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Additional Information

Hausdorff-Young type inequalities for vector-valued Dirichlet series

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Abstract

We study Hausdorff-Young type inequalities for vector-valued Dirichlet series which allow to compare the norm of a Dirichlet series in the Hardy space $\mathcal{H}_p(X)$ with the q -norm of its coefficients. In order to obtain inequalities completely analogous to the scalar case, a Banach space must satisfy the restrictive notion of Fourier type/cotype. We show that variants of these inequalities hold for the much broader range of spaces enjoying type/cotype. We also consider Hausdorff-Young type inequalities for functions defined on the infinite torus \mathbb{T}^∞ or the boolean cube $\{-1, 1\}^\infty$. As a fundamental tool we show that type and cotype are equivalent to hypercontractive homogeneous polynomial type and cotype, a result of independent interest.

1 Introduction

The Hilbert space of Dirichlet series \mathcal{H}_2 was first defined in [17] as those $\sum a_n n^{-s}$ for which $(a_n)_n \in \ell_2$. This was later extended by Bayart, who in [1] defined a whole scale of Hardy spaces of Dirichlet series \mathcal{H}_p for $1 \leq p \leq \infty$. Unlike the Hilbert space case, there is no general principle that allows to decide whether or not a Dirichlet series belongs to a given Hardy space just by looking at the size of the coefficients, but the classical Hausdorff-Young inequalities are a useful tool in this purpose. For each $1 \leq p \leq \infty$ the spaces \mathcal{H}_p and $H_p(\mathbb{T}^\infty)$ (precise definitions are given below) are isometrically isomorphic. A rather straightforward computation (using, for example, standard interpolation arguments) shows that Hausdorff-Young inequalities also hold for these spaces and this immediately gives (here r' denotes the conjugate of $1 \leq r \leq \infty$ so that $\frac{1}{r} + \frac{1}{r'} = 1$)

$$\left\| \sum a_n n^{-s} \right\|_{\mathcal{H}_{p'}} \leq C \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}} \quad (1)$$

for every $1 \leq p \leq 2$ and

$$\left(\sum_{n=1}^{\infty} |a_n|^q \right)^{\frac{1}{q}} \leq C \left\| \sum a_n n^{-s} \right\|_{\mathcal{H}_q} \quad (2)$$

for all $2 \leq q \leq \infty$.

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Hardy spaces $\mathcal{H}_p(X)$ of vector-valued Dirichlet series (that is, the coefficients a_n belong to some Banach space X) have been defined and studied in [4, 12]. Here the problem becomes more complicated. Once again, each one of these spaces is isometrically isomorphic to the corresponding $H_p(\mathbb{T}^\infty, X)$, but in this case the Hausdorff-Young inequalities do not hold for an arbitrary Banach space. Fourier type and cotype are the notions to get vector-valued Hausdorff-Young inequalities, and for spaces enjoying those properties (again, see below for the definition) we easily get in Propositions 2.1 and 2.2 inequalities that are analogous to (1) and (2). However, these properties are very restrictive in the sense that a Banach space has Fourier type or cotype with exponents p or q which are generally worse than those for the usual (Rademacher) type and cotype. Furthermore, the actual values of p and q are often unknown.

In Theorems 3.5 and 3.7 we show that Banach spaces with cotype q (respectively type p) satisfy variants of Hausdorff-Young inequalities which relate the norm of a Dirichlet series with a weighted ℓ_q norm of the coefficients (respectively, a weighted ℓ_p norm). Analogous inequalities are obtained for functions on \mathbb{T}^∞ and Walsh functions (i.e., functions on the infinite Boolean cube).

The main tool for these results is a polynomial reformulation of type and cotype (Theorems 3.1 and 3.2). More precisely, we prove that Rademacher cotype is equivalent to the notion of hypercontractive homogeneous cotype defined in [5] (see Section 3 for the definition), answering in the positive a conjecture that was open for a few years. An equivalence with an inequality concerning Walsh polynomials is also established. Analogously, Theorem 3.2 shows the corresponding results for type and its hypercontractive homogeneous version. We feel that these results together with the techniques introduced to prove them are interesting on their own and can find further applications. In [10] variants of vector-valued Bonhnenblust-Hille inequality with operators are shown to hold for Banach lattices nontrivial cotype. In [5], results regarding monomial convergence sets and multipliers for Hardy spaces were presented for Banach spaces with nontrivial cotype and local unconditional structure or with Fourier cotype. As mentioned in Remark 3.4, thanks to Theorem 3.1 all these results readily extend to Banach spaces with nontrivial cotype.

The proof of Theorems 3.1 and 3.2 are the most technical part of the article and are developed in Section 4. A crucial part is to show that it suffices to prove the desired inequality for tetrahedral polynomials, where the variables appear with at most power 1. We feel that this methodology (to reduce an inequality for general polynomials to an inequality for tetrahedral polynomials) can be useful in different situations and is, then, interesting on its own.

2 Definitions and first results

We denote by dz the normalized Lebesgue measure on the infinite dimensional polytorus $\mathbb{T}^\infty = \prod_{k=1}^\infty \mathbb{T}$, i.e., the countable product measure of the normalized Lebesgue measure on \mathbb{T} . For any multi index $\alpha = (\alpha_1, \dots, \alpha_n, 0, \dots) \in \mathbb{Z}^{(\mathbb{N})}$ (all finite sequences in \mathbb{Z}) the α th Fourier coefficient $\hat{f}(\alpha)$ of $f \in L_1(\mathbb{T}^\infty, X)$ is given by

$$\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(z) z^{-\alpha} dz.$$

For $1 \leq p < \infty$, the X -valued Hardy space on \mathbb{T}^∞ is the subspace of $L_p(\mathbb{T}^\infty, X)$ defined as

$$H_p(\mathbb{T}^\infty, X) = \left\{ f \in L_p(\mathbb{T}^\infty, X) : \hat{f}(\alpha) = 0, \quad \forall \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \right\}$$

(where $\mathbb{N}_0^{(\mathbb{N})}$ stands for the set of α s in $\mathbb{Z}^{(\mathbb{N})}$ with $\alpha_i \geq 0$ for every i). Observe that each $f \in H_p(\mathbb{T}^\infty, X)$ is uniquely determined by its Fourier coefficients. With this in mind we consider the X -valued Bohr transform \mathfrak{B}_X that to each f assigns the Dirichlet series $\sum a_n n^{-s}$ where $a_n = \hat{f}(\alpha)$ if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime number decomposition of n . Then the Hardy space $\mathcal{H}_p(X)$ of Dirichlet series in X is defined

as the image of $H_p(\mathbb{T}^\infty, X)$ under the Bohr transform \mathfrak{B}_X . This vector space of Dirichlet series together with the norm

$$\|D\|_{\mathcal{H}_p(X)} = \|\mathfrak{B}_X^{-1}(D)\|_{H_p(\mathbb{T}^\infty, X)} \quad (3)$$

forms a Banach space. In other words, Bohr's transform gives the isometric identification

$$\mathcal{H}_p(X) = H_p(\mathbb{T}^\infty, X) \text{ for } 1 \leq p < \infty.$$

A detailed account on this identification can be found in [9] or [25].

There are many equivalent definitions of Fourier type and cotype (see [15]). Let us give the ones that are more akin to our framework. Given $1 \leq p \leq 2$, we say that X has *Fourier type* p if there is a constant $C > 0$ such that for each choice of finitely many vectors $x_1, \dots, x_N \in X$ we have

$$\left(\int_{\mathbb{T}} \left\| \sum_{k=1}^N x_k z^k \right\|^{p'} dz \right)^{\frac{1}{p'}} \leq C \left(\sum_{k=1}^N \|x_k\|^p \right)^{\frac{1}{p}}.$$

For $2 \leq q < \infty$, X has *Fourier cotype* q if there is a constant $C > 0$ such that for each choice of finitely many vectors $x_1, \dots, x_N \in X$ we have

$$\left(\sum_{k=1}^N \|x_k\|^q \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{T}} \left\| \sum_{k=1}^N x_k z^k \right\|^{q'} dz \right)^{\frac{1}{q'}}.$$

We refer to the comments after Proposition 2.2 regarding the equivalence of these two concepts and also their connection with (4) and (5) below. It was shown in [5, Proposition 2.4] that a Banach space X has Fourier cotype $q \geq 2$ if and only if there exists $C > 0$ such that for every finite family $(x_\alpha)_{\alpha \in \mathbb{N}_0^{(N)}}$ we have

$$\left(\sum_{\alpha} \|x_\alpha\|^q \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{T}^n} \left\| \sum_{\alpha} x_\alpha z^\alpha \right\|^{q'} dz \right)^{\frac{1}{q'}}. \quad (4)$$

The proof of [5, Proposition 2.4]) also works to show that X has Fourier type $1 \leq p \leq 2$ if and only if there exists $C > 0$ such that for every finite family $(x_\alpha)_{\alpha \in \mathbb{N}_0^{(N)}}$ in X we have

$$\left(\int_{\mathbb{T}^n} \left\| \sum_{\alpha} x_\alpha z^\alpha \right\|^{p'} dz \right)^{\frac{1}{p'}} \leq C \left(\sum_{\alpha} \|x_\alpha\|^p \right)^{\frac{1}{p}}. \quad (5)$$

A straightforward argument using the Bohr transform (see (3)) allows to reformulate (4) and (5) in terms of Dirichlet series as

$$\left(\sum_{n=1}^N \|a_n\|_X^q \right)^{1/q} \leq C \left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_{q'}(X)} \quad (6)$$

and

$$\left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_p(X)} \leq C \left(\sum_{n=1}^N \|a_n\|_X^p \right)^{\frac{1}{p}}, \quad (7)$$

respectively, for every X -valued Dirichlet polynomial $\sum_{n=1}^N a_n n^{-s}$. Note that (7) and the density of the finite sequences in $\ell_p(X)$ (the space of p -summing sequences in X) show that the operator $\ell_p(X) \rightarrow \mathcal{H}_p(X)$ given by $(a_n) \rightsquigarrow \sum a_n n^{-s}$ is continuous. Analogously, by (6) and the density of the Dirichlet polynomials in $\mathcal{H}_{q'}(X)$ (see [9, 24.2.1e]), the operator $\mathcal{H}_{q'}(X) \rightarrow \ell_q(X)$ given by $\sum a_n n^{-s} \rightsquigarrow (a_n)$ is also continuous. This gives the equivalence between the first and third statements in each of the following two results. The equivalence between the second and third statements is a straightforward consequence of the definition of the Hardy spaces of Dirichlet series.

Proposition 2.1. *Let X be a Banach space. For $2 \leq q < \infty$ and $C \geq 1$, the following statements are equivalent:*

(a) X has Fourier cotype q with constant C ;

(b) every Dirichlet series $D = \sum a_n n^{-s} \in \mathcal{H}_{q'}(X)$ satisfies

$$\left(\sum_{n=1}^{\infty} \|a_n\|_X^q \right)^{1/q} \leq C \|D\|_{\mathcal{H}_{q'}(X)};$$

(c) every $f \in H_{q'}(\mathbb{T}^{\infty}, X)$ satisfies

$$\left(\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \|\hat{f}(\alpha)\|_X^q \right)^{1/q} \leq C \|f\|_{H_{q'}(\mathbb{T}^{\infty}, X)}.$$

Proposition 2.2. *Let X be a Banach space. For $1 \leq p \leq 2$ and $C \geq 1$, the following statements are equivalent:*

(a) X has Fourier type p with constant C ;

(b) for every $(a_n)_n \in \ell_p(X)$ the Dirichlet series $D = \sum a_n n^{-s}$ converges in $\mathcal{H}_p(X)$ and

$$\|D\|_{\mathcal{H}_p(X)} \leq C \left(\sum_{n=1}^{\infty} \|a_n\|_X^p \right)^{\frac{1}{p}};$$

(c) for every $(x_\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \in \ell_p(X)$ there is a function $f \in H_p(\mathbb{T}^{\infty}, X)$ and so that $\hat{f}(\alpha) = x_\alpha$ for every α and

$$\|f\|_{H_p(\mathbb{T}^{\infty}, X)} \leq C \left(\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \|\hat{f}(\alpha)\|_X^p \right)^{\frac{1}{p}}.$$

As a matter of fact, Fourier type and cotype can be seen as particular cases in the more general theory of Fourier type with respect to groups (see [15], whose notation we follow now, for an excellent survey on this and related subjects). Within this setting Fourier type p (as we have defined it) is Fourier type p with respect to \mathbb{Z} , and our Fourier cotype q is Fourier type q' with respect to \mathbb{T} . Then [15, Theorem 6.6] implies that X has Fourier type p if and only if it has Fourier cotype p' , and hence both concepts are equivalent. However, we have preferred to deal with them separately because we later work with other notions of type and cotype (which are not equivalent to each other) and in this way the relationship between these and the new ones becomes more apparent.

On the other hand, this abstract point of view allows a proof of Propositions 2.1 and 2.2 based on known results on Fourier type on groups. We only sketch here the arguments. Regarding Proposition 2.1, simply note that the statement (c) is Fourier type q' with respect to \mathbb{T}^{∞} . Then the equivalence between (a) and (c) follows from [15, Theorem 6.14].

The argument for Proposition 2.2 is slightly longer. First of all X has Fourier type p if and only if X^* has Fourier type p with respect to \mathbb{T} [15, Theorem 6.3], and this happens if and only if X^* has Fourier type p with respect to \mathbb{T}^{∞} by [15, Theorem 6.14]. Again by [15, Theorem 6.3], this is equivalent to X having type p with respect to the dual group of \mathbb{T}^{∞} , which is $\mathbb{Z}^{(\mathbb{N})}$, and this is Proposition 2.2–(c).

Propositions 2.1 and 2.2 provide Hausdorff-Young inequalities for vector valued Dirichlet series which are analogous to the original inequalities. However as mentioned in the introduction, Fourier type (or

cotype) is a very restrictive property on the geometry of a Banach space. We work with the much weaker notions of type and cotype.

A Banach space X is said to have *cotype* $2 \leq q < \infty$ if there is a constant $C \geq 1$ such that for every $N \in \mathbb{N}$ and every $x_1, \dots, x_N \in X$ we have

$$\left(\sum_{n=1}^N \|x_n\|^q \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{T}^N} \left\| \sum_{n=1}^N x_n z_n \right\|^q dz \right)^{\frac{1}{q}}, \quad (8)$$

and *type* $1 \leq p \leq 2$ if there is a constant $C \geq 1$ such that for every $N \in \mathbb{N}$ and every $x_1, \dots, x_N \in X$ we have

$$\left(\int_{\mathbb{T}^N} \left\| \sum_{n=1}^N x_n z_n \right\|^p dz \right)^{\frac{1}{p}} \leq C \left(\sum_{n=1}^N \|x_n\|^p \right)^{\frac{1}{p}}. \quad (9)$$

We denote the best constants in these inequalities by $C_q(X)$ and $T_p(X)$ respectively. Let us note that (see e.g. [9, Theorem 6.8]) the $\|\cdot\|_{L_r}$ -norms appearing at (8) and (9) can be replaced by any other $\|\cdot\|_{L_s}$ -norm at the only expense of modifying the constant.

Usually, type and cotype are defined in terms of Rademacher functions. It is well known that the definitions given above are equivalent to their Rademacher versions. Actually, this equivalence can be seen as a particular case of Lemma 4.2, since linear combinations of Rademacher functions are just 1-homogeneous Walsh polynomials.

For spaces with finite cotype, translating [4, Theorem 1.1] to our setting provides a lower estimate of the norm of a Dirichlet series in terms of its coefficients. More precisely, for a Banach space X with cotype q , $\sigma > 1/q'$ and $1 \leq p \leq \infty$ there is a constant $C \geq 1$ such that

$$\sum_{n=1}^{\infty} \frac{\|a_n\|_X}{n^\sigma} \leq C \|D\|_{\mathcal{H}_p(X)}, \quad (10)$$

for every $D \in \mathcal{H}_p(X)$. In Corollary 3.6, we prove that for every $\delta > 0$ there is a constant $C \geq 1$ such that

$$\left(\sum_{n=1}^{\infty} \frac{\|a_n\|_X^q}{n^\delta} \right)^{\frac{1}{q}} \leq C \|D\|_{\mathcal{H}_p(X)},$$

for every $D \in \mathcal{H}_p(X)$. Notice that this inequality is stronger than (10) since taking $\delta = \sigma - 1/q'$ and applying Hölder's inequality to the left-hand side of (10) we get

$$\sum_{n=1}^{\infty} \frac{\|a_n\|_X}{n^\sigma} = \sum_{n=1}^{\infty} \frac{\|a_n\|_X}{n^{\delta/q}} \frac{1}{n^{(\delta+1)/q'}} \leq \left(\sum_{n=1}^{\infty} \frac{\|a_n\|_X^q}{n^\delta} \right)^{\frac{1}{q}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{\delta+1}} \right)^{\frac{1}{q'}} \leq C \left(\sum_{n=1}^{\infty} \frac{\|a_n\|_X^q}{n^\delta} \right)^{\frac{1}{q}}.$$

Following [24, Section 5.4] (see also [13, Chapter 13]), we consider $\{-1, 1\}^\infty$ with the probability measure given by the infinite product of the uniform probability $(\delta_1 + \delta_{-1})/2$. For $\varepsilon = (\varepsilon_n)_n \in \{-1, 1\}^\infty$ and $A \subset \mathbb{N}$ finite we denote

$$\varepsilon_A = \prod_{n \in A} \varepsilon_n.$$

A finite sum $\sum_A x_A \varepsilon_A$ will be called a *Walsh polynomial*. Due to the probabilistic nature of the measure space, when dealing with $L_p(\{-1, 1\}^\infty, X)$, we will write \mathbb{E} (expected value) rather than integrals. For $f \in L_1(\{-1, 1\}^\infty, X)$, the corresponding Walsh-Fourier coefficients are defined by

$$\hat{f}(A) = \mathbb{E}[f(\varepsilon) \varepsilon_A].$$

With this at hand we may introduce another notion of type/cotype. A Banach space X has Walsh type p if there is a constant $C > 0$ such that for every n and every family $\{x_A : A \subseteq [n]\} \subset X$ (here and all through the text we denote $[n] = \{1, \dots, n\}$ for each $n \in \mathbb{N}$) we have

$$\left(\mathbb{E} \left\| \sum_A x_A \varepsilon_A \right\|^{p'} \right)^{\frac{1}{p'}} \leq C \left(\sum_A \|x_A\|^p \right)^{\frac{1}{p}},$$

and has Walsh cotype q if here is a constant $C > 0$ such that for every n and every family $\{x_A : A \subseteq [n]\} \subset X$ we have

$$\left(\sum_A \|x_A\|^q \right)^{\frac{1}{q}} \leq C \left(\mathbb{E} \left\| \sum_A x_A \varepsilon_A \right\|^{q'} \right)^{\frac{1}{q'}}.$$

Standard density arguments allow us to reformulate these concepts as inequalities analogous to Proposition 2.2(c) and Proposition 2.1(c). Indeed, X has Walsh type p if and only if there is $C \geq 1$ so that

$$\|f\|_{L_{p',((-1,1)^\infty, X)}} \leq C \left(\sum_{\substack{A \subset \mathbb{N} \\ A \text{ finite}}} \|\widehat{f}(A)\|^p \right)^{\frac{1}{p}}. \quad (11)$$

Analogously, for X with Walsh cotype q , we have

$$\left(\sum_{\substack{A \subset \mathbb{N} \\ A \text{ finite}}} \|\widehat{f}(A)\|^q \right)^{\frac{1}{q}} \leq C \|f\|_{L_{q',((-1,1)^\infty, X)}}. \quad (12)$$

Once again, these notions of type/cotype sit in a more general framework, namely that of type/cotype with respect to an orthonormal system (we refer again to [15]). The concepts of Walsh type p and Walsh cotype p' coincide (see [15, Theorem 7.14]). To our best knowledge it is not known whether or not these are the same as Fourier type and cotype.

We end this section addressing the notion of K -convexity which is closely related to the concepts of type/cotype. A Banach space X is said to be K -convex if the Rademacher projection is bounded. More precisely, the mapping defined on the finite sums in $L_2((-1, 1)^\infty, X)$ by

$$P_1 \left(\sum_A x_A \varepsilon_A \right) = \sum_{|A|=1} x_A \varepsilon_A$$

extends to bounded linear operator $P_1 : L_2((-1, 1)^\infty, X) \rightarrow L_2((-1, 1)^\infty, X)$.

If X is K -convex, we can also define for each m the projection $P_m : L_2((-1, 1)^\infty, X) \rightarrow L_2((-1, 1)^\infty, X)$, which on finite sums is given by $P_m \left(\sum_A x_A \varepsilon_A \right) = \sum_{|A|=m} x_A \varepsilon_A$. By [23, Theorem 2.1] or [13, Theorem 13.16], there exists $K > 1$ such that

$$\|P_m\| \leq K^m \quad (13)$$

for every m . Also, a Banach space is K -convex if and only if it has nontrivial type (see e.g. [13, Theorem 13.3]).

3 Type, cotype and Hausdorff-Young inequalities

In this section we present a polynomial reformulation of type/cotype and use it to prove Hausdorff-Young inequalities for Dirichlet series. We also provide a slightly stronger result for spaces enjoying uniform \mathbb{C} -convexity.

3.1 A polynomial reformulation of type and cotype

In [5] the notion of *hypercontractive homogeneous cotype* was introduced, as an extension of the ‘usual’ (or, to be more accurate, Rademacher) cotype. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n, 0, 0, \dots) \in \mathbb{N}_0^{(\mathbb{N})}$ we write $|\alpha| = \alpha_1 + \dots + \alpha_n$. With this notation, a Banach space X has *hypercontractive homogeneous cotype* q if there exists $C > 0$ such that for every $m \in \mathbb{N}$ and every finite family $(x_\alpha)_{|\alpha|=m}$ we have

$$\left(\sum_{|\alpha|=m} \|x_\alpha\|^q \right)^{\frac{1}{q}} \leq C^m \left(\int_{\mathbb{T}^n} \left\| \sum_{|\alpha|=m} x_\alpha z^\alpha \right\|^2 dz \right)^{\frac{1}{2}}. \quad (14)$$

Then, different conditions were presented in [5] which ensure that a Banach space X enjoys this property. As a consequence of Theorem 3.1 (and Remark 3.3) below we see that, actually, every Banach space with Rademacher cotype has hypercontractive homogeneous cotype.

Theorem 3.1. *For a Banach space X and $2 \leq q < \infty$ the following statements are equivalent:*

- (a) X has cotype q ;
- (b) there exists $C > 0$ such that for every $m \in \mathbb{N}$ and every finite family $(x_\alpha)_{|\alpha| \leq m}$

$$\left(\sum_{|\alpha| \leq m} \|x_\alpha\|^q \right)^{\frac{1}{q}} \leq C^m \left(\int_{\mathbb{T}^n} \left\| \sum_{|\alpha| \leq m} x_\alpha z^\alpha \right\|^q dz \right)^{\frac{1}{q}}; \quad (15)$$

- (c) there exists $C > 0$ such that for every $m \in \mathbb{N}$ and every finite family $\{x_A : |A| \leq m\} \subset X$ we have

$$\left(\sum_A \|x_A\|^q \right)^{\frac{1}{q}} \leq C^m \left(\mathbb{E} \left\| \sum_A x_A \varepsilon_A \right\|^q \right)^{\frac{1}{q}}. \quad (16)$$

The proof of this theorem is rather technical, so we postpone it to Section 4. Although it was not considered in [5], for our purposes we also need a *hypercontractive homogeneous type*. This, again, turns out to be equivalent to the usual concept of Rademacher type, as follows from the next theorem.

Theorem 3.2. *For a Banach space X and $1 \leq p \leq 2$ the following statements are equivalent:*

- (a) X has type p ;
- (b) there exists $C > 0$ such that for every $m \in \mathbb{N}$ and every finite family $(x_\alpha)_{|\alpha| \leq m}$

$$\left(\int_{\mathbb{T}^n} \left\| \sum_{|\alpha| \leq m} x_\alpha z^\alpha \right\|^p dz \right)^{\frac{1}{p}} \leq C^m \left(\sum_{|\alpha| \leq m} \|x_\alpha\|^p \right)^{\frac{1}{p}}; \quad (17)$$

- (c) there exists $C > 0$ such that for every $m \in \mathbb{N}$ and every finite family $\{x_A : |A| \leq m\} \subset X$ we have

$$\left(\mathbb{E} \left\| \sum_A x_A \varepsilon_A \right\|^p \right)^{\frac{1}{p}} \leq C^m \left(\sum_A \|x_A\|^p \right)^{\frac{1}{p}}. \quad (18)$$

The proof follows essentially the same trends as that of Theorem 3.1, so it is only sketched at the end of Section 4.

Remark 3.3. We want to stress the fact that the L_2 -norm in (14), the L_q -norm in (15) and the L_p -norm in (17) can be replaced by any other L_r norm (in some cases with different constants, but still of exponential growth in m). For homogeneous polynomials this is an immediate consequence of the polynomial Kahane inequality shown in [5, Proposition 1.2]: for $1 \leq s \leq r < \infty$, if $\{x_\alpha: \alpha \in \mathbb{N}_0^{(N)}, |\alpha| = m\} \subset X$ is a family with only finitely many nonzero elements, then

$$\left\| \sum x_\alpha z^\alpha \right\|_{L_r(\mathbb{T}^\infty, X)} \leq \left(\frac{r}{s}\right)^{\frac{m}{2}} \left\| \sum x_\alpha z^\alpha \right\|_{L_s(\mathbb{T}^\infty, X)} \quad (19)$$

These inequalities extend to general polynomials of degree m proceeding as in [9, Theorem 8.10].

Once we know that (19) holds for polynomials of degree m (not necessarily homogeneous), Lemma 4.2 below gives an analogous inequality for Walsh polynomials: for any family $\{x_A: A \subset \mathbb{N}, |A| \leq m\} \subset X$ with only finitely many nonzero elements we have

$$\left(\mathbb{E} \left\| \sum_A x_A \varepsilon_A \right\|^r\right)^{\frac{1}{r}} \leq \left((1 + \sqrt{2}) \sqrt{\frac{r}{s}}\right)^m \left(\mathbb{E} \left\| \sum_A x_A \varepsilon_A \right\|^s\right)^{\frac{1}{s}}. \quad (20)$$

This inequality should be compared to [24, Corollary 5.5], from which the homogeneous case of (20) constants can be deduced, with better constants. As a consequence, the exponents p or q in the expectations in inequalities (16) and (18) can be also replaced by any other exponent r .

As we pointed out earlier, the proof of Theorem 3.1 is given in Section 4, but let us sketch here the main ideas.

Structure of the proof. To begin with, let us recall that an X -valued polynomial of n variables is a function $P: \mathbb{C}^n \rightarrow X$ given by a finite sum

$$P(z) = \sum_{\alpha \in \Lambda \subseteq \mathbb{N}_0^n} x_\alpha z^\alpha,$$

where $x_\alpha \in X$ for every α . The degree of the polynomial is the maximum over Λ of $|\alpha| = \alpha_1 + \dots + \alpha_n$. A polynomial is m -homogeneous if $|\alpha| = m$ for every $\alpha \in \Lambda$. A quick thought shows that X has cotype q (see (8)) if and only if

$$\left(\sum_\alpha \|x_\alpha\|^q\right)^{\frac{1}{q}} \leq C \|P\|_{L_q}$$

for every polynomial of degree 1 (note that the constant C does not depend on the number of variables n). Also, (15) can be reformulated as

$$\left(\sum_\alpha \|x_\alpha\|^q\right)^{\frac{1}{q}} \leq C^m \|P\|_{L_q} \quad (21)$$

for every polynomial of degree m (here C depends neither on n nor on m). We begin the proof of Theorem 3.1 by showing that the inequality we aim at holds for a specific, easier to handle, class of polynomials: tetrahedral. These are polynomials where no power bigger than 1 appears or, in other words, the monomials involved consist only of products of different variables. More precisely, a tetrahedral polynomial of n variables is of the form

$$\sum_{\alpha \in \{0,1\}^n} x_\alpha z^\alpha.$$

Then, the first step towards the proof of Theorem 3.1 is to show in Lemma 4.1 that (21) (or, equivalently, (15)) holds for m -homogeneous tetrahedral polynomials. The second step is to show that the same

inequality holds for every homogeneous polynomial. It is this second step (to pass from tetrahedral homogeneous to arbitrary homogeneous polynomials) that requires some work. In order to achieve this, we have to translate our results to the Walsh setting.

Note that, given $A \subseteq [n]$ we can define $\alpha = (\alpha_i)_i \in \{0,1\}^{[n]}$ as $\alpha_i = 1$ if $i \in A$ and 0 if $i \notin A$. With this idea, to each finite set we can associate a tetrahedral multi-index (and vice-versa). Denoting $z_A = \prod_{i \in A} z_i$ for each $A \subseteq [n]$ we may rewrite each tetrahedral polynomial as $\sum_{A \subseteq [n]} \chi_A z_A$. In this way, we have a straightforward identification between tetrahedral and Walsh polynomials:

$$\sum_{\alpha \in \{0,1\}^n} \chi_\alpha z^\alpha = \sum_{A \subseteq [n]} \chi_A z_A \longleftrightarrow \sum_{A \subseteq [n]} \chi_A \varepsilon_A.$$

In Lemma 4.2 we relate the $L_p(\mathbb{T}^\infty, X)$ -norm of the tetrahedral polynomial $\sum_{A \subseteq [n]} \chi_A z_A$ with the $L_p(\{-1,1\}^\infty, X)$ -norm of the Walsh polynomial $\sum_{A \subseteq [n]} \chi_A \varepsilon_A$ and in this way obtain that (16) holds for homogeneous Walsh polynomials. Then Lemma 4.3 shows how to pass from homogeneous to arbitrary Walsh polynomials, completing the proof of (a) implies (c). Finally, to deduce from (c) that (21) holds for homogeneous polynomials goes through a convoluted description of a polynomial given in Lemma 4.4. To pass from homogeneous to arbitrary polynomials is rather standard, and this finishes the proof.

Remark 3.4. Following exactly the same arguments as in [10, Theorem 5.3] (see also [9, Proposition 25.29]) it can be shown that if Y is a cotype q space and $v : X \rightarrow Y$ is an $(r, 1)$ -summing operator, then there exists a constant $C > 0$ so that

$$\left(\sum_{\alpha} \|v(c_\alpha)\|_Y^{\frac{qrm}{q+(m-1)r}} \right)^{\frac{q+(m-1)r}{qrm}} \leq C^m \sup_{z \in \mathbb{D}^n} \|P(z)\|_X$$

for every X -valued polynomial $P(z) = \sum_{\alpha} c_\alpha z^\alpha$ of n variables of degree m . With this at hand, the estimates in [10, Theorem 1.6–(2) and Theorem 5.4–(2)] hold for Banach spaces with cotype q .

Every $f \in H_p(\mathbb{T}^\infty, X)$ defines a formal power series in infinitely many variables $\sum_{\alpha} \hat{f}(\alpha) z^\alpha$. The set of z s for which the power series of every f in $H_p(\mathbb{T}^\infty, X)$ converges is called the set of monomial convergence:

$$\text{mon } H_p(\mathbb{T}^\infty, X) = \left\{ z \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha} \|\hat{f}(\alpha) z^\alpha\|_X < \infty \text{ for all } f \in H_p(\mathbb{T}^\infty, X) \right\}.$$

Then, the equivalence between cotype and polynomial cotype given Theorem 3.1 combined with [5, (16)] shows that, if we denote $\text{cot}(X) = \inf\{q : X \text{ has cotype } q\}$, then

$$\ell_{\text{cot}(X)'} \cap B_{c_0} \subseteq \text{mon } H_p(\mathbb{T}^\infty, X) \subseteq \ell_{\text{cot}(X)'+\varepsilon} \cap B_{c_0}$$

for every $\varepsilon > 0$. If X attains its optimal cotype (that is, if X has cotype $\text{cot}(X)$), then [5, Theorem 3.1] gives

$$\text{mon } H_p(\mathbb{T}^\infty, X) = \ell_{\text{cot}(X)'} \cap B_{c_0}. \quad (22)$$

Also, let us recall that a sequence $b = (b_n)_n$ is an ℓ_1 -multiplier of $\mathcal{H}_p(X)$ if $\sum_{n=1}^{\infty} \|a_n\|_X |b_n| < \infty$ for every $\sum a_n n^{-s}$ in $\mathcal{H}_p(X)$. As an immediate consequence of (22) (see [5, Theorem 4.3]) we have that if X has cotype $\text{cot}(X)$, then a multiplicative b (that is, $b_{mn} = b_n b_m$ for every m, n) is an ℓ_1 -multiplier of $\mathcal{H}_p(X)$ if and only if $b \in \ell_{\text{cot}(X)'}$.

3.2 Hausdorff-Young inequalities

Let us note that Fourier cotype, as formulated in (4), implies (15) with universal constants, independent of m . Under the much weaker assumption of cotype, the exponential dependence on m (as C^m) in Theorem 3.1 still allows us to carry estimations from the polynomial to the Dirichlet setting at a reasonable

price. An analogous situation holds for Banach spaces with type p . We obtain inequalities, not only for Dirichlet series, but also for functions defined on \mathbb{T}^∞ or $\{-1, 1\}^\infty$, as in Proposition 2.1 or (12). Comparing what we obtain with those inequalities, we gather that the r factor in the following theorem is the price we pay for loosening the hypothesis of Fourier or Walsh to just Rademacher cotype. Let us recall that the number of prime divisors of $n \in \mathbb{N}$, counted with multiplicity is denoted by $\Omega(n)$.

Theorem 3.5. *For a Banach space X and $2 \leq q < \infty$ the following statements are equivalent:*

- (a) X has cotype q ;
- (b) for some (every) $1 \leq p < \infty$, there exist constants $C \geq 1$ and $0 < r < 1$ such that every vector-valued Dirichlet series $D = \sum a_n n^{-s} \in \mathcal{H}_p(X)$ satisfies

$$\left(\sum_{n=1}^{\infty} r^{\Omega(n)} \|a_n\|^q \right)^{\frac{1}{q}} \leq C \|D\|_{\mathcal{H}_p(X)}.$$

- (c) for some (every) $1 \leq p < \infty$, there exist constants $C \geq 1$ and $0 < r < 1$ such that every function $f \in H_p(\mathbb{T}^\infty, X)$ satisfies

$$\left(\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} r^{|\alpha|} \|\widehat{f}(\alpha)\|^q \right)^{\frac{1}{q}} \leq C \|f\|_{H_p(\mathbb{T}^\infty, X)};$$

In addition, the next statement (d) implies (a),(c) and (b) and is equivalent to them whenever X is K -convex:

- (d) for some (every) $1 < p < \infty$, there exist constants $C \geq 1$ and $0 < r < 1$ such that every function $f \in L_p(\{-1, 1\}^\infty, X)$ satisfies

$$\left(\sum_{\substack{A \subset \mathbb{N} \\ A \text{ finite}}} r^{|A|} \|\widehat{f}(A)\|^q \right)^{\frac{1}{q}} \leq C \|f\|_{L_p(\{-1, 1\}^\infty, X)}.$$

Proof. Observe that (c) and (b) are equivalent via Bohr's transform. The fact that (c) \Rightarrow (a) follows by noticing that, given $x_1, \dots, x_N \in X$ and defining $P(z) = \sum_{n=1}^N x_n z_n$, the sum at the left-hand side becomes $r^{1/q} \left(\sum \|x_n\|^q \right)^{1/q}$. The same argument proves that (d) \Rightarrow (a) invoking Theorem 3.1.

Next we see that (a) \Rightarrow (c). Set $f \in H_p(\mathbb{T}^\infty, X)$ and for every $m \in \mathbb{N}$ let f_m be its m -homogeneous projection (see [4, Proposition 2.5]). By Theorem 3.1, there is a constant $c \geq 1$ such that for every (finite) m -homogeneous polynomial $P = \sum x_\alpha z^\alpha$ we have

$$\left(\sum_{|\alpha|=m} \|x_\alpha\|^q \right)^{\frac{1}{q}} \leq c^m \|P\|_p.$$

Since polynomials are dense in $H_p(\mathbb{T}^\infty, X)$ and the m -homogeneous projection is a contraction, a straightforward density argument yields

$$\left(\sum_{|\alpha|=m} \|\widehat{f}(\alpha)\|^q \right)^{\frac{1}{q}} \leq c^m \|f_m\|_p \leq c^m \|f\|_p. \quad (23)$$

Taking $r < 1/c^q$ we get

$$\left(\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} r^{|\alpha|} \|\widehat{f}(\alpha)\|^q \right)^{\frac{1}{q}} = \left(\sum_{m=1}^{\infty} r^m \sum_{|\alpha|=m} \|\widehat{f}(\alpha)\|^q \right)^{\frac{1}{q}} \leq \left(\sum_{m=1}^{\infty} (rc^q)^m \right)^{\frac{1}{q}} \|f\|_p \leq C \|f\|_p,$$

which completes the argument.

We finally show that (a) \Rightarrow (d) for K -convex spaces. First assume that $p = 2$. In this case (13) gives a constant K so that

$$\|f_m\|_2 \leq K^m \|f\|_2 \quad (24)$$

for every $f \in L_2(\{-1, 1\}^\infty, X)$. This enables us to proceed exactly as in (23) to get the desired result. For the general case when $1 < p < \infty$, it only remains to show that an inequality analogous to (24) holds. On the one hand, if $2 \leq p < \infty$, using (20) we get

$$\|f_m\|_p \leq C^m \|f_m\|_2 \leq (CK)^m \|f\|_2 \leq (CK)^m \|f\|_p,$$

for some constant $C > 0$.

On the other hand, it is a well-known fact that if X is K -convex, so is X^* (see for example [13, Corollary 13.7 and Theorem 13.15]). Therefore, for $1 < p \leq 2$,

$$\begin{aligned} \|f_m\|_p &= \sup_{\substack{g \in L_{p'}(X^*) \\ \|g\|_{p'}=1}} \mathbb{E}[g(\varepsilon)(f_m(\varepsilon))] = \sup_{\substack{g \in L_{p'}(X^*) \\ \|g\|_{p'}=1}} \mathbb{E}[g_m(\varepsilon)(f(\varepsilon))] \\ &\leq \sup_{\substack{g \in L_{p'}(X^*) \\ \|g\|_{p'}=1}} \|g_m\|_{p'} \|f\|_p \leq \sup_{\substack{g \in L_{p'}(X^*) \\ \|g\|_{p'}=1}} \widetilde{K}^m \|g\|_{p'} \|f\|_p \leq \widetilde{K}^m \|f\|_p, \end{aligned}$$

for some constant $\widetilde{K} > 0$. □

As a consequence of Theorem 3.5 we obtain the following result mentioned in the previous section.

Corollary 3.6. *Let X be a Banach space with cotype q and set $1 \leq p \leq \infty$. For every $\delta > 0$ there is a constant $C \geq 1$ such that every $D = \sum a_n n^{-s} \in \mathcal{H}_p(X)$ satisfies*

$$\left(\sum_{n=1}^{\infty} \frac{\|a_n\|^q}{n^\delta} \right)^{\frac{1}{q}} \leq C \|D\|_{\mathcal{H}_p(X)}.$$

Proof. From Theorem 3.5 we know that there exist constants $C \geq 1$ and $0 < r < 1$ such that every $D = \sum a_n n^{-s} \in \mathcal{H}_p(X)$ satisfies

$$\left(\sum_{n=1}^{\infty} r^{\Omega(n)} \|a_n\|^q \right)^{\frac{1}{q}} \leq C \|D\|_{\mathcal{H}_p(X)}.$$

Fix $\delta > 0$ and let $k \in \mathbb{N}$ be such that $1/p_k^\delta \leq r$ where p_k denotes the k -th prime number. Notice that if $p_k = 2$ we are done since we would get $1/n^\delta \leq r^{\Omega(n)}$. However if $p_k > 2$ we must deal with the first k primes where the estimation by r fails. This procedure is analogous to [8, Lemma 2] so we only sketch the proof. Fix $D = \sum a_n n^{-s} \in \mathcal{H}_p(X)$ and consider $f = \mathfrak{B}_X^{-1}(D) \in H_p(\mathbb{T}^\infty, X)$. For $\alpha_1, \dots, \alpha_k \in \mathbb{N}_0$ define

$$f_{\alpha_1, \dots, \alpha_k}(z) = \int_{\mathbb{T}^k} f(\omega_1, \dots, \omega_k, z_{k+1}, z_{k+2}, \dots) \omega_1^{-\alpha_1} \dots \omega_k^{-\alpha_k} d\omega.$$

An easy computation shows that $f_{\alpha_1, \dots, \alpha_k} \in H_p(\mathbb{T}^\infty, X)$ and $\|f_{\alpha_1, \dots, \alpha_k}\|_p \leq \|f\|_p$. Moreover, for $\beta \in \mathbb{N}_0^{(N)}$ we have that

$$\widehat{f}_{\alpha_1, \dots, \alpha_k}(\beta) = \begin{cases} \widehat{f}(\alpha_1, \dots, \alpha_k, \beta_{k+1}, \beta_{k+2}, \dots) & \text{if } \beta = (0, \dots, 0, \beta_{k+1}, \beta_{k+2}, \dots) \\ 0 & \text{otherwise.} \end{cases}$$

Applying (c) of Theorem 3.5 to $f_{\alpha_1, \dots, \alpha_k}$ we get

$$\left(\sum_{\beta \in \mathbb{N}_0^{(N)}} r^{|\beta|} \|\widehat{f}(\alpha_1, \dots, \alpha_k, \beta)\|^q \right)^{\frac{1}{q}} \leq C \|f\|_p.$$

Therefore we deduce

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\|a_n\|^q}{n^\delta} &\leq \sum_{\alpha_1, \dots, \alpha_k \geq 0} p_1^{-\alpha_1 \delta} \dots p_k^{-\alpha_k \delta} \sum_{\beta \in \mathbb{N}_0^{(N)}} r^{|\beta|} \|\widehat{f}(\alpha_1, \dots, \alpha_k, \beta)\|^q \\ &\leq C^q \sum_{\alpha_1, \dots, \alpha_k \geq 0} p_1^{-\alpha_1 \delta} \dots p_k^{-\alpha_k \delta} \|f\|_p^q = C^q \left(\prod_{j=1}^k \frac{1}{1 - 1/p_j^\delta} \right) \|f\|_p^q, \end{aligned}$$

which completes the proof. \square

We now turn our attention to spaces with nontrivial type, and get an analogous result (compare it also with Proposition 2.2 and (11)). The proof follows essentially the same lines as that of Theorem 3.5, so we only sketch it, pointing out the differences.

Theorem 3.7. *For a Banach space X and for $1 \leq p \leq 2$ the following statements are equivalent:*

- (a) X has type p ;
- (b) for some (every) $1 \leq q < \infty$ there exist constants $R, C \geq 1$ such that every X -valued Dirichlet series $D = \sum a_n n^{-s}$ satisfies

$$\|D\|_{\mathcal{H}_q(X)} \leq C \left(\sum_{n=1}^{\infty} R^{\Omega(n)} \|a_n\|^p \right)^{\frac{1}{p}};$$

- (c) for some (every) $1 \leq q < \infty$ there exist constants $C, R \geq 1$ and such that every function $f \in H_1(\mathbb{T}^\infty, X)$ satisfies

$$\|f\|_{H_q(\mathbb{T}^\infty, X)} \leq C \left(\sum_{\alpha \in \mathbb{N}_0^{(N)}} R^{|\alpha|} \|\widehat{f}(\alpha)\|^p \right)^{\frac{1}{p}};$$

- (d) for some (every) $1 \leq q < \infty$ there exist constants $C, R \geq 1$ such that every function $f \in L_1(\{-1, 1\}^\infty, X)$ satisfies

$$\|f\|_{L_q(\{-1, 1\}^\infty, X)} \leq C \left(\sum_{\substack{A \subset \mathbb{N} \\ A \text{ finite}}} R^{|A|} \|\widehat{f}(A)\|^p \right)^{\frac{1}{p}}.$$

The preceding inequalities should be understood as follows: if the sum at the right-hand side is finite, then the Dirichlet series (or the function) belongs to the corresponding space and its norm is controlled by the sum. But if the sum does not converge, then nothing can be said about the series or the function.

Proof. The implications (b) \Leftrightarrow (c), (c) \Rightarrow (a) and (d) \Rightarrow (a) follow as in Theorem 3.5. Regarding (a) \Rightarrow (c) notice that for $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\|f\|_{H_q(\mathbb{T}^\infty, X)} \leq \sum_{m=0}^{\infty} \|f_m\|_{H_q(\mathbb{T}^\infty, X)}.$$

From Theorem 3.2 and a density argument as in Theorem 3.5 we get

$$\|f\|_{H_q(\mathbb{T}^\infty, X)} \leq \sum_{m=0}^{\infty} C^m \left(\sum_{|\alpha|=m} \|\widehat{f}(\alpha)\|^p \right)^{\frac{1}{p}}.$$

Taking $R > C^p$ and applying Hölder inequality we deduce

$$\begin{aligned} \|f\|_{H_q(\mathbb{T}^\infty, X)} &\leq \sum_{m=0}^{\infty} \left(\frac{C}{R^{1/p}} \right)^m \left(R^m \sum_{|\alpha|=m} \|\widehat{f}(\alpha)\|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{m=0}^{\infty} \left(\frac{C}{R^{1/p}} \right)^{p'm} \right)^{\frac{1}{p'}} \left(\sum_{m=0}^{\infty} R^m \sum_{|\alpha|=m} \|\widehat{f}(\alpha)\|^p \right)^{\frac{1}{p}} \leq \widetilde{C} \left(\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} R^{|\alpha|} \|\widehat{f}(\alpha)\|^p \right)^{\frac{1}{p}}. \end{aligned}$$

The implication (a) \Rightarrow (d) follow in the same way. \square

3.3 Uniform \mathbb{C} -convexity

A Banach space X is q -uniformly \mathbb{C} -convex [16] (for $q \geq 2$) if there exists $\lambda > 0$ such that

$$\left(\|x\|^q + \lambda \|y\|^q \right)^{\frac{1}{q}} \leq \max_{z \in \mathbb{T}} \|x + zy\|,$$

for all $x, y \in X$ and q -uniformly PL-convex (see [6] or [24, Chapter 11]) if

$$\|x\|^q + \lambda \|y\|^q \leq \int_{\mathbb{T}} \|x + zy\|^q dz, \quad (25)$$

for all $x, y \in X$. In fact these two concepts are equivalent (see [21]) and provide an analytic version of the more familiar geometric property known as q -uniform convexity. A Banach space X is q -uniformly convex (for $q \geq 2$) if there exists $\lambda > 0$ such that

$$\|x\|^q + \lambda \|y\|^q \leq \mathbb{E} \|x + \varepsilon y\|^q.$$

It is easy to check that q -uniform convexity implies q -uniform PL-convexity.

In [3, Proposition 2.1] it is proven that q -uniform \mathbb{C} -convexity is equivalent to either of the following conditions:

(a) there exists $\lambda > 0$ such that for every analytic function $f : \mathbb{D} \rightarrow X$ we have

$$\|f(0)\|^q + \lambda \|f'(0)\|^q \leq \sup_{|z|<1} \|f(z)\|^q. \quad (26)$$

(b) there exists $\lambda > 0$ such that for every analytic function $f : \mathbb{D} \rightarrow X$ we have

$$\|f(0)\|^q + \lambda \|f'(0)\|^q \leq \sup_{0 < r < 1} \int_{\mathbb{T}} \|f(rz)\|^q dz. \quad (27)$$

Let us note that for every such function the mapping $r \in [0, 1[\rightsquigarrow \|f(r \bullet)\|_{H_q(\mathbb{T}, X)}$ is increasing and, then, the supremum at the right-hand side of (27) is in fact a limit as $r \rightarrow 1^-$. With this, if $f : \mathbb{C} \rightarrow X$ is entire, then

$$\|f(0)\|^q + \lambda \|f'(0)\|^q \leq \int_{\mathbb{T}} \|f(z)\|^q dz. \quad (28)$$

Since taking $f(z) = x + zy$ for given x and y gives (25), the equivalence with q -uniform \mathbb{C} -convexity is maintained.

Using (26) Blasco proved in [2, Theorem 2.4] that q -uniformly \mathbb{C} -convex spaces have positive q -Bohr radius. That is, there exists $\rho > 0$ such that

$$\left(\sum_{n=0}^{\infty} \|x_n\|^q \rho^{qn} \right)^{\frac{1}{q}} \leq \sup_{|z|<1} \|f(z)\|, \quad (29)$$

for every analytic function $f = \sum_n x_n z^n$ on \mathbb{D} . Replacing (26) by (28) in his argument we deduce that for q -uniformly \mathbb{C} -convex spaces there exists $\rho > 0$ such that

$$\left(\sum_{n=0}^{\infty} \|x_n\|^q \rho^{qn} \right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{T}} \|f(z)\|^q dz \right)^{\frac{1}{q}}, \quad (30)$$

for every entire function $f = \sum_n x_n z^n$. The following theorem extends this fact to several variables.

Theorem 3.8. *Let X be a q -uniformly \mathbb{C} -convex Banach space. Then there exists $\rho > 0$ such that for every n and every polynomial $P = \sum x_\alpha z^\alpha$ of n variables with values in X we have*

$$\left(\sum_{\alpha} \|x_\alpha\|^q \rho^{|\alpha|q} \right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{T}^n} \|P(z)\|^q dz \right)^{\frac{1}{q}}.$$

Proof. We proceed by induction on n , the number of variables. The case $n = 1$ follows from (30).

Suppose now that the result holds for $n - 1$ and take some polynomial

$$P(z) = \sum_{\alpha \in F} x_\alpha z^\alpha,$$

for $z \in \mathbb{C}^n$ (where $F \subseteq \mathbb{N}_0^n$ is finite). Then we can write

$$\sum_{\alpha} \|x_\alpha\|^q \rho^{|\alpha|q} = \sum_{k=0}^N \rho^{qk} \sum_{\substack{\alpha \in F \\ \alpha_n = k}} \|x_\alpha\|^q \rho^{(|\alpha| - \alpha_n)q}.$$

Applying the inductive hypothesis to each polynomial

$$z \in \mathbb{C}^{n-1} \rightsquigarrow \sum_{\substack{\alpha \in F \\ \alpha_n = k}} x_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}},$$

we have

$$\begin{aligned} \sum_{\alpha} \|x_\alpha\|^q \rho^{|\alpha|q} &\leq \sum_{k=0}^N \rho^{qk} \int_{\mathbb{T}^{n-1}} \left\| \sum_{\substack{\alpha \in F \\ \alpha_n = k}} c_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \right\|^q d(z_1, \dots, z_{n-1}) \\ &= \int_{\mathbb{T}^{n-1}} \sum_{k=0}^N \left\| \sum_{\substack{\alpha \in F \\ \alpha_n = k}} c_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \right\|^q \rho^{qk} d(z_1, \dots, z_{n-1}). \end{aligned}$$

Finally, for each fixed $(z_1, \dots, z_{n-1}) \in \mathbb{T}^{n-1}$ we may consider the polynomial $\mathbb{C} \rightarrow X$ given by

$$z \rightsquigarrow \sum_{k=0}^n \left(\sum_{\substack{\alpha \in F \\ \alpha_n = k}} c_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \right) z^k$$

and then use the case $n = 1$ of the induction to conclude

$$\begin{aligned} & \sum_{k=0}^N \left\| \sum_{\substack{\alpha \in F \\ \alpha_n = k}} x_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \right\|^q \rho^{qk} \\ & \leq \int_{\mathbb{T}} \left\| \sum_{k=0}^N \left(\sum_{\substack{\alpha \in F \\ \alpha_n = k}} x_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \right) z_n^k \right\|^q dz_n = \int_{\mathbb{T}} \left\| \sum_{\alpha \in F} x_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} z_n^{\alpha_n} \right\|^q dz_n. \end{aligned}$$

Fubini's theorem completes the proof. \square

Let us note that Theorem 3.8 can be reformulated as

$$\left(\sum_{n \leq x} \|a_n\|^q \rho^{q\Omega(n)} \right)^{\frac{1}{q}} \leq \left\| \sum_{n \leq x} a_n n^{-s} \right\|_{\mathcal{H}_q(X)} \leq \left\| \sum_{n \leq x} a_n n^{-s} \right\|_{\mathcal{H}_\infty(X)}, \quad (31)$$

for every Dirichlet polynomial. This gives a version of the q -Bohr radius for vector-valued Dirichlet series (although for better constants one should proceed to the multivariate setting directly from (29)). Also, the first inequality in (31) gives the equivalence (b) of Theorem 3.5 with constant $C = 1$ (taking $r = \rho^q$ and $p = q$). Hence, for q -uniformly \mathbb{C} -convex Banach spaces we have better Hausdorff-Young inequalities than those for general spaces with cotype q .

4 Proof of Theorem 3.1

We face now the proof to Theorem 3.1. Let us recall that we are aiming at inequalities like (15) for every polynomial of n variables of degree m . We begin by showing that such an inequality indeed holds for homogeneous tetrahedral polynomials. But before we get into that let us note that given a vector space V , a family $\{v_A : A \subseteq [n], |A| = m\} \subseteq V$ (where $n, m \in \mathbb{N}$) and $k \in \mathbb{N}$ we have

$$\begin{aligned} & \sum_{\substack{B \subseteq [n] \\ |B|=k}} \sum_{\substack{A_1 \subseteq B \\ |A_1|=1}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-1}} v_{A_1 \cup A_2} = \sum_{\substack{B \subseteq [n] \\ |B|=k}} \sum_{\substack{A \subseteq [n] \\ |A|=m \\ |A \cap B|=1}} v_A = \sum_{\substack{A \subseteq [n] \\ |A|=m}} \sum_{\substack{B \subseteq [n] \\ |B|=k \\ |A \cap B|=1}} v_A \\ & = \sum_{\substack{A \subseteq [n] \\ |A|=m}} |\{B \subseteq [n] : |B| = k, |A \cap B| = 1\}| v_A = m \binom{n-m}{k-1} \sum_{\substack{A \subseteq [n] \\ |A|=m}} v_A. \end{aligned} \quad (32)$$

This is essentially equations (3.13) through (3.16) from [26]. Once we have this we can prove that the inequality we aim at holds for homogeneous tetrahedral polynomials.

Lemma 4.1. *Let X be a Banach space X of cotype $2 \leq q < \infty$. Then for every $m, n \in \mathbb{N}$ and every family $\{x_A : A \subseteq [n], |A| = m\} \subseteq X$ we have*

$$\left(\sum_A \|x_A\|^q \right)^{\frac{1}{q}} \leq (4^{1/q} C_q(X))^m \left(\int_{\mathbb{T}^n} \left\| \sum_A x_A z_A \right\|^q dz \right)^{\frac{1}{q}}. \quad (33)$$

Proof. We prove this by induction on m . The case $m = 1$ is trivial just comparing (8) with (33) for $m = 1$.

For the inductive step, let $\{x_A : A \subseteq [n], |A| = m\}$ be a family of vectors in X . Taking n larger if necessary, we may assume $n = km$ for some $k \in \mathbb{N}$. Note that, in this case, $\binom{n-m}{k-1} = \binom{(k-1)m}{k-1}$. Let's recall the following version of Stirling's formula, stated in [14, (9.15)]:

$$\sqrt{2\pi n}^{n+\frac{1}{2}} e^{-n} e^{(12n+1)^{-1}} < n! < \sqrt{2\pi n}^{n+\frac{1}{2}} e^{-n} e^{(12n)^{-1}}.$$

From this, a straightforward (but tedious) computation yields

$$\frac{1}{2} \leq \frac{\binom{n}{k}}{m \binom{n-m}{k-1}} \leq 4.$$

Using (32) for $v_A = \|x_A\|^q$ we get

$$\sum_{\substack{A \subseteq [n] \\ |A|=m}} \|x_A\|^q = \frac{1}{m \binom{n-m}{k-1}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \sum_{\substack{A_1 \subseteq B \\ |A_1|=1}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-1}} \|x_{A_1 \cup A_2}\|^q \leq \frac{4}{\binom{n}{k}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \sum_{\substack{A_1 \subseteq B \\ |A_1|=1}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-1}} \|x_{A_1 \cup A_2}\|^q.$$

For a fixed A_1 we can apply the inductive hypothesis to the family $\{x_{A_1 \cup A_2} : A_2 \subseteq B^c, |A_2| = m-1\}$. Let \mathbb{T}^{B^c} denote $|B^c|$ copies of the torus indexed in B^c . We get

$$\begin{aligned} \sum_{\substack{A \subseteq [n] \\ |A|=m}} \|x_A\|^q &\leq \frac{4^m C_q(X)^{q(m-1)}}{\binom{n}{k}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \sum_{\substack{A_1 \subseteq B \\ |A_1|=1}} \int_{\mathbb{T}^{B^c}} \left\| \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-1}} x_{A_1 \cup A_2} z_{A_2} \right\|^q dz \\ &\leq \frac{4^m C_q(X)^{q(m-1)}}{\binom{n}{k}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \int_{\mathbb{T}^{B^c}} \sum_{\substack{A_1 \subseteq B \\ |A_1|=1}} \left\| \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-1}} x_{A_1 \cup A_2} z_{A_2} \right\|^q dz. \end{aligned}$$

Notice that we are integrating over variables z_i whose index i always lies in B^c , while A_1 is always included in B . In some sense, the variables z_{A_1} remain unused. So, by the cotype inequality (8) we obtain

$$\sum_{\substack{A \subseteq [n] \\ |A|=m}} \|x_A\|^q \leq \frac{4^m C_q(X)^{qm}}{\binom{n}{k}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \int_{\mathbb{T}^n} \left\| \sum_{\substack{A_1 \subseteq B \\ |A_1|=1}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-1}} x_{A_1 \cup A_2} z_{A_1} z_{A_2} \right\|^q dz. \quad (34)$$

Regarding the last expression, observe that

$$\begin{aligned} &\int_{\mathbb{T}} \left(\sum_{j=0}^{\min(k,m)} \sum_{\substack{A_1 \subseteq B \\ |A_1|=j}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-j}} x_{A_1 \cup A_2} \omega^{|A_1|} z_{A_1} z_{A_2} \right) \bar{\omega} d\omega \\ &= \sum_{j=0}^{\min(k,m)} \int_{\mathbb{T}} \omega^{j-1} d\omega \sum_{\substack{A_1 \subseteq B \\ |A_1|=j}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-j}} x_{A_1 \cup A_2} z_{A_1} z_{A_2} = \sum_{\substack{A_1 \subseteq B \\ |A_1|=1}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-1}} x_{A_1 \cup A_2} z_{A_1} z_{A_2}. \end{aligned} \quad (35)$$

Using (35) in (34) and applying Jensen's inequality we have

$$\sum_{\substack{A \subseteq [n] \\ |A|=m}} \|x_A\|^q \leq \frac{4^m C_q(X)^{qm}}{\binom{n}{k}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \int_{\mathbb{T}} \int_{\mathbb{T}^n} \left\| \sum_{j=0}^{\min(k,m)} \sum_{\substack{A_1 \subseteq B \\ |A_1|=j}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-j}} x_{A_1 \cup A_2} \omega^{|A_1|} z_{A_1} z_{A_2} \right\|^q dz d\omega.$$

Finally, by rotation invariance $\omega^{|\Lambda_1|} z_{\Lambda_1}$ may be replaced by z_{Λ_1} . We get

$$\begin{aligned} \sum_{\substack{A \subseteq [n] \\ |\Lambda|=m}} \|x_A\|^q &\leq \frac{4^m C_q(X)^{qm}}{\binom{n}{k}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \int_{\mathbb{T}^n} \left\| \sum_{j=0}^{\min(k,m)} \sum_{\substack{A_1 \subseteq B \\ |A_1|=j}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-j}} x_{A_1 \cup A_2} z_{A_1} z_{A_2} \right\|^q dz \\ &= \frac{4^m C_q(X)^{qm}}{\binom{n}{k}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \int_{\mathbb{T}^n} \left\| \sum_{\substack{A \subseteq [n] \\ |A|=m}} x_A z_A \right\|^q dz = 4^m C_q(X)^{qm} \int_{\mathbb{T}^n} \left\| \sum_{\substack{A \subseteq [n] \\ |A|=m}} x_A z_A \right\|^q dz. \quad \square \end{aligned}$$

In order to deal with Walsh polynomials in Theorem 3.1 (c) we need two lemmas. The first one shows that tetrahedral Steinhaus polynomials and their Walsh counterparts have equivalent norms up to exponential constants. The argument translates estimates from the scalar to the Banach setting applying a theorem of Pełczyński. This result can also be proven using [20, Proposition 6.3.1] and checking the hypothesis by hand.

Lemma 4.2. *Let X be a Banach space and set $1 \leq q < \infty$. For every tetrahedral polynomial P of degree m and n variables we have*

$$(1 + \sqrt{2})^{-m} (\mathbb{E} \|P(\varepsilon)\|_X^q)^{\frac{1}{q}} \leq \left(\int_{\mathbb{T}^n} \|P(z)\|_X^q dz \right)^{\frac{1}{q}} \leq (1 + \sqrt{2})^m (\mathbb{E} \|P(\varepsilon)\|_X^q)^{\frac{1}{q}}. \quad (36)$$

Proof. In [18, p. 2764] it is shown that for every polynomial $Q : \mathbb{C}^n \rightarrow \mathbb{C}$ of degree m , we have

$$\sup_{z \in \mathbb{T}^n} |Q(z)| \leq (1 + \sqrt{2})^m \sup_{x \in [-1,1]^n} |Q(x)|.$$

If we assume Q to be tetrahedral, we observe as in [11] that

$$\sup_{x \in [-1,1]^n} |Q(x)| = \sup_{\varepsilon \in \{-1,1\}^n} |Q(\varepsilon)|,$$

since Q is affine in every coordinate. Thus,

$$\sup_{\varepsilon \in \{-1,1\}^n} |Q(\varepsilon)| \leq \sup_{z \in \mathbb{T}^n} |Q(z)| \leq (1 + \sqrt{2})^m \sup_{\varepsilon \in \{-1,1\}^n} |Q(\varepsilon)|.$$

Equivalently, for every finite choice of scalars $\{c_A\}_{|A| \leq m} \subseteq \mathbb{C}$ we have

$$\sup_{\varepsilon \in \{-1,1\}^n} \left| \sum_{|A| \leq m} c_A \varepsilon_A \right| \leq \sup_{z \in \mathbb{T}^n} \left| \sum_{|A| \leq m} c_A z_A \right| \leq (1 + \sqrt{2})^m \sup_{\varepsilon \in \{-1,1\}^n} \left| \sum_{|A| \leq m} c_A \varepsilon_A \right|, \quad (37)$$

where for simplicity we also used Walsh notation for the variable z . Consider the sets of characters $\{\varepsilon_A\}_{|A| \leq m}$ and $\{z_A\}_{|A| \leq m}$ of the compact abelian groups $\{-1,1\}^n$ and \mathbb{T}^n respectively. Since these sets satisfy (37), the conditions of [22, Theorem 1] are met. So we get

$$\begin{aligned} (1 + \sqrt{2})^{-m} \left\| \sum_{|A| \leq m} x_A \varepsilon_A \right\|_{L^q(\{-1,1\}^n, X)} &\leq \left\| \sum_{|A| \leq m} x_A z_A \right\|_{L^q(\mathbb{T}^n, X)} \\ &\leq (1 + \sqrt{2})^m \left\| \sum_{|A| \leq m} x_A \varepsilon_A \right\|_{L^q(\{-1,1\}^n, X)}, \end{aligned}$$

for every choice of vectors $\{x_A\}_{|A| \leq m} \subseteq X$. This concludes the proof since it is equivalent to (36). \square

The following lemma estimates the norm of the homogeneous projection of a Walsh polynomial and can be found in [19, Lemma 2] (see also [7, Lemma 3.2.4]). A proof is included since the constant is not explicitly computed there, and we need it to grow exponentially on the degree of the polynomial (i.e., to be of the form B^m for some $B > 0$).

Lemma 4.3. *For every Banach space X there is a constant $B > 0$ such that for every $1 \leq q < \infty$ and every X -valued Walsh polynomial P of degree m , its k -homogeneous projection P_k satisfies*

$$(\mathbb{E}\|P_k(\varepsilon)\|_X^q)^{\frac{1}{q}} \leq B^m (\mathbb{E}\|P(\varepsilon)\|_X^q)^{\frac{1}{q}}.$$

Proof. For each m we consider the functions $\{1, t, \dots, t^m\}$ in $L_2(0, 1)$. We show that there are polynomials $\{p_1^{(m)}, \dots, p_{m+1}^{(m)}\}$ of degree at most m such that

$$\int_0^1 t^{i-1} p_j^{(m)}(t) dt = \delta_{ij},$$

for every $1 \leq i, j \leq m+1$. Indeed, writing $p_j^{(m)}(t) = \sum_{k=1}^{m+1} a_{kj}^{(m)} t^{k-1}$ we get

$$\delta_{ij} = \int_0^1 t^{i-1} p_j^{(m)}(t) dt = \sum_{k=1}^{m+1} a_{kj}^{(m)} \int_0^1 t^{i+k-2} dt = \sum_{k=1}^{m+1} \frac{1}{i+k-1} a_{kj}^{(m)},$$

for every $1 \leq i, j \leq m+1$. In other words, we obtain the matrix identity

$$I = HA,$$

where H is the well-known Hilbert matrix and A is the matrix defined by the coefficients $a_{ij}^{(m)}$. Thus, we have $A = H^{-1}$, which provides a specific formula for the polynomials $p_j^{(m)}$. Notice that $|a_{ij}^{(m)}|$ can be easily bounded by the condition number of H , which is smaller than C^m for some $C > 1$ (see [28, Equation 3.35]). Alternatively, using the explicit formula in [27] for the elements of H^{-1} , it is easy to check that there is a constant $C > 1$ so that $\sup_{i,j} |a_{ij}^{(m)}| \leq C^m$. Therefore, taking $B = 2C$ we get

$$\sup_{0 < t < 1} |p_j^{(m)}(t)| \leq (m+1)C^m \leq B^m.$$

Notice that, if P is a polynomial of degree m , then

$$P_k(\varepsilon) = \int_0^1 P(t\varepsilon) p_{k+1}^{(m)}(t) dt,$$

for every $0 \leq k \leq m$. So we get

$$(\mathbb{E}\|P_k(\varepsilon)\|_X^q)^{\frac{1}{q}} \leq \int_0^1 (\mathbb{E}\|P(t\varepsilon) p_{k+1}^{(m)}(t)\|_X^q)^{\frac{1}{q}} dt \leq B^m \int_0^1 (\mathbb{E}\|P(t\varepsilon)\|_X^q)^{\frac{1}{q}} dt.$$

Now, [7, Lemma 3.2.3] gives

$$(\mathbb{E}\|P(t\varepsilon)\|_X^q)^{\frac{1}{q}} \leq (\mathbb{E}\|P(\varepsilon)\|_X^q)^{\frac{1}{q}}$$

for every $0 \leq t \leq 1$, and this completes the proof. \square

For the last ingredient in the proof of Theorem 3.1, we need a rather convoluted description of a polynomial in terms of the parity of the exponents of the variables. Fix an even $m \in \mathbb{N}$. Given $A \subseteq [n]$ we define

$$\Lambda_A = \{\alpha \in \mathbb{N}_0^n : |\alpha| = m, \alpha_i \text{ is odd if and only if } i \in A\}.$$

Since m is even, it is clear that $\Lambda_A \neq \emptyset$ if and only if A has even cardinality between 0 and m . In the rest of this discussion we only consider A with $\Lambda_A \neq \emptyset$. Note that for any $\varepsilon \in \{-1, 1\}^n$ and $z \in \mathbb{T}^n$, we have

$$(\varepsilon z)^\alpha = \varepsilon_A z^\alpha$$

for every $\alpha \in \Lambda_A$, where, as always, $\varepsilon_A = \prod_{i \in A} \varepsilon_i$.

Now, for an m -homogeneous polynomial of n variables $P(z) = \sum_{|\alpha|=m} x_\alpha z^\alpha$ we write

$$P_A(z) = \sum_{\alpha \in \Lambda_A} x_\alpha z^\alpha.$$

With this notation, we clearly have

$$P(\varepsilon z) = \sum_{A \subseteq [n]} \varepsilon_A P_A(z).$$

As we can see from the expression above, $P(\varepsilon z)$ regarded as a polynomial on ε is tetrahedral. Also, we may write $P(\varepsilon z)$ as the sum of its homogeneous components (as a function of ε). As we have already mentioned, each A considered has even cardinality between 0 and m . So, if we define

$$\mathcal{A}_k = \{A \subseteq [n] : |A| = 2k\},$$

we can write

$$P(\varepsilon z) = \sum_{k=0}^{m/2} \sum_{A \in \mathcal{A}_k} \varepsilon_A P_A(z). \quad (38)$$

Note that, whenever i belongs to some A , the exponents of z_i are odd for every monomial in $P_A(z)$. Also, since m is even, given $\alpha \in \Lambda_A$, we have that $\sum_{i \in A} \alpha_i$ must be even and greater than $|A| = 2k$. We then define

$$\Lambda_{A,l} = \{\alpha \in \Lambda_A : \sum_{i \in A} \alpha_i = 2l\},$$

which allows us to write, for $A \in \mathcal{A}_k$,

$$P_A(z) = \sum_{l=k}^{m/2} \sum_{\alpha \in \Lambda_{A,l}} x_\alpha z^\alpha = \sum_{l=k}^{m/2} P_{A,l}(z). \quad (39)$$

Note that $P_{A,l}(z)$ is the $2l$ -homogeneous component of the polynomial $P_A(z)$ regarded as a function of the variables z_i with $i \in A$ (that is, the variables with odd exponents). In other words, the polynomial $P_{A,l}(z)$ consists of the monomials $x_\alpha z^\alpha$ of $P_A(z)$ where the sum of the odd exponents equals $2l$.

To conclude our description of P , for $\alpha \in \Lambda_A$ define exponents β, γ and 1_A by

$$\beta_i = \begin{cases} 0 & \text{if } i \in A \\ \frac{\alpha_i}{2} & \text{if } i \in A^c \end{cases}, \quad \gamma_i = \begin{cases} \frac{\alpha_i - 1}{2} & \text{if } i \in A \\ 0 & \text{if } i \in A^c \end{cases} \quad \text{and} \quad 1_{A,i} = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \in A^c \end{cases},$$

for every $1 \leq i \leq n$. Note that $\alpha = 2\beta + 2\gamma + 1_A$ where $\beta \in \mathbb{N}_0^n$ is supported in A^c and $\gamma, 1_A \in \mathbb{N}_0^n$ are supported in A . Moreover, for $\alpha \in \Lambda_{A,l}$ we have

$$|\beta| = \sum_{i=1}^n \beta_i = \sum_{i \in A^c} \frac{\alpha_i}{2} = \frac{|\alpha|}{2} - \sum_{i \in A} \frac{\alpha_i}{2} = \frac{m}{2} - l,$$

and

$$|\gamma| = \sum_{i=1}^n \gamma_i = \sum_{i \in A} \frac{\alpha_i - 1}{2} = \frac{2l - |A|}{2} = l - k.$$

Denote the set of all the exponents β supported in A^c with $|\beta| = m/2 - l$ by $B_{A,l}$ and the set of all the exponents γ supported in A with $|\gamma| = l - k$ by $\Gamma_{A,l}$. We get

$$P_{A,l}(z) = \sum_{\alpha \in \Lambda_{A,l}} x_\alpha z^\alpha = \sum_{\gamma \in \Gamma_{A,l}} \sum_{\beta \in B_{A,l}} x_{2\beta+2\gamma+1_A} z^{2\beta+2\gamma+1_A} = \sum_{\gamma \in \Gamma_{A,l}} \left(\sum_{\beta \in B_{A,l}} x_{2\beta+2\gamma+1_A} z^{2\beta} \right) z^{2\gamma+1_A}. \quad (40)$$

Gathering (38), (39) and (40) we get the full description of $P(\varepsilon z)$ proving the following lemma.

Lemma 4.4. *For an even $m \in \mathbb{N}$, an m -homogeneous polynomial in n variables*

$$P(z) = \sum_{|\alpha|=m} x_\alpha z^\alpha,$$

and $\varepsilon \in \{-1, 1\}^n$ we have

$$P(\varepsilon z) = \sum_{k=0}^{m/2} \sum_{A \in \mathcal{A}_k} \sum_{l=k}^{m/2} \sum_{\gamma \in \Gamma_{A,l}} \sum_{\beta \in B_{A,l}} x_{2\beta+2\gamma+1_A} \varepsilon_A z^{2\beta+2\gamma+1_A}.$$

With the same argument we may deduce a similar formula when m is odd. For every m -homogeneous polynomial in n variables P and $\varepsilon \in \{-1, 1\}^n$, we get

$$P(\varepsilon z) = \sum_{k=0}^{(m-1)/2} \sum_{A \in \mathcal{A}'_k} \sum_{l=k}^{(m-1)/2} \sum_{\gamma \in \Gamma'_{A,l}} \sum_{\beta \in B'_{A,l}} x_{2\beta+2\gamma+1_A} \varepsilon_A z^{2\beta+2\gamma+1_A},$$

where

$$\begin{aligned} \mathcal{A}'_k &= \{A \subseteq [n] : |A| = 2k + 1\}, \\ \Gamma'_{A,l} &= \{\gamma \in \mathbb{N}_0^n : \sum_{i \in A} \gamma_i = l - k \text{ and } \gamma_i = 0 \text{ for } i \in A^c\}, \text{ and} \\ B'_{A,l} &= \{\beta \in \mathbb{N}_0^n : \sum_{i \in A} \beta_i = (m-1)/2 - l \text{ and } \beta_i = 0 \text{ for } i \in A\}. \end{aligned}$$

We are now in position to give the proof of Theorem 3.1.

Proof of Theorem 3.1. First notice that (b) \Rightarrow (a) follows immediately by taking $m = 1$ in (b). Next we show that (a) \Rightarrow (c). Let $\{x_A : A \subseteq [n], |A| \leq m\}$ be a family of vectors in X . Applying Lemma 4.1 to the subfamilies $\{x_A : A \subseteq [n], |A| = k\}$ for each $0 \leq k \leq m$ and denoting $C = 4^{1/q} C_q(X)$ we get

$$\sum_{A \subseteq [n]} \|x_A\|^q = \sum_{k=0}^m \sum_{\substack{A \subseteq [n] \\ |A|=k}} \|x_A\|^q \leq \sum_{k=0}^m C^{qk} \int_{\mathbb{T}^n} \left\| \sum_{\substack{A \subseteq [n] \\ |A|=k}} x_A z_A \right\|^q dz.$$

Using Lemmas 4.2 and 4.3 we obtain

$$\begin{aligned} \sum_{A \subseteq [n]} \|x_A\|^q &\leq \sum_{k=0}^m ((1 + \sqrt{2})C)^{qk} \mathbb{E} \left\| \sum_{\substack{A \subseteq [n] \\ |A|=k}} x_A \varepsilon_A \right\|^q \leq B^m \sum_{k=0}^m ((1 + \sqrt{2})C)^{qk} \mathbb{E} \left\| \sum_{A \subseteq [n]} x_A \varepsilon_A \right\|^q \\ &\leq (m+1)((1 + \sqrt{2})BC)^{qm} \mathbb{E} \left\| \sum_{A \subseteq [n]} x_A \varepsilon_A \right\|^q \leq (20BC_q(X))^{qm} \mathbb{E} \left\| \sum_{A \subseteq [n]} x_A \varepsilon_A \right\|^q. \end{aligned}$$

This gives (16) and completes the argument.

It only remains to show that (c) \Rightarrow (b). As a first step we show that the inequality holds for homogeneous polynomials. Let C_ε be the constant provided by (c) and B the constant in Lemma 4.3. Our aim is to show that if $C = \max\{C_\varepsilon^2, B^4\}$, then

$$\left(\sum_{|\alpha|=m} \|x_\alpha\|^q \right)^{\frac{1}{q}} \leq C^m \left(\int_{\mathbb{T}^n} \|P(z)\|^q dz \right)^{\frac{1}{q}} \quad (41)$$

for every m -homogeneous polynomial $P(z) = \sum_{|\alpha|=m} x_\alpha z^\alpha$ of n variables. We proceed by induction on m . The case $m = 1$ is the well known equivalence between Rademacher and Steinhaus averages, which is a particular case of Lemma 4.2. We fix some $m \geq 2$ and suppose that (41) holds for every k -homogeneous polynomial with $k < m$. We may assume that m is even (being the case when m is odd completely analogous). Fix an m -homogeneous polynomial in n variables

$$P(z) = \sum_{|\alpha|=m} x_\alpha z^\alpha.$$

Since our goal involves estimating an integral of $P(z)$, we take advantage of the rotation invariance and work with $P(\varepsilon z)$, but this requires some preparation. For a fixed $1 \leq k \leq m/2$ and $A \subseteq [n]$ with $|A| = 2k$, take $k \leq l \leq m/2$ and define P_A and $P_{A,l}$ as in (39). Intuitively, $P_{A,l}$ detaches the z_i 's with odd exponent from the z_i 's with even exponent. This enables us to use the inductive hypothesis twice (once for the odd and once for the even part) to assemble the polynomials $P_{A,l}$. Let \mathbb{T}^{A^c} denote $|A^c|$ copies of the torus indexed in A^c . We get

$$\begin{aligned} \sum_{\gamma \in \Gamma_{A,l}} \sum_{\beta \in B_{A,l}} \|x_{2\beta+2\gamma+1_A}\|^q &\leq \sum_{\gamma \in \Gamma_{A,l}} C^{q(m/2-l)} \int_{\mathbb{T}^{A^c}} \left\| \sum_{\beta \in B_{A,l}} x_{2\beta+2\gamma+1_A} z^\beta \right\|^q dz \\ &\leq C^{q(m/2-l)} \int_{\mathbb{T}^{A^c}} \sum_{\gamma \in \Gamma_{A,l}} \left\| \sum_{\beta \in B_{A,l}} x_{2\beta+2\gamma+1_A} z^\beta \right\|^q dz \\ &= C^{q(m/2-l)} \int_{\mathbb{T}^{A^c}} \sum_{\gamma \in \Gamma_{A,l}} \left\| \sum_{\beta \in B_{A,l}} x_{2\beta+2\gamma+1_A} z^{2\beta} \right\|^q dz, \end{aligned}$$

where the last step follows by a change of variables. Since β is supported in A^c , the variables z_i with $i \in A$ do not appear in the expression above. So, we are still able to introduce them by applying the inductive hypothesis again. We obtain

$$\begin{aligned} \sum_{\gamma \in \Gamma_{A,l}} \sum_{\beta \in B_{A,l}} \|x_{2\beta+2\gamma+1_A}\|^q &\leq C^{q(m/2-l)} C^{q(l-k)} \int_{\mathbb{T}^n} \left\| \sum_{\gamma \in \Gamma_{A,l}} \left(\sum_{\beta \in B_{A,l}} x_{2\beta+2\gamma+1_A} z^{2\beta} \right) z^\gamma \right\|^q dz \\ &= C^{q(m/2-k)} \int_{\mathbb{T}^n} \left\| \sum_{\gamma \in \Gamma_{A,l}} \left(\sum_{\beta \in B_{A,l}} x_{2\beta+2\gamma+1_A} z^{2\beta} \right) z^{2\gamma} \right\|^q dz \\ &= C^{q(m/2-k)} \int_{\mathbb{T}^n} \left\| z^{1_A} \sum_{\gamma \in \Gamma_{A,l}} \left(\sum_{\beta \in B_{A,l}} x_{2\beta+2\gamma+1_A} z^{2\beta} \right) z^{2\gamma} \right\|^q dz \\ &= C^{q(m/2-k)} \int_{\mathbb{T}^n} \|P_{A,l}(z)\|^q dz, \end{aligned} \quad (42)$$

where in the last step we used (40). Since $P_{A,l}$ is the $2l$ -homogeneous component of P_A regarded as a function depending only on the variables z_i with $i \in A$, we have

$$\int_{\mathbb{T}^n} \|P_{A,l}(z)\|^q dz \leq \int_{\mathbb{T}^n} \|P_A(z)\|^q dz. \quad (43)$$

From (42) and (43), we deduce

$$\begin{aligned} \sum_{l=k}^{m/2} \sum_{\gamma \in \Gamma_{\lambda, l}} \sum_{\beta \in B_{\lambda, l}} \|x_{2\beta+2\gamma+1_{\lambda}}\|^q &\leq \left(\frac{m}{2} - k\right) C^{q(m/2-k)} \int_{\mathbb{T}^n} \|P_{\lambda, l}(z)\|^q dz \\ &\leq m C^{q(m/2-k)} \int_{\mathbb{T}^n} \|P_{\lambda}(z)\|^q dz. \end{aligned}$$

Finally we deal with $P(\varepsilon z)$ using (41) and (38). Taking Lemma 4.4 into consideration (and the definition of C) we get

$$\begin{aligned} \sum_{|\alpha|=m} \|x_{\alpha}\|^q &= \sum_{k=1}^{m/2} \sum_{A \in \mathcal{A}_k} \sum_{l=k}^{m/2} \sum_{\gamma \in \Gamma_{\lambda, l}} \sum_{\beta \in B_{\lambda, l}} \|x_{2\beta+2\gamma+1_{\lambda}}\|^q \\ &\leq m \int_{\mathbb{T}^n} \sum_{k=1}^{m/2} C^{q(m/2-k)} \sum_{A \in \mathcal{A}_k} \|P_A(z)\|^q dz \\ &\leq m \int_{\mathbb{T}^n} \sum_{k=1}^{m/2} C^{q(m/2-k)} C_{\varepsilon}^{2qk} \mathbb{E} \left\| \sum_{A \in \mathcal{A}_k} \varepsilon_A P_A(z) \right\|^q dz \\ &\leq m C^{qm/2} \int_{\mathbb{T}^n} \sum_{k=1}^{m/2} \mathbb{E} \left\| \sum_{A \in \mathcal{A}_k} \varepsilon_A P_A(z) \right\|^q dz. \end{aligned}$$

Using now Lemma 4.3, we have (note that $m^2 \leq B^{qm}$ for every m , since $B > 2$)

$$\begin{aligned} \sum_{|\alpha|=m} \|x_{\alpha}\|^q &\leq m C^{qm/2} B^{qm} \int_{\mathbb{T}^n} \sum_{k=1}^{m/2} \mathbb{E} \|P(\varepsilon z)\|^q dz \\ &\leq m^2 C^{qm/2} B^{qm} \mathbb{E} \int_{\mathbb{T}^n} \|P(\varepsilon z)\|^q dz \leq C^{qm} \int_{\mathbb{T}^n} \|P(z)\|^q dz. \end{aligned}$$

Hence (41) (and then (15)) holds for every m -homogeneous polynomial. To finish the argument, take an arbitrary polynomial of degree m

$$P(z) = \sum_{|\alpha| \leq m} x_{\alpha} z^{\alpha}.$$

For $0 \leq k \leq m$, denote by P_k its k -homogeneous projection. We have

$$\begin{aligned} \sum_{|\alpha| \leq m} \|x_{\alpha}\|^q &= \sum_{k=0}^m \sum_{|\alpha|=k} \|x_{\alpha}\|^q \leq \sum_{k=0}^m C^{qk} \int_{\mathbb{T}^n} \|P_k(z)\|^q dz \leq \sum_{k=0}^m C^{qk} \int_{\mathbb{T}^n} \|P(z)\|^q dz \\ &\leq (m+1) C^{qm} \int_{\mathbb{T}^n} \|P(z)\|^q dz \leq (2C)^{qm} \int_{\mathbb{T}^n} \|P(z)\|^q dz. \end{aligned}$$

This concludes the argument. □

We end this paper with a few comments regarding the proof of Theorem 3.2. In order to get Theorem 3.1 we used Lemmas 4.1, 4.2, 4.3 and 4.4. For the proof of Theorem 3.2 we essentially have to reverse all the inequalities. We use Lemmas 4.2 and 4.4 in the same way as before whereas Lemma 4.3 is simply replaced by the triangular inequality. Regarding Lemma 4.1 we replace it by the following analogous result.

Lemma 4.5. *Let X be a Banach space X of type $1 \leq p \leq 2$. Then for every $m, n \in \mathbb{N}$ and every family $\{x_A: A \subseteq [n], |A| = m\} \subseteq X$ we have*

$$\left(\int_{\mathbb{T}^n} \left\| \sum_A x_{AZ_A} \right\|^p dz \right)^{\frac{1}{p}} \leq (4T_p(X))^m \left(\sum_A \|x_A\|^p \right)^{\frac{1}{p}}.$$

Sketch of the proof. This is deduced by induction on m . Proceeding as in Lemma 4.1 for the inductive step we have

$$\left(\int_{\mathbb{T}^n} \left\| \sum_{\substack{A \subseteq [n] \\ |A|=m}} x_{AZ_A} \right\|^p dz \right)^{\frac{1}{p}} \leq \frac{4}{\binom{n}{k}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \left(\int_{\mathbb{T}^n} \left\| \sum_{\substack{A_1 \subseteq B \\ |A_1|=1}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-1}} x_{A_1 \cup A_2 Z_{A_1} Z_{A_2}} \right\|^p dz \right)^{\frac{1}{p}}.$$

Applying the type p inequality and the inductive hypothesis we get

$$\begin{aligned} \left(\int_{\mathbb{T}^n} \left\| \sum_{\substack{A \subseteq [n] \\ |A|=m}} x_{AZ_A} \right\|^p dz \right)^{\frac{1}{p}} &\leq (4T_p(X))^m \frac{1}{\binom{n}{k}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \left(\sum_{\substack{A_1 \subseteq B \\ |A_1|=1}} \sum_{\substack{A_2 \subseteq B^c \\ |A_2|=m-1}} \|x_{A_1 \cup A_2}\|^p \right)^{\frac{1}{p}} \\ &\leq (4T_p(X))^m \frac{1}{\binom{n}{k}} \sum_{\substack{B \subseteq [n] \\ |B|=k}} \left(\sum_{\substack{A \subseteq [n] \\ |A|=m}} \|x_A\|^p \right)^{\frac{1}{p}} \\ &= (4T_p(X))^m \left(\sum_{\substack{A \subseteq [n] \\ |A|=m}} \|x_A\|^p \right)^{\frac{1}{p}}. \quad \square \end{aligned}$$

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