On Grothendieck Sets

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Abstract: We call a subset \( \mathcal{M} \) of an algebra of sets \( \mathcal{A} \) a Grothendieck set for the Banach space \( ba(\mathcal{A}) \) of bounded finitely additive scalar-valued measures on \( \mathcal{A} \) equipped with the variation norm if each sequence \( \{\mu_n\}_{n=1}^{\infty} \) in \( ba(\mathcal{A}) \) which is pointwise convergent on \( \mathcal{M} \) is weakly convergent in \( ba(\mathcal{A}) \), i.e., if there is \( \mu \in ba(\mathcal{A}) \) such that \( \mu_n(A) \to \mu(A) \) for every \( A \in \mathcal{M} \) then \( \mu_n \to \mu \) weakly in \( ba(\mathcal{A}) \).

A subset \( \mathcal{M} \) of an algebra of sets \( \mathcal{A} \) is called a Nikodým set for \( ba(\mathcal{A}) \) if each sequence \( \{\mu_n\}_{n=1}^{\infty} \) in \( ba(\mathcal{A}) \) which is pointwise bounded on \( \mathcal{M} \) is bounded in \( ba(\mathcal{A}) \). We prove that if \( \Sigma \) is a \( \sigma \)-algebra of subsets of a set \( \Omega \) which is covered by an increasing sequence \( \{\Sigma_n : n \in \mathbb{N}\} \) of subsets of \( \Sigma \) there exists \( p \in \mathbb{N} \) such that \( \Sigma_p \) is a Grothendieck set for \( ba(\mathcal{A}) \). This statement is the exact counterpart for Grothendieck sets of Valdivia’s result for Nikodým sets.

Keywords: property (G); rainwater set; property (N); Nikodým set; property (VHS)

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1. Introduction

With a different terminology, Valdivia showed in [1] that if a \( \sigma \)-algebra \( \Sigma \) of subsets of a set \( \Omega \) is covered by an increasing sequence \( \{\Sigma_n : n \in \mathbb{N}\} \) of subsets, there is \( p \in \mathbb{N} \) such that \( \Sigma_p \) is a Nikodým set for \( ba(\Sigma) \). We prove that if \( \Sigma \) is covered by an increasing sequence \( \{\Sigma_n : n \in \mathbb{N}\} \) of subsets of \( \Sigma \) there is \( p \in \mathbb{N} \) such that \( \Sigma_p \) is a Grothendieck set for \( ba(\mathcal{A}) \) (definitions below). This statement is both the exact counterpart for Grothendieck sets of Valdivia’s result for Nikodým sets and a refinement of Grothendieck’s classic result stating that the Banach space \( \ell_\infty(\Sigma) \) of bounded scalar-valued \( \Sigma \)-measurable functions defined on \( \Omega \) equipped with the supremum-norm is a Grothendieck space. Our previous result applies easily to Banach space theory to extend some well-known results. For example, Phillip’s lemma can be read as follows. If \( \{\Sigma_n : n \in \mathbb{N}\} \) is an increasing sequence of subsets of \( \Sigma \) covering \( \Sigma \), there is \( p \in \mathbb{N} \) such that if \( \{\mu_n\}_{n=1}^{\infty} \subseteq ba(\Sigma) \) verifies \( \lim_{n \to \infty} \mu_n(A) = 0 \) for every \( A \in \Sigma_p \) and \( \{A_k : k \in \mathbb{N}\} \) is a sequence of pairwise disjoint elements of \( \Sigma \), then \( \lim_{n \to \infty} \sum_{k=1}^{\infty} |\mu_n(A_k)| = 0 \).

2. Preliminaries

In what follow we use the notation of [2] (Chapter 5). Let \( \mathcal{R} \) be a ring of subsets of a nonempty set \( \Omega \), \( \chi_A \) be the characteristic function of the set \( A \in \mathcal{R} \) and let \( \ell_0^\infty(\mathcal{R}) = \text{span} \{\chi_A : A \in \mathcal{R}\} \) denote the linear space of all \( \mathbb{K} \)-valued \( \mathcal{R} \)-simple functions, \( \mathbb{K} \) being the scalar field of real or complex numbers. Since \( A \cap B \in \mathcal{R} \) whenever \( A, B \in \mathcal{R} \), for each \( f \in \ell_0^\infty(\mathcal{R}) \) there are pairwise disjoint
sets $A_1, \ldots, A_m \in \mathcal{R}$ and nonzero $a_1, \ldots, a_m \in \mathbb{K}$, with $a_i \neq a_j$ if $i \neq j$ such that $f = \sum_{i=1}^{m} a_i \chi_{A_i}$, with $f = \chi_{\emptyset}$ if $f = 0$. Unless otherwise stated, we shall assume $\ell^0_0(\mathcal{R})$ equipped with the norm $\|f\|_\infty = \sup \{|f(\omega)| : \omega \in \Omega\}$. If $Q = \{x_{\emptyset} : A \in \mathcal{R}\}$ is the absolutely convex hull of $\{\chi_{A_1} : A \in \mathcal{R}\}$, then there exists an equivalent norm defined on $\ell^0_0(\mathcal{R})$ by the gauge of $Q$, namely $\|f\|_Q = \inf \{\lambda > 0 : f \in \lambda Q\}$. For if $f \in \ell^0_0(\mathcal{R})$ with $\|f\|_\infty \leq 1$, it can be shown that $f \in 4Q$ (cf. [2] (Proposition 5.1.1)), hence $\|f\|_\infty \leq \|f\|_Q \leq 4 \|f\|_\infty$.

The dual of $\ell^0_0(\mathcal{R})$ is the Banach space $b_0(\mathcal{R})$ of bounded finitely additive scalar-valued measures on $\mathcal{R}$, which we shall assume to be equipped with the variation norm

$$\|\mu\| = \sup \sum_{i=1}^{n} |\mu(A_i)|,$$

where the supremum is taken over all finite sequences of pairwise disjoint members of $\mathcal{R}$. This is the dual of the supremum-norm $\|\cdot\|_\infty$ of $\ell^0_0(\mathcal{R})$. An equivalent norm is given by $\|\mu\| = \sup \{|\mu(A)| : A \in \mathcal{R}\}$, which is the dual norm of the gauge $\|\cdot\|_Q$. We shall also consider the Banach space $b_0(\mathcal{R})^*$ equipped with the bidual norm $\|\cdot\|$ of $\|\cdot\|_\infty$. The completion of the normed space $(\ell^0_0(\mathcal{R}), \|\cdot\|_\infty)$ is the Banach space $\ell_\infty(\mathcal{R})$ of all bounded $\mathcal{R}$-measurable functions.

The Banach space $\ell_\infty(\mathcal{R})$ embeds isometrically into $b_0(\mathcal{R})^*$, hence each characteristic function $\chi_A$ in $\ell^0_0(\mathcal{R})$ with $A \in \mathcal{R}$ can be considered as a bounded linear functional on $b_0(\mathcal{R})$ defined by evaluation $\langle \chi_A, \mu \rangle = \mu(A)$. So, we may write $\{\chi_A : A \in \mathcal{R}\} \subseteq b_0(\mathcal{R})^*$, where $b_0(\mathcal{R})^*$ stands for the unit sphere of $b_0(\mathcal{R})^*$, and the set $\{\chi_A : A \in \mathcal{R}\}$ regarded as a topological subspace of $b_0(\mathcal{R})^*$ (weak*), is the same as $\{\chi_A : A \in \mathcal{R}\}$ regarded as a topological subspace of $\ell^0_0(\mathcal{R})$ (weak).

A subfamily $\mathcal{F}$ of an algebra of sets $\mathcal{A}$ is called a Nikodým set for $b_0(\mathcal{A})$ (cf. [3]) if each set $\{\mu_{a_0} : a_0 \in \Lambda\}$ in $b_0(\mathcal{A})$ which is pointwise bounded on $\mathcal{F}$ is bounded in $b_0(\mathcal{A})$, i.e., if $\sup_{a_0 \in \Lambda} |\mu_{a_0}(A)| < \infty$ for each $A \in \mathcal{F}$ implies that $\sup_{a_0 \in \Lambda} |\mu_{a_0}| < \infty$. The algebra $\mathcal{A}$ is said to have property $(N)$ if the whole family $\mathcal{A}$ is a Nikodým set for $b_0(\mathcal{A})$. Nikodým’s classic boundedness theorem establishes that every $\sigma$-algebra has property $(N)$. An algebra $\mathcal{A}$ is said to have property $(G)$ if $\ell_\infty(\mathcal{A})$ is a Grothendieck space, i.e., if each weak* convergent sequence in $b_0(\mathcal{A})$ is weakly convergent in the Banach space $b_0(\mathcal{A})$. The fact that every $\sigma$-algebra has property $(G)$ is also due to Grothendieck. Every countable algebra lacks property $(N)$, and the algebra $\mathcal{G}$ of Jordan-measurable subsets of the real interval $[0, 1]$ has property $(N)$ but fails property $(G)$ (cf. [4] (Propositions 3.2 and 3.3) and [5]). Let us recall that a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $b_0(\mathcal{A})$ is uniformly exhaustive if for each sequence $\{A_i : i \in \mathbb{N}\}$ of pairwise disjoint elements of $\mathcal{A}$ it holds that $\lim_{k \to \infty} \sup_{n \in \mathbb{N}} |\mu_n(A_k)| = 0$. We shall use the following result, originally stated in [4] (2.3 Definition).

**Theorem 1.** An algebra of sets $\mathcal{A}$ has property $(G)$ if and only if every bounded sequence $\{\mu_n\}_{n=1}^{\infty}$ in $b_0(\mathcal{A})$ which converges pointwise on $\mathcal{A}$ is uniformly exhaustive.

An algebra $\mathcal{A}$ is said to have property $(VHS)$ if every sequence $\{\mu_n\}_{n=1}^{\infty}$ in $b_0(\mathcal{A})$ which converges pointwise on $\mathcal{A}$ is uniformly exhaustive. It should be mentioned that $(VHS) \iff (N) \land (G)$, where the proof of the non-trivial implication can be found in [6] (see also [7] (Theorem 4.2)). For later use we introduce the following definition.

**Definition 1.** A subfamily $\mathcal{M}$ of an algebra of sets $\mathcal{A}$ will be called a Grothendieck set for $b_0(\mathcal{A})$ if each sequence $\{\mu_n\}_{n=1}^{\infty}$ in $b_0(\mathcal{A})$ which is pointwise convergent on $\mathcal{M}$ is weakly convergent in $b_0(\mathcal{A})$, i.e., if there is $\mu \in b_0(\mathcal{A})$ such that $\mu_n(A) \to \mu(A)$ for every $A \in \mathcal{M}$, then $\mu_n \to \mu$ weakly in $b_0(\mathcal{A})$.

If an algebra $\mathcal{A}$ contains a Grothendieck subset for $b_0(\mathcal{A})$, clearly $\mathcal{A}$ has property $(G)$. Grothendieck sets are closely related to the so-called Rainwater sets (defined below) for $b_0(\mathcal{A})$, and the study of the Rainwater sets for $b_0(\mathcal{A})$ leads to Theorem 4 below, from which the following result is a straightforward corollary.
Theorem 2. If $\Sigma$ is a $\sigma$-algebra of subsets of a set $\Omega$ which is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets of $\Sigma$ there exists $p \in \mathbb{N}$ such that $\Sigma_p$ is a Grothendieck set for $ba(\Sigma)$.

Indeed, in [1] (Theorem 1) Valdivia showed that if a $\sigma$-algebra $\Sigma$ of subsets of a set $\Omega$ is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets (subfamilies) of $\Sigma$, there exists some $p \in \mathbb{N}$ such that $\Sigma_p$ is a Nikodym set for $ba(\Sigma)$ or, equivalently, that given an increasing sequence $\{E_n : n \in \mathbb{N}\}$ of linear subspaces of $\ell^p_0(\Sigma)$ covering $\ell^p_0(\Sigma)$, there exists $p \in \mathbb{N}$ such that $E_p$ is dense and barreled (see also [8] (Theorem 3)).

As a consequence of Theorem 4 we show that if a $\sigma$-algebra $\Sigma$ is covered by an increasing sequence $\{\Sigma_n : n \in \mathbb{N}\}$ of subsets, there exists some $p \in \mathbb{N}$ such that $\{\chi_A : A \in \Sigma_p\}$, regarded as a subset of the dual unit ball of $ba(\Sigma)$, is also a Rainwater set for $ba(\Sigma)$. This easily implies Theorem 2. In the last section we give some applications of Theorem 2 to classic Banach space theory which seems to have gone unnoticed so far. Let us point out that some results of this paper hold for Boolean algebras [9] (Theorem 12.35).

3. Rainwater Sets for $ba(\mathcal{A})$

A subset $X$ of the dual closed unit ball $B_{\ell^p_0}$ of a Banach space $E$ is called a Rainwater set for $E$ if every bounded sequence $\{x_n\}_{n=1}^{\infty}$ of $E$ that converges pointwise on $X$, i.e., such that $x^*x_n \to x^*x$ for each $x^* \in X$, converges weakly in $E$ (cf. [10]). Rainwater’s classic theorem [11] asserts that the set of the extreme points of the closed dual unit ball of a Banach space $E$ is a Rainwater set for $E$. According to [12] (Corollary 11), each James boundary of $E$ is a Rainwater set for $E$. As regards the Banach space $C(X)$ of real-valued continuous functions over a compact Hausdorff space $X$ equipped with the supremum norm, if $K = \text{Ext} B_{C(X)}$ is the set of the extreme points of the compact subset $B_{C(X)}$ of $C(X)^*$ (weak*), the Arens-Kelly theorem asserts that $K = \{ \pm \delta_x : x \in X\}$ (see [13]). By the Lebesgue dominated convergence theorem, if $\{f_n\}_{n=1}^{\infty}$ is a norm-bounded sequence in $C(X)$ (with respect to the supremum-norm) then $f_n \to f$ weakly in $C(X)$ if and only if $f_n(x) \to f(x)$ for every $x \in X$, that is, $(f_n, \mu) \to (f, \mu)$ for every $\mu \in C(X)^*$ if and only if $(f_n, \delta_v) \to (f, \delta_v)$ for each $v \in K$ (see [14] (IV.6.4 Corollary)). This is Rainwater’s theorem for $C(X)$. In [10] the weak $K$-analyticity of the Banach space $C^b(X)$ of real-valued continuous and bounded functions defined on a completely regular space equipped with the supremum norm is characterized in terms of certain Rainwater sets for $C^b(X)$. The next theorem, based on [3] (Proposition 4.1), exhibits a connection between Rainwater sets and property (G). We include it for future reference and provide a proof for the sake of completeness.

Theorem 3. Let $\mathcal{A}$ be an algebra of sets. The following are equivalent
1. $\mathcal{A}$ has property (G).
2. $\{\chi_A : A \in \mathcal{A}\}$ is a Rainwater set for $ba(\mathcal{A})$, considered as a subset of $ba(\mathcal{A})^*$.
3. The unit ball of $\ell^p_0(\mathcal{A})$ is a Rainwater set for $ba(\mathcal{A})$.
4. The unit ball of $\ell^p_0(\mathcal{A})$ is a Rainwater set for $ba(\mathcal{A})$.

Proof. 1 $\Rightarrow$ 2. Assume that $\mathcal{A}$ has property (G) and let $\{\mu_n\}_{n=1}^{\infty}$ be a bounded sequence in $ba(\mathcal{A})$ and $\mu \in ba(\mathcal{A})$ such that $\langle \chi_A, \mu_n \rangle \to \langle \chi_A, \mu \rangle$ for each $A \in \mathcal{A}$, i.e., such that $\mu_n(\mathcal{A}) \to \mu(\mathcal{A})$ for each $A \in \mathcal{A}$. By Theorem 1 the sequence $M = \{\mu_n : n \in \mathbb{N}\}$ is (bounded and) uniformly exhaustive on $\mathcal{A}$, so [15] (Corollary 5.2) produces a nonnegative real-valued finitely-additive measure $\lambda$ on $\mathcal{A}$ such that $\lim_{n \to \infty} \sup_{A \in \mathcal{A}} |\mu_n(A)| = 0$. Hence, [14] (4.9.12 Theorem) shows that $M$ is relatively weakly sequentially compact. Given that $\mu_n(\mathcal{A}) \to \mu(\mathcal{A})$ for each $A \in \mathcal{A}$, necessarily $\mu$ is the only possible weakly adherent point of the sequence $\{\mu_n\}_{n=1}^{\infty}$. So we get that $\mu_n \to \mu$ weakly in $ba(\mathcal{A})$, which shows that $\{\chi_A : A \in \mathcal{A}\}$ is a Rainwater set for $ba(\mathcal{A})$.

2 $\Rightarrow$ 3. If $B_{ba(\mathcal{A})^*}$ denotes the second dual ball of the closed unit ball $B_{ba(\mathcal{A})}$ of $\ell^p_0(\mathcal{A})$ and $B_0$ stands for the unit ball of $\ell^p_0(\mathcal{A})$, from the relations $\{\chi_A : A \in \mathcal{A}\} \subseteq B_0 \subseteq B_{ba(\mathcal{A})^*}$ it follows that $B_0$ is a also Rainwater set for $ba(\mathcal{A})$.

3 $\Rightarrow$ 4 is obvious.
Theorem 4. Assume that $\mathcal{M}$ is a Nikodým set for $ba(\mathcal{A})$ and let $\{\mathcal{M}_n : n \in \mathbb{N}\}$ be an increasing covering of $\mathcal{M}$ by subsets of $\mathcal{M}$. If $\{\chi_A : A \in \mathcal{M}\}$ is a Rainwater set for $ba(\mathcal{A})$, there exists some $p \in \mathbb{N}$ such that $\{\chi_A : A \in \mathcal{M}_p\}$ is a Rainwater set for $ba(\mathcal{A})$.

Proof. Assume that $\{\chi_A : A \in \mathcal{M}\}$ is a Rainwater set for $ba(\mathcal{A})$. First we claim that

$$\{\chi_A : A \in \mathcal{A}\} \subseteq \bigcup_{n=1}^{\infty} n \cdot abx \{\chi_A : A \in \mathcal{M}_n\}^{\|\cdot\|_\infty}$$

Let us proceed by contradiction. Assume otherwise that there exists $B \in \mathcal{A}$ such that $\chi_B \notin n \cdot abx \{\chi_A : A \in \mathcal{M}_n\}^{\|\cdot\|_\infty}$ for all $n \in \mathbb{N}$. In this case the separation theorem provides $\mu_n \in ba(\mathcal{A})$ with $|\mu_n(B)| = 1$ such that

$$\sup \left\{ \|f, \mu_n\| : f \in abx \{\chi_A : A \in \mathcal{M}_n\}^{\|\cdot\|_\infty} \right\} \leq \frac{1}{n}$$

So, in particular it holds that

$$\sup \{\|\mu_n(A)\| : A \in \mathcal{M}_n\} \leq \frac{1}{n}$$

for every $n \in \mathbb{N}$. If $\mathcal{M} \subseteq \mathcal{M}_n$ for every $n \geq k$. Consequently $|\mu_n(M)| \leq \frac{1}{n}$ for $n \geq k$, which shows that $\mu_n(M) \to 0$. Since $\mathcal{M}$ is a Nikodým set and $\{\mu_n\}_{n=1}^{\infty}$ is pointwise bounded on $\mathcal{M}$, it follows that $\{\mu_n\}_{n=1}^{\infty}$ is bounded in $ba(\mathcal{A})$. So, the fact that $\mu_n(M) \to 0$ for all $\mathcal{M} \subseteq \mathcal{M}_n$ along with the assumption that $\mathcal{M}$ is a Rainwater set leads to $\mu_n \to 0$ weakly in $ba(\mathcal{A})$. This is a contradiction, since $\langle \chi_B, \mu_n \rangle = 1$ for every $n \in \mathbb{N}$. The claim is proved.

Set $Q := \{\chi_A : A \in \mathcal{A}\}$. Since we are assuming that $\mathcal{M}$ is a Nikodým set for $ba(\mathcal{A})$, the larger set $\mathcal{A}$ is also a Nikodým set for $ba(\mathcal{A})$, which implies that $\ell_0^\infty(\mathcal{A})$ is a metrizable barrelled space, hence a Baire-like space (see [17]). On the other hand, as a consequence of the previous claim, the family $\{W_n\}_{n=1}^{\infty}$ with

$$W_n := n \cdot abx \{\chi_A : A \in \mathcal{M}_n\}^{\|\cdot\|_\infty}$$

is an increasing sequence of closed absolutely convex sets covering $\ell_0^\infty(\mathcal{A})$. So, there exists $p \in \mathbb{N}$ such that

$$Q \subseteq p \cdot abx \{\chi_A : A \in \mathcal{M}_p\}^{\|\cdot\|_\infty},$$

which shows that

$$abx \{\chi_A : A \in \mathcal{M}_p\}^{\|\cdot\|_\infty}$$

is a Rainwater set for $ba(\mathcal{A})$.

We claim that this implies that $\{\chi_A : A \in \mathcal{M}_p\}$ is a Rainwater set for $ba(\mathcal{A})$. In order to establish the claim it suffices to show that $abx \{\chi_A : A \in \mathcal{M}_p\}$ is a Rainwater set for $ba(\mathcal{A})$. So, let $\{\lambda_n\}_{n=1}^{\infty}$ be a bounded sequence in $ba(\mathcal{A})$ such that $\langle u, \lambda_n \rangle \to 0$ for every $u \in abx \{\chi_A : A \in \mathcal{M}_p\}$. Let us show that $\langle v, \lambda_n \rangle \to 0$ for each $v \in abx \{\chi_A : A \in \mathcal{M}_p\}^{\|\cdot\|_\infty}$. If $v \in abx \{\chi_A : A \in \mathcal{M}_p\}^{\|\cdot\|_\infty}$ there exists a
sequence \( \{ u_k \}_{k=1}^{\infty} \) in \( \text{abx} \{ \chi_A : A \in \mathcal{M}_p \} \) such that \( \| u_k - v \|_\infty \to 0 \). Consequently, given \( \epsilon > 0 \) there is \( k ( \epsilon ) \in \mathbb{N} \) with

\[
\left\| u_{k(\epsilon)} - v \right\|_\infty < \frac{\epsilon}{2 \left( 1 + \sup_{n \in \mathbb{N}} |\lambda_n| \right)}.
\]

Let \( n ( \epsilon ) \in \mathbb{N} \) be such that

\[
\left| \left< u_{k(\epsilon)}, \lambda_n \right> \right| < \frac{\epsilon}{2}
\]

for every \( n \geq n ( \epsilon ) \). Consequently, one has

\[
|\langle v, \lambda_n \rangle| \leq |\langle v - u_{k(\epsilon)}, \lambda_n \rangle| + |\langle u_{k(\epsilon)}, \lambda_n \rangle| \leq \left\| u_{k(\epsilon)} - v \right\|_\infty |\lambda_n| + \left| \left< u_{k(\epsilon)}, \lambda_n \right> \right| < \epsilon
\]

for all \( n \geq n_0 ( \epsilon ) \). This proves that \( \langle v, \lambda_n \rangle \to 0 \) for each \( v \in \text{abx} \{ \chi_A : A \in \mathcal{M}_p \} \). Since we have shown before that \( \text{abx} \{ \chi_A : A \in \mathcal{M}_p \} \) is a Rainwater set for \( ba (A) \), we get that \( \lambda_n \to 0 \) weakly in \( ba (A) \). Therefore the absolutely convex set \( \text{abx} \{ \chi_A : A \in \mathcal{M}_p \} \) is a Rainwater set for \( ba (A) \), a stated.

\[\square\]

**Corollary 1.** Let \( A \) be an algebra of sets with property \((VH)\). If \( \{ A_n : n \in \mathbb{N} \} \) is an increasing covering of \( A \) consisting of subsets of \( A \), there is some \( p \in \mathbb{N} \) such that \( \{ \chi_A : A \in A_p \} \) is a Rainwater set for \( ba (A) \).

**Proof.** This is a straightforward consequence of the Theorem 4 for \( M = A \), since as mentioned earlier an algebra \( A \) has property \((VH)\) if and only if \( A \) has both properties \((N)\) and \((G)\) (this also can be found in [7] (Theorem 4.2)). So, on the one hand \( A \) is a Nikodým set for \( ba (A) \) and, on the other hand, according to Theorem 3, the family \( \{ \chi_A : A \in A \} \) is a Rainwater set for \( ba (A) \). \[\square\]

**Proof of Theorem 2.** If \( \Sigma \) is a \( \sigma \)-algebra of subsets of a set \( \Omega \) which is covered by an increasing sequence \( \{ \Sigma_n : n \in \mathbb{N} \} \) of \( \Sigma \)-subsets, Corollary 1 and Valdivia’s result [1] provide an index \( p \in \mathbb{N} \) such that \( \Sigma_p \) is a Nikodým set for \( ba (\Sigma) \) at the same time that \( \{ \chi_A : A \in \Sigma_p \} \) is a Rainwater set for \( ba (\Sigma) \). If \( \{ \mu_n \}_{n=1}^{\infty} \) verifies that \( \mu_n (A) \to \mu (A) \) for every \( A \in \Sigma_p \) the sequence \( \{ \mu_n \}_{n=1}^{\infty} \) is bounded in \( ba (\Sigma) \) since \( \Sigma_p \) is a Nikodým set for \( ba (\Sigma) \). But then \( \mu_n \to \mu \) weakly in \( ba (\Sigma) \) due to \( \{ \chi_A : A \in \Sigma_p \} \) is a Rainwater set for \( ba (\Sigma) \). Consequently \( \Sigma_p \) is a Grothendieck for \( ba (\Sigma) \) and we are done.

\[\square\]

**Corollary 2.** If \( \{ A_n : n \in \mathbb{N} \} \) is an increasing sequence of subsets of \( \Sigma = 2^\mathbb{N} \) covering \( 2^\mathbb{N} \), there exists some \( p \in \mathbb{N} \) such that each sequence \( \{ \mu_n \}_{n=1}^{\infty} \) in \( ba (2^\mathbb{N}) \) that converges pointwise on \( A_p \) converges weakly in \( ba (2^\mathbb{N}) = \ell_\infty \).

**Proof.** Apply Theorem 2 to the \( \sigma \)-algebra \( 2^\mathbb{N} \). \[\square\]

We complete our study of Rainwater sets for \( ba (A) \) with the following result. Note that if \( X^{\text{w}^*} \) (weak* closure) with \( X \subseteq \overline{B}_{ba (A)} \) is a Rainwater set for \( ba (A) \) then \( X \) could not be a Rainwater set for \( ba (A) \). However the following property holds.

**Theorem 5.** Let \( A \) be an algebra of sets. Assume that \( \{ \chi_A : A \in A \} \) is a Grothendieck set for \( ba (A) \).

If \( \{ \chi_A : A \in M \} \) is a \( G_\delta \)-dense subset of \( \{ \chi_A : A \in A \} \) under the relative weak* topology of \( ba (A) \) or, which is the same, under the relative weak topology of \( \ell_0^\infty (A) \), then \( \{ \chi_A : A \in M \} \) is a Grothendieck set for \( ba (A) \).

**Proof.** Let \( \{ \mu_n \}_{n=1}^{\infty} \) be a sequence in \( ba (A) \) such that \( \mu_n (Q) \to 0 \) for every \( Q \in M \). Given \( B \in A \), let us define \( G_n := \{ C : C \subseteq A, \mu_n (C) = \mu_n (B) \} \). Then one has that \( \chi_B \in \bigcap_{n=1}^{\infty} G_n \), so that \( G := \bigcap_{n=1}^{\infty} G_n \) is a nonempty intersection of countably many zero-sets of \( \{ \chi_A : A \in A \} \), hence a non-empty \( G_\delta \)-set in \( \{ \chi_A : A \in A \} \) in the relative weak topology of \( \ell_0^\infty (A) \). According to the hypothesis \( G \) meets \( \{ \chi_A : A \in M \} \). Hence there exists \( M_B \in M \) such that \( \chi_{M_B} \in G \cap \{ \chi_A : A \in M \} \), which means that \( \mu_n (M_B) = \mu_n (B) \) for every \( n \in \mathbb{N} \). Since \( \mu_n (M_B) \to 0 \), it follows that \( \mu_n (B) \to 0 \). So, we conclude

\[\square\]
that $\mu_n(B) \to 0$ for every $B \in \mathcal{A}$. Putting together that (i) $\{\chi_A: A \in \mathcal{A}\}$ is a Grothendieck set for $ba(\mathcal{A})$, and (ii) $\mu_n(B) \to 0$ for all $B \in \mathcal{A}$, we get that $\mu_n \to 0$ weakly in $ba(\mathcal{A})$. Thus $\{\chi_A: A \in \mathcal{M}\}$ is a Grothendieck set for $ba(\mathcal{A})$. ☐

4. Application to Banach Spaces

Theorem 2 facilitates the extension of various classic theorems of Banach space theory. As a sample, we include three of them: namely, the Phillips lemma about convergence in $ba(\Sigma)$, Nikodým’s pointwise convergence theorem in $ca(\Sigma)$ and the usual characterization of weak convergence in $ca(\Sigma)$, the linear subspace of $ba(\Sigma)$ consisting of the countably additive measures in $\Sigma$ (see [18] (Chapter 7)).

**Proposition 1.** Let $\Sigma$ be a $\sigma$-algebra of subsets of a set $\Omega$. If $\{\Sigma_n: n \in \mathbb{N}\}$ is an increasing sequence of subsets of $\Sigma$ covering $\Sigma$, there exists some $p \in \mathbb{N}$ enjoying the following property. If $\{\mu_n\}_{n=1}^\infty \subseteq ba(\Sigma)$ verifies $\lim_{n \to \infty} \mu_n(A) = 0$ for every $A \in \Sigma_p$ and $\{A_k: k \in \mathbb{N}\}$ is a sequence of pairwise disjoint elements of $\Sigma$, then

$$\lim_{n \to \infty} \sum_{k=1}^\infty |\mu_n(A_k)| = 0. \tag{1}$$

**Proof.** According to Theorem 2 there is $p \in \mathbb{N}$ such that $\Sigma_p$ is Grothendieck set for $ba(\Sigma)$. So, if $\lim_{n \to \infty} \mu_n(A) = 0$ for every $A \in \Sigma_p$, then $\mu_n \to 0$ weakly in $ba(\Sigma)$. In particular, $\mu_n(A) \to 0$ for every $A \in \Sigma$. Hence, (1) holds by Phillip’s classic theorem. ☐

**Proposition 2.** Let $\Sigma$ be a $\sigma$-algebra of subsets of a set $\Omega$. If $\{\Sigma_n: n \in \mathbb{N}\}$ is an increasing sequence of subsets of $\Sigma$ covering $\Sigma$, there exists some $p \in \mathbb{N}$ such that if $\{\mu_n\}_{n=1}^\infty \subseteq ca(\Sigma)$ verifies $\mu_n(A) \to \mu(A)$ for every $A \in \Sigma_p$ then the set $\{\mu_n: n \in \mathbb{N}\}$ is uniformly exhaustive and $\mu \in ca(\Sigma)$.

**Proposition 3.** Let $\Sigma$ be a $\sigma$-algebra of subsets of a set $\Omega$. If $\{\Sigma_n: n \in \mathbb{N}\}$ is an increasing sequence of subsets of $\Sigma$ covering $\Sigma$, there exists some $p \in \mathbb{N}$ such that $\mu_n \to \mu$ weakly in $ca(\Sigma)$ if and only if $\mu_n(A) \to \mu(A)$ for every $A \in \Sigma_p$.

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