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On Grothendieck Sets

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Abstract: We call a subset $\mathcal{M}$ of an algebra of sets $\mathcal{A}$ a *Grothendieck set* for the Banach space $ba(\mathcal{A})$ of bounded finitely additive scalar-valued measures on $\mathcal{A}$ equipped with the variation norm if each sequence $\{\mu_n\}_{n=1}^{\infty}$ in $ba(\mathcal{A})$ which is pointwise convergent on $\mathcal{M}$ is weakly convergent in $ba(\mathcal{A})$, i. e., if there is $\mu \in ba(\mathcal{A})$ such that $\mu_n(A) \to \mu(A)$ for every $A \in \mathcal{M}$ then $\mu_n \to \mu$ weakly in $ba(\mathcal{A})$. A subset $\mathcal{M}$ of an algebra of sets $\mathcal{A}$ is called a *Nikodým set* for $ba(\mathcal{A})$ if each sequence $\{\mu_n\}_{n=1}^{\infty}$ in $ba(\mathcal{A})$ which is pointwise bounded on $\mathcal{M}$ is bounded in $ba(\mathcal{A})$. We prove that if Σ is a σ-algebra of subsets of a set Ω which is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets of Σ there exists $p \in \mathbb{N}$ such that Σ_p is a Grothendieck set for $ba(\mathcal{A})$. This statement is the exact counterpart for Grothendieck sets of a classic result of Valdivia asserting that if a σ-algebra Σ is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets, there is $p \in \mathbb{N}$ such that Σ_p is a Nikodým set for $ba(\Sigma)$. This also refines the Grothendieck result stating that for each σ-algebra Σ the Banach space $\ell_\infty(\Sigma)$ is a Grothendieck space. Some applications to classic Banach space theory are given.

Keywords: property (G); rainwater set; property (N); Nikodým set; property (VHS)

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1. Introduction

With a different terminology, Valdivia showed in [1] that if a σ-algebra Σ of subsets of a set Ω is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets, there is $p \in \mathbb{N}$ such that Σ_p is a Nikodým set for $ba(\Sigma)$. We prove that if Σ is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets of Σ there is $p \in \mathbb{N}$ such that Σ_p is a Grothendieck set for $ba(\mathcal{A})$ (definitions below). This statement is both the exact counterpart for Grothendieck sets of Valdivia's result for Nikodým sets and a refinement of Grothendieck's classic result stating that the Banach space $\ell_\infty(\Sigma)$ of bounded scalar-valued Σ-measurable functions defined on Ω equipped with the supremum-norm is a Grothendieck space. Our previous result applies easily to Banach space theory to extend some well-known results. For example, Phillip's lemma can be read as follows. If $\{\Sigma_n : n \in \mathbb{N}\}$ is an increasing sequence of subsets of Σ covering Σ, there is $p \in \mathbb{N}$ such that if $\{\mu_n\}_{n=1}^{\infty} \subseteq ba(\Sigma)$ verifies $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \Sigma_p$ and $\{A_k : k \in \mathbb{N}\}$ is a sequence of pairwise disjoint elements of Σ, then $\lim_{n\to\infty} \sum_{k=1}^{\infty} |\mu_n(A_k)| = 0$.

2. Preliminaries

In what follow we use the notation of [2] (Chapter 5). Let $\mathcal{R}$ be a ring of subsets of a nonempty set Ω, χ_A be the characteristic function of the set $A \in \mathcal{R}$ and let $\ell_0^{\infty}(\mathcal{R}) = \operatorname{span}\{\chi_A : A \in \mathcal{R}\}$ denote the linear space of all $\mathbb{K}$-valued $\mathcal{R}$-simple functions, $\mathbb{K}$ being the scalar field of real or complex numbers. Since $A \cap B \in \mathcal{R}$ and $A \Delta B \in \mathcal{R}$ whenever $A, B \in \mathcal{R}$, for each $f \in \ell_0^{\infty}(\mathcal{R})$ there are pairwise disjoint sets $A_1, \ldots, A_m \in \mathcal{R}$ and nonzero $a_1, \ldots, a_m \in \mathbb{K}$, with $a_i \neq a_j$ if $i \neq j$ such that $f = \sum_{i=1}^m a_i \chi_{A_i}$, with $f = \chi_\emptyset$ if $f = 0$. Unless otherwise stated we shall assume $\ell_0^\infty(\mathcal{R})$ equipped with the norm $\|f\|_\infty = \sup\{|f(\omega)| : \omega \in \Omega\}$. If $Q = \text{abx}\{\chi_A : A \in \mathcal{R}\}$ is the absolutely convex hull of $\{\chi_A : A \in \mathcal{R}\}$, an equivalent norm is defined on $\ell_0^\infty(\mathcal{R})$ by the *gauge* of Q, namely $\|f\|_Q = \inf\{\lambda > 0 : f \in \lambda Q\}$. For if $f \in \ell_0^\infty(\mathcal{R})$ with $\|f\|_\infty \leq 1$, it can be shown that $f \in 4Q$ (*cf.* [2] (Proposition 5.1.1)), hence $\|\cdot\|_\infty \leq \|\cdot\|_Q \leq 4\|\cdot\|_\infty$.

The dual of $\ell_0^\infty(\mathcal{R})$ is the Banach space $ba(\mathcal{R})$ of bounded finitely additive scalar-valued measures on $\mathcal{R}$, which we shall assume to be equipped with the variation norm

$$|\mu| = \sup \sum_{i=1}^n |\mu(E_i)|,$$

where the supremum is taken over all finite sequences of pairwise disjoint members of $\mathcal{R}$. This is the dual of the supremum-norm $\|\cdot\|_\infty$ of $\ell_0^\infty(\mathcal{R})$. An equivalent norm is given by $\|\mu\| = \sup\{|\mu(A)| : A \in \mathcal{R}\}$, which is the dual norm of the gauge $\|\cdot\|_Q$. We shall also consider the Banach space $ba(\mathcal{R})^*$ equipped with the bidual norm $\|\cdot\|$ of $\|\cdot\|_\infty$. The completion of the normed space $(\ell_0^\infty(\mathcal{R}), \|\cdot\|_\infty)$ is the Banach space $\ell_\infty(\mathcal{R})$ of all bounded $\mathcal{R}$-measurable functions.

The Banach space $\ell_\infty(\mathcal{R})$ embeds isometrically into $ba(\mathcal{R})^*$, hence each characteristic function χ_A in $\ell_0^\infty(\mathcal{R})$ with $A \in \mathcal{R}$ can be considered as a bounded linear functional on $ba(\mathcal{R})$ defined by evaluation $\langle \chi_A, \mu \rangle = \mu(A)$. So, we may write $\{\chi_A : A \in \mathcal{R}\} \subseteq S_{ba(\mathcal{R})^*}$, where $S_{ba(\mathcal{R})^*}$ stands for the unit sphere of $ba(\mathcal{R})^*$, and the set $\{\chi_A : A \in \mathcal{R}\}$, regarded as a topological subspace of $ba(\mathcal{R})^*$ (weak*), is the same as $\{\chi_A : A \in \mathcal{R}\}$ regarded as a topological subspace of $\ell_0^\infty(\mathcal{R})$ (weak).

A subfamily $\mathcal{F}$ of an algebra of sets $\mathcal{A}$ is called a *Nikodým set* for $ba(\mathcal{A})$ (*cf.* [3]) if each set $\{\mu_\alpha : \alpha \in \Lambda\}$ in $ba(\mathcal{A})$ which is pointwise bounded on $\mathcal{F}$ is bounded in $ba(\mathcal{A})$, i. e., if $\sup_{\alpha\in\Lambda} |\mu_\alpha(A)| < \infty$ for each $A \in \mathcal{F}$ implies that $\sup_{\alpha\in\Lambda} |\mu_\alpha| < \infty$. The algebra $\mathcal{A}$ is said to have *property* (N) if the whole family $\mathcal{A}$ is a Nikodým set for $ba(\mathcal{A})$. Nikodým's classic boundedness theorem establishes that every σ-algebra has property (N). An algebra $\mathcal{A}$ is said to have *property* (G) if $\ell_\infty(\mathcal{A})$ is a Grothendieck space, i. e., if each weak* convergent sequence in $ba(\mathcal{A})$ is weakly convergent in the Banach space $ba(\mathcal{A})$. The fact that every σ-algebra has property (G) is also due to Grothendieck. Every countable algebra lacks property (N), and the algebra $\mathfrak{J}$ of Jordan-measurable subsets of the real interval $[0,1]$ has property (N) but fails property (G) (*cf.* [4] (Propositions 3.2 and 3.3) and [5]). Let us recall that a sequence $\{\mu_n\}_{n=1}^\infty$ in $ba(\mathcal{A})$ is *uniformly exhaustive* if for each sequence $\{A_i : i \in \mathbb{N}\}$ of pairwise disjoint elements of $\mathcal{A}$ it holds that $\lim_{k\to\infty} \sup_{n\in\mathbb{N}} |\mu_n(A_k)| = 0$. We shall use the following result, originally stated in [4] (2.3 Definition).

Theorem 1. *An algebra of sets $\mathcal{A}$ has property (G) if and only if every bounded sequence $\{\mu_n\}_{n=1}^\infty$ in $ba(\mathcal{A})$ which converges pointwise on $\mathcal{A}$ is uniformly exhaustive.*

An algebra $\mathcal{A}$ is said to have *property* (VHS) if every sequence $\{\mu_n\}_{n=1}^\infty$ in $ba(\mathcal{A})$ which converges pointwise on $\mathcal{A}$ is uniformly exhaustive. It should be mentioned that $(VHS) \Leftrightarrow (N) \wedge (G)$, where the proof of the non-trivial implication can be found in [6] (see also [7] (Theorem 4.2)). For later use we introduce the following definition.

Definition 1. *A subfamily $\mathcal{M}$ of an algebra of sets $\mathcal{A}$ will be called a Grothendieck set for $ba(\mathcal{A})$ if each sequence $\{\mu_n\}_{n=1}^\infty$ in $ba(\mathcal{A})$ which is pointwise convergent on $\mathcal{M}$ is weakly convergent in $ba(\mathcal{A})$, i. e., if there is $\mu \in ba(\mathcal{A})$ such that $\mu_n(A) \to \mu(A)$ for every $A \in \mathcal{M}$ then $\mu_n \to \mu$ weakly in $ba(\mathcal{A})$.*

If an algebra $\mathcal{A}$ contains a Grothendieck subset for $ba(\mathcal{A})$, clearly $\mathcal{A}$ has property (G). Grothendieck sets are closely related to the so-called Rainwater sets (defined below) for $ba(\mathcal{A})$, and the study of the Rainwater sets for $ba(\mathcal{A})$ leads to Theorem 4 below, from which the following result is a straightforward corollary.

Theorem 2. *If Σ is a σ-algebra of subsets of a set Ω which is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets of Σ there exists $p \in \mathbb{N}$ such that Σ_p is a Grothendieck set for $ba(\Sigma)$.*

Indeed, in [1] (Theorem 1) Valdivia showed that if a σ-algebra Σ of subsets of a set Ω is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets (subfamilies) of Σ, there exists some $p \in \mathbb{N}$ such that Σ_p is a Nikodým set for $ba(\Sigma)$ or, equivalently, that given an increasing sequence $\{E_n : n \in \mathbb{N}\}$ of linear subspaces of $\ell_0^\infty(\Sigma)$ covering $\ell_0^\infty(\Sigma)$, there exists $p \in \mathbb{N}$ such that E_p is dense and barrelled (see also [8] (Theorem 3)).

As a consequence of Theorem 4 we show that if a σ-algebra Σ is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets, there exists some $p \in \mathbb{N}$ such that $\{\chi_A : A \in \Sigma_p\}$, regarded as a subset of the dual unit ball of $ba(\Sigma)$, is also a Rainwater set for $ba(\Sigma)$. This easily implies Theorem 2. In the last section we give some applications of Theorem 2 to classic Banach space theory which seems to have gone unnoticed so far. Let us point out that some results of this paper hold for Boolean algebras [9] (Theorem 12.35).

3. Rainwater Sets for $ba(\mathcal{A})$

A subset X of the dual closed unit ball B_{E^*} of a Banach space E is called a *Rainwater set* for E if every *bounded* sequence $\{x_n\}_{n=1}^\infty$ of E that converges pointwise on X, i. e., such that $x^* x_n \to x^* x$ for each $x^* \in X$, converges weakly in E (*cf.* [10]). Rainwater's classic theorem [11] asserts that the set of the extreme points of the closed dual unit ball of a Banach space E is a Rainwater set for E. According to [12] (Corollary 11), each *James boundary* of E is a Rainwater set for E. As regards the Banach space $C(X)$ of real-valued continuous functions over a *compact Hausdorff space* X equipped with the supremum norm, if $K = \operatorname{Ext} B_{C(X)^*}$ is the set of the extreme points of the compact subset $B_{C(X)^*}$ of $C(X)^*$ (weak*), the Arens-Kelly theorem asserts that $K = \{\pm\delta_x : x \in X\}$ (see [13]). By the Lebesgue dominated convergence theorem, if $\{f_n\}_{n=1}^\infty$ is a norm-bounded sequence in $C(X)$ (with respect to the supremum-norm) then $f_n \to f$ weakly in $C(X)$ if and only if $f_n(x) \to f(x)$ for every $x \in X$, that is, $\langle f_n, \mu\rangle \to \langle f, \mu\rangle$ for every $\mu \in C(X)^*$ if and only if $\langle f_n, \delta_v\rangle \to \langle f, \delta_v\rangle$ for each $v \in K$ (see [14] (IV.6.4 Corollary)). This is Rainwater's theorem for $C(X)$. In [10] the weak K-analyticity of the Banach space $C^b(X)$ of real-valued continuous and bounded functions defined on a completely regular space X equipped with the supremum norm is characterized in terms of certain Rainwater sets for $C^b(X)$. The next theorem, based on [3] (Proposition 4.1), exhibits a connection between Rainwater sets and property (G). We include it for future reference and provide a proof for the sake of completeness.

Theorem 3. *Let $\mathcal{A}$ be an algebra of sets. The following are equivalent*

1. *$\mathcal{A}$ has property (G).*
2. *$\{\chi_A : A \in \mathcal{A}\}$ is a Rainwater set for $ba(\mathcal{A})$, considered as a subset of $ba(\mathcal{A})^*$.*
3. *The unit ball of $\ell_0^\infty(\mathcal{A})$ is a Rainwater set for $ba(\mathcal{A})$.*
4. *The unit ball of $\ell_\infty(\mathcal{A})$ is a Rainwater set for $ba(\mathcal{A})$.*

Proof. 1 $\Rightarrow$ 2. Assume that $\mathcal{A}$ has property (G) and let $\{\mu_n\}_{n=1}^\infty$ be a bounded sequence in $ba(\mathcal{A})$ and $\mu \in ba(\mathcal{A})$ such that $\langle \chi_A, \mu_n\rangle \to \langle \chi_A, \mu\rangle$ for each $A \in \mathcal{A}$. i. e., such that $\mu_n(A) \to \mu(A)$ for each $A \in \mathcal{A}$. By Theorem 1 the sequence $M = \{\mu_n : n \in \mathbb{N}\}$ is (bounded and) uniformly exhaustive on $\mathcal{A}$, so [15] (Corollary 5.2) produces a nonnegative real-valued finitely-additive measure λ on $\mathcal{A}$ such that $\lim_{\lambda(E)\to 0} \sup_{n\in\mathbb{N}} |\mu_n(E)| = 0$. Hence, [14] (4.9.12 Theorem]) shows that M is relatively weakly sequentially compact. Given that $\mu_n(A) \to \mu(A)$ for each $A \in \mathcal{A}$, necessarily μ is the only possible weakly adherent point of the sequence $\{\mu_n\}_{n=1}^\infty$. So we get that $\mu_n \to \mu$ weakly in $ba(\mathcal{A})$, which shows that $\{\chi_A : A \in \mathcal{A}\}$ is a Rainwater set for $ba(\mathcal{A})$.

2 $\Rightarrow$ 3. If $B_{ba(\mathcal{A})^*}$ denotes the second dual ball of the closed unit ball $B_{\ell_\infty(\mathcal{A})}$ of $\ell_\infty(\mathcal{A})$ and B_0 stands for the unit ball of $\ell_0^\infty(\mathcal{A})$, from the relations $\{\chi_A : A \in \mathcal{A}\} \subseteq B_0 \subseteq B_{ba(\mathcal{A})^*}$ it follows that B_0 is a also Rainwater set for $ba(\mathcal{A})$.

3 $\Rightarrow$ 4 is obvious.

4 ⇒ 1. If $\mu_n \to \mu$ in $ba(\mathcal{A})$ under the weak* topology $\sigma(ba(\mathcal{A}), \ell_\infty(\mathcal{A}))$ of $ba(\mathcal{A})$ then $\{\mu_n\}_{n=1}^\infty$ is a bounded sequence in $ba(\mathcal{A})$. Given that $\langle \mu_n, f\rangle \to \langle \mu, f\rangle$ for every $f \in B_{\ell_\infty(\mathcal{A})}$ and given the hypothesis that $B_{\ell_\infty(\mathcal{A})}$ is a Rainwater set for $ba(\mathcal{A})$, we have that $\mu_n \to \mu$ weakly in $ba(\mathcal{A})$. Consequently $\mathcal{A}$ has property (G). □

Example 1. *If $\mathcal{Z}$ stands for the algebra generated by the sets of density zero in $\mathbb{N}$, then $\{\chi_A : A \in \mathcal{Z}\}$ is not a Rainwater set for $ba(\mathcal{Z})$. This follows from the previous theorem and from the fact that $\mathcal{Z}$ does not have property (G) (see [16]).*

Theorem 4. *Assume that $\mathcal{A}$ is an algebra of sets. Let $\mathcal{M}$ be a Nikodým subset for $ba(\mathcal{A})$ and let $\{\mathcal{M}_n : n \in \mathbb{N}\}$ be an increasing covering of $\mathcal{M}$ by subsets of $\mathcal{M}$. If $\{\chi_A : A \in \mathcal{M}\}$ is a Rainwater set for $ba(\mathcal{A})$, there exists some $p \in \mathbb{N}$ such that $\{\chi_A : A \in \mathcal{M}_p\}$ is a Rainwater set for $ba(\mathcal{A})$.*

Proof. Assume that $\{\chi_A : A \in \mathcal{M}\}$ is a Rainwater set for $ba(\mathcal{A})$. First we claim that

$$\{\chi_A : A \in \mathcal{A}\} \subseteq \bigcup_{n=1}^\infty n \cdot \overline{\operatorname{abx}\{\chi_A : A \in \mathcal{M}_n\}}^{\|\cdot\|_\infty}$$

Let us proceed by contradiction. Assume otherwise that there exists $B \in \mathcal{A}$ such that $\chi_B \notin n \cdot \overline{\operatorname{abx}\{\chi_A : A \in \mathcal{M}_n\}}^{\|\cdot\|_\infty}$ for all $n \in \mathbb{N}$. In this case the separation theorem provides $\mu_n \in ba(\mathcal{A})$ with $|\mu_n(B)| = 1$ such that

$$\sup\left\{|\langle f, \mu_n\rangle| : f \in \overline{\operatorname{abx}\{\chi_A : A \in \mathcal{M}_n\}}^{\|\cdot\|_\infty}\right\} \le \frac{1}{n}$$

So, in particular it holds that

$$\sup\{|\mu_n(A)| : A \in \mathcal{M}_n\} \le \frac{1}{n}$$

for every $n \in \mathbb{N}$. If $M \in \mathcal{M}$ there is $k \in \mathbb{N}$ such that $M \subseteq \mathcal{M}_n$ for every $n \ge k$. Consequently $|\mu_n(M)| \le \frac{1}{n}$ for $n \ge k$, which shows that $\mu_n(M) \to 0$. Since $\mathcal{M}$ is a Nikodým set and $\{\mu_n\}_{n=1}^\infty$ is pointwise bounded on $\mathcal{M}$, it follows that $\{\mu_n\}_{n=1}^\infty$ is bounded in $ba(\mathcal{A})$. So, the fact that $\mu_n(M) \to 0$ for all $M \in \mathcal{M}$ along with the assumption that $\mathcal{M}$ is a Rainwater set leads to $\mu_n \to 0$ weakly in $ba(\mathcal{A})$. This is a contradiction, since $\langle \chi_B, \mu_n\rangle = \mu_n(B) = 1$ for every $n \in \mathbb{N}$. The claim is proved.

Set $Q := \{\chi_A : A \in \mathcal{A}\}$. Since we are assuming that $\mathcal{M}$ is a Nikodým set for $ba(\mathcal{A})$, the larger set $\mathcal{A}$ is also a Nikodým set for $ba(\mathcal{A})$, which implies that $\ell_0^\infty(\mathcal{A})$ is a metrizable barrelled space, hence a Baire-like space (see [17]). On the other hand, as a consequence of the previous claim, the family $\{W_n\}_{n=1}^\infty$ with

$$W_n := n \cdot \overline{\operatorname{abx}\{\chi_A : A \in \mathcal{M}_n\}}^{\|\cdot\|_\infty}$$

is an increasing sequence of closed absolutely convex sets covering $\ell_0^\infty(\mathcal{A})$. So, there exists $p \in \mathbb{N}$ such that

$$Q \subseteq p \cdot \overline{\operatorname{abx}\{\chi_A : A \in \mathcal{M}_p\}}^{\|\cdot\|_\infty},$$

which shows that

$$\overline{\operatorname{abx}\{\chi_A : A \in \mathcal{M}_p\}}^{\|\cdot\|_\infty}$$

is a Rainwater set for $ba(\mathcal{A})$.

We claim that this implies that $\{\chi_A : A \in \mathcal{M}_p\}$ is a Rainwater set for $ba(\mathcal{A})$. In order to establish the claim it suffices to show that $\operatorname{abx}\{\chi_A : A \in \mathcal{M}_p\}$ is a Rainwater set for $ba(\mathcal{A})$. So, let $\{\lambda_n\}_{n=1}^\infty$ be a bounded sequence in $ba(\mathcal{A})$ such that $\langle u, \lambda_n\rangle \to 0$ for every $u \in \operatorname{abx}\{\chi_A : A \in \mathcal{M}_p\}$. Let us show that $\langle v, \lambda_n\rangle \to 0$ for each $v \in \overline{\operatorname{abx}\{\chi_A : A \in \mathcal{M}_p\}}^{\|\cdot\|_\infty}$. If $v \in \overline{\operatorname{abx}\{\chi_A : A \in \mathcal{M}_p\}}^{\|\cdot\|_\infty}$ there exists a sequence $\{u_k\}_{k=1}^{\infty}$ in abx $\{\chi_A : A \in \mathcal{M}_p\}$ such that $\|u_k - v\|_\infty \to 0$. Consequently, given $\epsilon > 0$ there is $k(\epsilon) \in \mathbb{N}$ with

$$\left\|u_{k(\epsilon)} - v\right\|_\infty < \frac{\epsilon}{2\left(1 + \sup_{n\in\mathbb{N}} |\lambda_n|\right)}.$$

Let $n(\epsilon) \in \mathbb{N}$ be such that

$$\left|\left\langle u_{k(\epsilon)}, \lambda_n\right\rangle\right| < \frac{\epsilon}{2}$$

for every $n \geq n(\epsilon)$. Consequently, one has

$$|\langle v, \lambda_n\rangle| \leq \left|\left\langle v - u_{k(\epsilon)}, \lambda_n\right\rangle\right| + \left|\left\langle u_{k(\epsilon)}, \lambda_n\right\rangle\right| \leq \left\|u_{k(\epsilon)} - v\right\|_\infty |\lambda_n| + \left|\left\langle u_{k(\epsilon)}, \lambda_n\right\rangle\right| < \epsilon$$

for all $n \geq n_0(\epsilon)$. This proves that $\langle v, \lambda_n\rangle \to 0$ for each $v \in \overline{\text{abx}\{\chi_A : A \in \mathcal{M}_p\}}^{\|\cdot\|_\infty}$. Since we have shown before that $\overline{\text{abx}\{\chi_A : A \in \mathcal{M}_p\}}^{\|\cdot\|_\infty}$ is a Rainwater set for $ba(\mathcal{A})$, we get that $\lambda_n \to 0$ weakly in $ba(\mathcal{A})$. Therefore the absolutely convex set abx $\{\chi_A : A \in \mathcal{M}_p\}$ is a Rainwater set for $ba(\mathcal{A})$, a stated. □

Corollary 1. *Let $\mathcal{A}$ be an algebra of sets with property (VHS). If $\{\mathcal{A}_n : n \in \mathbb{N}\}$ is an increasing covering of $\mathcal{A}$ consisting of subsets of $\mathcal{A}$, there is some $p \in \mathbb{N}$ such that $\{\chi_A : A \in \mathcal{A}_p\}$ is a Rainwater set for ba $(\mathcal{A})$.*

Proof. This is a straightforward consequence of the Theorem 4 for $\mathcal{M} = \mathcal{A}$, since as mentioned earlier an algebra $\mathcal{A}$ has property (VHS) if and only if $\mathcal{A}$ has both properties (N) and (G) (this also can be found in [7] (Theorem 4.2)). So, on the one hand $\mathcal{A}$ is a Nikodým set for $ba(\mathcal{A})$ and, on the other hand, according to Theorem 3, the family $\{\chi_A : A \in \mathcal{A}\}$ is a Rainwater set for $ba(\mathcal{A})$. □

Proof of Theorem 2. If Σ is a σ-algebra of subsets of a set Ω which is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets of Σ, Corollary 1 and Valdivia's result [1] provide an index $p \in \mathbb{N}$ such that Σ_p is a Nikodým set for $ba(\Sigma)$ at the same time that $\{\chi_A : A \in \Sigma_p\}$ is a Rainwater set for $ba(\Sigma)$. If $\{\mu_n\}_{n=1}^{\infty}$ verifies that $\mu_n(A) \to \mu(A)$ for every $A \in \Sigma_p$, the sequence $\{\mu_n\}_{n=1}^{\infty}$ is bounded in $ba(\Sigma)$ since Σ_p is a Nikodým set for $ba(\Sigma)$. But then $\mu_n \to \mu$ weakly in $ba(\Sigma)$ due to $\{\chi_A : A \in \Sigma_p\}$ is a Rainwater set for $ba(\Sigma)$. Consequently Σ_p is a Grothendieck for $ba(\Sigma)$ and we are done. □

Corollary 2. *If $\{\Lambda_n : n \in \mathbb{N}\}$ is an increasing sequence of subsets of $\Sigma = 2^{\mathbb{N}}$ covering $2^{\mathbb{N}}$, there exists some $p \in \mathbb{N}$ such that each sequence $\{\mu_n\}_{n=1}^{\infty}$ in ba $(2^{\mathbb{N}})$ that converges pointwise on Λ_p converges weakly in ba $(2^{\mathbb{N}}) = \ell_\infty^*$.*

Proof. Apply Theorem 2 to the σ-algebra $2^{\mathbb{N}}$. □

We complete our study of Rainwater sets for $ba(\mathcal{A})$ with the following result. Note that if $\overline{X}^{w^*}$ (weak* closure) with $X \subseteq B_{ba(\mathcal{A})^*}$ is a Rainwater set for $ba(\mathcal{A})$ then X could not be a Rainwater set for $ba(\mathcal{A})$. However the following property holds.

Theorem 5. *Let $\mathcal{A}$ be an algebra of sets. Assume that $\{\chi_A : A \in \mathcal{A}\}$ is a Grothendieck set for ba $(\mathcal{A})$. If $\{\chi_A : A \in \mathcal{M}\}$ is a G_δ-dense subset of $\{\chi_A : A \in \mathcal{A}\}$ under the relative weak* topology of ba $(\mathcal{A})^*$ or, which is the same, under the relative weak topology of $\ell_0^\infty(\mathcal{A})$, then $\{\chi_A : A \in \mathcal{M}\}$ is a Grothendieck set for ba $(\mathcal{A})$.*

Proof. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence in $ba(\mathcal{A})$ such that $\mu_n(Q) \to 0$ for every $Q \in \mathcal{M}$. Given $B \in \mathcal{A}$, let us define $G_n := \{\chi_C : C \in \mathcal{A}, \mu_n(C) = \mu_n(B)\}$. Then one has that $\chi_B \in \bigcap_{n=1}^{\infty} G_n$, so that $G := \bigcap_{n=1}^{\infty} G_n$ is a nonempty intersection of countably many zero-sets of $\{\chi_A : A \in \mathcal{A}\}$, hence a non-empty G_δ-set in $\{\chi_A : A \in \mathcal{A}\}$ in the relative weak topology of $\ell_0^\infty(\mathcal{A})$. According to the hypothesis G meets $\{\chi_A : A \in \mathcal{M}\}$. Hence there exists $M_B \in \mathcal{M}$ such that $\chi_{M_B} \in G \cap \{\chi_A : A \in \mathcal{M}\}$, which means that $\mu_n(M_B) = \mu_n(B)$ for every $n \in \mathbb{N}$. Since $\mu_n(M_B) \to 0$, it follows that $\mu_n(B) \to 0$. So, we conclude that $\mu_n(B) \to 0$ for every $B \in \mathcal{A}$. Putting together that (i) $\{\chi_A : A \in \mathcal{A}\}$ is a Grothendieck set for $ba(\mathcal{A})$, and (ii) $\mu_n(B) \to 0$ for all $B \in \mathcal{A}$, we get that $\mu_n \to 0$ weakly in $ba(\mathcal{A})$. Thus $\{\chi_A : A \in \mathcal{M}\}$ is a Grothendieck set for $ba(\mathcal{A})$. □

4. Application to Banach Spaces

Theorem 2 facilitates the extension of various classic theorems of Banach space theory. As a sample, we include three of them: namely, the Phillips lemma about convergence in $ba(\Sigma)$, Nikodým's pointwise convergence theorem in $ca(\Sigma)$ and the usual characterization of weak convergence in $ca(\Sigma)$, the linear subspace of $ba(\Sigma)$ consisting of the countably additive measures in Σ (see [18] (Chapter 7)).

Proposition 1. *Let Σ be a σ-algebra of subsets of a set Ω. If $\{\Sigma_n : n \in \mathbb{N}\}$ is an increasing sequence of subsets of Σ covering Σ, there exists some $p \in \mathbb{N}$ enjoying the following property. If $\{\mu_n\}_{n=1}^{\infty} \subseteq ba(\Sigma)$ verifies $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \Sigma_p$ and $\{A_k : k \in \mathbb{N}\}$ is a sequence of pairwise disjoint elements of Σ, then*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |\mu_n(A_k)| = 0. \tag{1}$$

Proof. According to Theorem 2 there is $p \in \mathbb{N}$ such that Σ_p is Grothendieck set for $ba(\Sigma)$. So, if $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \Sigma_p$, then $\mu_n \to 0$ weakly in $ba(\Sigma)$. In particular, $\mu_n(A) \to 0$ for every $A \in \Sigma$. Hence, (1) holds by Phillip's classic theorem. □

Proposition 2. *Let Σ be a σ-algebra of subsets of a set Ω. If $\{\Sigma_n : n \in \mathbb{N}\}$ is an increasing sequence of subsets of Σ covering Σ, there exists some $p \in \mathbb{N}$ such that if $\{\mu_n\}_{n=1}^{\infty} \subseteq ca(\Sigma)$ verifies that $\mu_n(A) \to \mu(A)$ for every $A \in \Sigma_p$ then the set $\{\mu_n : n \in \mathbb{N}\}$ is uniformly exhaustive and $\mu \in ca(\Sigma)$.*

Proposition 3. *Let Σ be a σ-algebra of subsets of a set Ω. If $\{\Sigma_n : n \in \mathbb{N}\}$ is an increasing sequence of subsets of Σ covering Σ, there exists some $p \in \mathbb{N}$ such that $\mu_n \to \mu$ weakly in $ca(\Sigma)$ if and only if $\mu_n(A) \to \mu(A)$ for every $A \in \Sigma_p$.*

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