometric on D, and then the classical McShane-Whitney Theorem provides a K-Lipschitz extension to D. Again an application of Theorem 4.6 —the other way round—, gives the result.

□

References

[1] E. Dahia, D. Achour, P. Rueda, E. A. Sánchez Pérez and R. Yahi, Factorization of Lipschitz operators on Banach function spaces, Mathematical Inequalities and Applications **21** (2018), no.4, 1091–1104.

[2] G. Beliakov, Optimization and aggregation functions, in: Fuzzy optimization: Recent advances and applications. W. A. Lodwick and J. Kacprzyk, eds. pp 77-108. Springer, Berlin, 2010.

[3] G. Beliakov, H. Bustince Sola and T. Calvo Sánchez, A practical guide to averaging functions, vol. 329. Heidelberg: Springer, 2016.

[4] G. Botelho, D. Pellegrino, and P. Rueda, A unified Pietsch domination theorem, J. Math. Anal. Appl. 365 (2010), no. 1, 269–276.

[5] G. Botelho, D. Pellegrino and P. Rueda, On Pietsch measures for summing operators and dominated polynomials. Linear Multilinear Algebra 62(7) (2014), 860–874.

[6] J. A. Chávez-Domínguez, Duality for Lipschitz p-summing operators, J. Funct. Anal. 261 (2011), 387–407.
[7] J. A. Chávez-Domínguez, Lipschitz p-convex and q-concave maps. arXiv preprint arXiv:1406.6357. (2014).
[8] A. Defant, Variants of the Maurey-Rosenthal theorem for quasi Köthe function spaces. Positivity. 5 (2001), 153–175.
[9] A. Defant and K. Floret, Tensor norms and operator ideals, Elsevier. Amsterdam, 1992.
[10] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators, Cambridge University Press. Cambridge, 1995.
[11] J. Dugundji, Topology, Allyn and Bacon Inc., 1966.
[12] J. Farmer and W. Johnson, Lipschitz p-summing operators, Proc. Amer. Math. Soc. 137, no. 9 (2009), 2989–2995.
[13] P. Juutinen, Absolutely minimizing Lipschitz extensions on a metric space, Annales Academiae Scientiarum Fennicae Mathematica 27 (2002), 57–67.
[14] C. Mustata, Extensions of semi-Lipschitz functions on quasi-metric spaces, Rev. Anal. Numer. Theor. Approx. 30, no. 1 (2001), 61–67.
[15] C. Mustata, On the extremal semi-Lipschitz functions, Rev. Anal. Numer. Theor. Approx. 31, no. 1 (2002), 103–108.
[16] D. Pellegrino and J. Santos, Absolutely summing multilinear operators: a panorama, Quaestiones Mathematicae 34(4) (2011), 447–478.
[17] S. Romaguera and M. Sanchis, Semi-Lipschitz functions and best approximation in quasi-metric spaces, Journal of Approximation Theory 103 (2000), 292–301.
[18] J. T. Schwartz, Nonlinear functional analysis. New York: Gordon and Breach Science, 1969.
[19] S. Willard, General Topology, Addison-Wesley, 1970.
[20] R. Yahi, D. Achour and P. Rueda, Absolutely summing Lipschitz conjugates, Mediterranean Journal of Mathematics 13, no. 4 (2016), 1949–1961.

J. Rodríguez-López , E. A. Sánchez Pérez, Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia. Spain.

E-mail address: jrlopez@mat.upv.es, easancpe@mat.upv.es