A RIEMANN MANIFOLD STRUCTURE ON THE SPECTRA OF WEIGHTED ALGEBRAS OF HOLOMORPHIC FUNCTIONS

DANIEL CARANDO, DOMINGO GARCÍA, MANUEL MAESTRE, AND PABLO SEVILLA-PERIS

To the memory of Goyo Sevilla, a good, honest man.

Abstract. In this paper we give general conditions on a countable family V of weights on an unbounded open set U in a complex Banach space X such that the weighted space $HV(U)$ of holomorphic functions on U has a Fréchet algebra structure. For that kind of weights it is shown that the spectrum of $HV(U)$ has a natural analytic manifold structure when X is a symmetrically regular Banach space, in particular when $X = \mathbb{C}^n$.

1. Introduction

Weighted algebras of entire functions on $\mathbb{C}$ have been studied for many years. Berenstein, Li and Vidras in [7], Braun in [12] and Meise and Taylor in [22] used some particular weights to describe these types of algebras, for instance of infraexponential type. Motivated by this approach and by the very recent study by Carando and Sevilla-Peris [14] of weighted algebras of entire functions, essentially of exponential type, we study general conditions on a countable family V of weights such that the space $HV(U)$ has a Fréchet algebra structure. Moreover its spectrum $\mathfrak{M}V(U)$ has a natural analytic manifold structure whenever X is a symmetrically regular complex Banach space. This structure is based in the classical one given for any open (connected) subset of $\mathbb{C}^n$ (e.g. see [21, H.7.Lemma]), which was extended to the space $H_b(U)$ of holomorphic functions of bounded type on the open set U in [2] and to weighted algebras of entire functions, with weights of exponential type, in [14]. The notation and the approach of the proofs are infinitely dimensional, but the Riemann analytic structure obtained in Section 2 is new even for $\mathfrak{M}V(\mathbb{C}^n)$, $n = 1, 2, \ldots$, where our construction works since any finite dimensional Banach space is symmetrically regular. The main reason to write the results in the setting of Banach spaces is that if X is non reflexive Banach space, then the Riemann structure on $\mathfrak{M}V(U)$ is obtained on the topological bidual of

The first author was partially supported by ANPCyT PICT 05 17-33042, UBACyT Grant X038 and ANPCyT PICT 06 00587. The three last authors were supported in part by MEC and FEDER Project MTM2008-03211. The third author was also supported by Prometeo 2008/101.

X. In the case of $U = X$, in Section 3, we prove (Theorem 3.7) that $\mathfrak{M}V(X)$ is a disjoint union of analytic copies of X^{**}. To do that, a key ingredient is to extend the concept of associated weight given in [4] to the bidual of X.

By a Fréchet algebra we will understand an algebra for which the respective topological vector space is a Fréchet space in which the product is continuous (these are sometimes also called B_0-algebras). For a Banach space X, a function $P : X \to \mathbb{C}$ is called an n-homogeneous polynomial if there exists a continousm n-linear mapping $L : X \times \cdots \times X \to \mathbb{C}$ such that $P(x) = L(x, \ldots, x)$ for every x. The space of all n-homogenous polynomials is denoted by $\mathscr{P}(^n X)$. A function f is called holomorphic if for every point x there exists $(P_n(x))_n$ (with each $P_n(x) \in \mathscr{P}(^n X)$) so that $f = \sum_n P_n(x)$ in some ball around x.

Also, for $x \in X$ and $r > 0$, $B_X(x, r)$ (resp. $\overline{B}_X(x, r)$) will denote the open (resp. the closed) ball centered at x with radius r. Given an open set $U \subseteq X$, by a weight we will understand any continuous function $v : U \to [0, \infty[$. Following [3, 4, 5, 6, 8, 9, 10, 11, 14, 18, 19, 20] we consider a countable family $V = (v_n)_n$ of weights and define the space

$$HV(U) = \{f : U \to \mathbb{C}\colon \text{ holom. } \|f\|_v = \sup_{x \in U} v(x)|f(x)| < \infty \text{ for all } v \in V\}.$$

It is worth mentioning that, since each $\| \cdot \|_v$ is a seminorm and the family V is countable, we are dealing with Fréchet spaces and (when that is the case) Fréchet algebras. Also, the fact that X is finite or infinite dimensional makes no difference in our study.

Given a weight v, the associated weight $\tilde{v}$ was defined in [4] by

$$\tilde{v}(x) = \frac{1}{\sup\{|f(x)| : f \text{ holomorphic },\ \|f\|_v \leq 1\}}.$$

It is well known that $v \leq \tilde{v}$ [4, Proposition 1.2] and that, if U is absolutely convex, then $\|f\|_v = \|f\|_{\tilde{v}}$ for every f [4, Observation 1.12].

A set $A \subseteq U$ is said to be a U-bounded set if it is bounded and $d(A, X \setminus U) > 0$. The space of holomorphic functions on U that are bounded on U-bounded sets is denoted by $H_b(U)$. Following [18], we will say that a family of weights satisfies condition I if for every U-bounded set A there exists some $v \in V$ such that $\inf_{x \in A} v(x) > 0$. If condition I holds, then $HV(U)$ is continuously included in $H_b(U)$, a fact which we denote by $HV(U) \hookrightarrow H_b(U)$.

We will also consider the following conditions: for each $v \in V$ there exist $s > 0$, $w \in V$ and $C > 0$ so that

$$\operatorname{supp} v + \overline{B}_X(0, s) \subseteq U \tag{1}$$

$$v(x) \leq Cw(x + y) \text{ for all } x \in \operatorname{supp} v \text{ and all } y \in X \text{ with } \|y\| \leq s. \tag{2}$$

We will say that a family of weights V has **good local control** if it satisfies condition I, conditions (1)-(2) and X^* is contained in $HV(U)$. Unless otherwise is stated, we will always assume that V has good local control. Our interest in this good local control will become apparent in next section.

2. The analytic structure of the spectrum

Given a Fréchet algebra $\mathscr{A}$, its spectrum is the set of all non-zero, continuous, linear and multiplicative functionals $\phi : \mathscr{A} \to \mathbb{C}$. Our aim in this section is to define an analytic structure on $\mathfrak{M}V(U)$, the spectrum of $HV(U)$. We will follow essentially the same trends as in [2] (see also [14] or [16, Section 6.3]).

It is known [14, Proposition 1] that $HV(U)$ is an algebra if and only if for every $v \in V$ there exist $w \in V$ and $C > 0$ so that $v(x) \leq C\tilde{w}(x)^2$ for every $x \in U$. Clearly, this holds if we can get C and w such that

$$v(x) \leq Cw(x)^2. \tag{3}$$

The good local control will be crucial for the existence of the analytic structure on $\mathfrak{M}V(U)$ (see Theorem 2.12 and the lemmas preceding it). Let us then present some examples of families that enjoy this property jointly with (3), for which our main results in this section apply.

Example 2.1. In [18, Example 14] a family of weights V is defined so that $HV(U) = H_b(U)$. Obviously, this family V has good local control.

If we consider entire functions, condition (1) is trivially satisfied. In this case, a standard way to define a family of weights such that $HV(X)$ is an algebra is to take a continuous and decreasing function $\varphi : [0, \infty[\to]0, \infty[$ such that $\lim_{t\to\infty} t^k \varphi(t) = 0$ for every k (this condition is needed to get that $X^* \hookrightarrow HV(X)$) and then define weights $v_n(x) = \varphi(\|x\|)^{1/n}$. If we define the family $V = \{v_n\}_n$, condition (2) translates into restrictions on the decreasing rate of φ.

Proposition 2.2. *V defined as above has good local control if and only if there exist $\alpha \geq 1$ and $s > 0$ such that*

$$\sup_{t\in\mathbb{R}} \frac{\varphi(t)^\alpha}{\varphi(t + s)} < \infty. \tag{4}$$

Proof. Let us assume first that for each n there exist $s, C > 0$ and m so that $v_n(x) \leq Cv_m(x+y)$ for every x and $\|y\| \leq s$ (i.e. V satisfies (2)). Given $t \in \mathbb{R}$, let us choose $x \in X$ with $\|x\| = t$ and put $y = \frac{s}{\|x\|}x$. Then $\|x+y\| = t+s$ and we have $\varphi(t)^{1/n} \leq C\varphi(t+s)^{1/m}$. Defining $\alpha = m/n$ we have (4).

On the other hand, let $K = \sup \varphi(t)^\alpha / \varphi(t+s)$. Given n we choose m such that $m/n \geq \alpha$ and t_0 such that $\varphi(t) < 1$ for all $t \geq t_0$. Then $\varphi(t)^{m/n} \leq \varphi(t)^\alpha \leq K\varphi(t+s)$ for every $t \geq t_0$. Taking $M = \sup_{t\in[0,t_0]} \varphi(t)^{m/n}/\varphi(t+s)$ and $C = \max(K^{1/m}, M^{1/m})$ we have

$$\varphi(t)^{1/n} \leq C\varphi(t+s)^{1/m}$$

for every t.

Finally, if $\|x\| = t$ and $\|y\| \leq s$ then $\|x+y\| \leq t+s$. Then, since φ is decreasing, we have $\varphi(t+s) \leq \varphi(\|x+y\|)$ and this gives $v_n(x) \leq Cv_m(x+y)$. ☐

In [14] weights are defined as above in terms of a function φ satisfying

$$\varphi(s)\varphi(t) \leq C\varphi(s+t) \tag{5}$$

for some constant $C > 0$ and all t, s. In that case an analytical structure is given to the spectrum of $HV(X)$. It is easy to check that if V satisfies (5) then it also satisfies (4). As a consequence, we have that $\varphi(t) = e^{-t^k}$ satisfy (4) for every k (the fact that $(s+t)^k \leq 2^k(t^k+s^k)$ implies that φ satisfies (5)). However, the converse is not true, as it is shown in the next example. Therefore, even for entire functions, our setting is more general than that of [14].

Example 2.3. The function $\varphi(t) = e^{-e^t}$ satisfies (4) but does not satisfy (5). Indeed, if we choose s and α so that $e^s \leq \alpha$ we have that

$$\frac{\left(e^{-e^t}\right)^\alpha}{e^{-e^{t+s}}} = \left(\frac{e^{e^s}}{e^\alpha}\right)^{e^t} \leq 1,$$

for every $t \geq 0$.

Example 2.4. Let $(a_n)_n$ be a sequence such that $a_n \geq 0$ for all n, $a_0 > 0$ and $\frac{a_n}{a_{n-1}} \leq \frac{1}{n}$ (equivalently $\frac{a_n}{a_k} \leq \frac{k!}{n!}$ for all $k \leq n$). Then we define

$$\varphi(t) = \Big(\sum_{n=0}^{\infty} a_n t^n\Big)^{-1}.$$

Let us see that $\varphi(t) \leq e^s \varphi(s+t)$ for all s, t (this obviously implies (4)).

$$\sum_{n=0}^{\infty} a_n (s+t)^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} s^k t^{n-k} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n \frac{a_{n-k}}{a_n} \frac{1}{k!} s^k t^{n-k}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} s^k a_{n-k} t^{n-k} = \Big(\sum_{k=0}^{\infty} \frac{1}{k!} s^k \Big) \Big(\sum_{n=k}^{\infty} a_{n-k} t^{n-k} \Big) = e^s \sum_{n=0}^{\infty} a_n t^n.$$

This clearly implies our claim.

Examples of sequences satisfying this condition can be constructed by taking $p_{n+1} \geq p_n > 1$ and defining $a_n = \left(\frac{1}{n!}\right)^{p_n}$ (e.g. $a_n = \left(\frac{1}{n!}\right)^{p}$ or $a_n = \left(\frac{1}{n!}\right)^{n}$). Obviously, for $a_n = \frac{1}{n!}$ we get $\varphi(t) = e^{-t}$.

Condition (4) implies that the function φ cannot decrease 'too fast', as next example shows.

Example 2.5. The function $\varphi(t) = e^{-e^{t^2}}$ does not satisfy (4). Indeed, for each fixed $\alpha \geq 1$ and $s > 0$ we have

$$\frac{\left(e^{-e^{t^2}}\right)^{\alpha}}{e^{-e^{(t+s)^2}}} = \frac{e^{-\alpha e^{t^2}}}{e^{-e^{t^2+s^2+2st}}} = \frac{e^{-\alpha e^{t^2}}}{e^{-e^{t^2} e^{s^2} e^{2st}}} = \left(\frac{e^{\alpha}}{e^{e^{s^2} e^{2st}}} \right)^{-e^{t^2}} = \left(\frac{e^{e^{s^2} e^{2st}}}{e^{\alpha}} \right)^{e^{t^2}}$$

and this tends to ∞.

Example 2.6. Let X be a complex Banach space and $\varphi : [0,\infty[\to]0,\infty[$ be an increasing and convex continuous function. Define the weights $v_\lambda(x) = e^{-\lambda\varphi(\|x\|/\lambda)}$ for $0 < \lambda \in \mathbb{Q}$ and the family of weights $V = \{v_\lambda\}$. Since

$$v_\lambda(x) v_\lambda(y) \leq v_{2\lambda}(x+y),$$

for every $x, y \in X$, then condition (2) is fulfilled, for example, considering $s = 1$ and $C = \max\{e^{\lambda\varphi(t/\lambda)} \ : 0 \leq t \leq 1\}$. Hence V has good local control. On the other hand φ is increasing, hence

$$2\lambda\varphi(\frac{\|x\|}{\lambda}) \geq 2\lambda\varphi(\frac{\|x\|}{2\lambda}),$$

for every $x \in X$, and we obtain that

$$v_\lambda(x) \leq (v_{\lambda/2}(x))^2,$$

for all $x \in X$. So, as condition (3) is also satisfied, $HV(X)$ is a Fréchet algebra.

Example 2.7. In [7, 12, 22] the following weighted algebra is studied:

$$A_p^0 = \{f \in H(\mathbb{C}) : \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{1}{n}p(z)} < \infty, \quad \text{for all} \ \ n \in \mathbb{N}\},$$

where $p : \mathbb{C} \to [0, \infty[$ has the following properties:

(i) p is continuous and subharmonic.

(ii) $\log(1+|z|^2) = o(p(z))$.

(iii) There exists $C \geq 1$ such that for all $y \in \mathbb{C}$

$$\sup_{|z-y|\leq 1} p(z) \leq C \inf_{|z-y|\leq 1} p(z) + C,$$

and hence

(iv) $p(x+y) \leq Cp(x) + C$ for all $x \in \mathbb{C}$ and all $y \in \mathbb{C}$ with $|y| \leq 1$.

In [7, 12], (iii) is replaced by

(iii') $p(2z) = O(p(z))$

that also implies (iv).

Define $v_n(x) = e^{-\frac{1}{n}p(z)}$ for $n \in \mathbb{N}$ and $V = (v_n)$. Since

$$e^{-\frac{1}{n}p(x)} \leq e^{\frac{1}{n}} e^{-\frac{1}{nC}p(x+y)}$$

for all $x \in \mathbb{C}$ and all $y \in \mathbb{C}$ with $|y| \leq 1$, we have that if $m \geq nC$, $m \in \mathbb{N}$,

$$v_n(x) \leq e^{\frac{1}{n}} v_m(x+y)$$

for all $x \in \mathbb{C}$ and all $y \in \mathbb{C}$ with $|y| \leq 1$, and then $A_p^0 = HV(\mathbb{C})$ is a Fréchet algebra and V has good local control (actually, V has what later is called excellent local control).

A more general algebras A_P^0 for families $P = (p_n)$ are given in [22, 1.2. Definition]. In a similar way it can be checked that A_P^0 is a Fréchet algebra and V has good (excellent) local control.

Let us now turn our attention to functions defined on open subsets of X. If $U \subseteq X$ is a bounded, open set and V is a family of bounded weights, it is easy to check that if V satisfies (1) then $HV(U) = H_b(U)$. The condition that the weights be bounded is an extra hypothesis, but it is actually fulfilled by all usual examples. Thus, we will always consider unbounded sets, as in the following examples.

Example 2.8. In $\mathbb{C}^2$ we consider $U = \mathbb{C} \times \mathbb{D}$ (where $\mathbb{D}$ is the open unit disk). Then we define functions ψ_n on $[0,1]$ letting $\psi_n \equiv 1$ on $[0,1/n]$, $\psi_n \equiv 0$ on $[1/(n+1),1]$ and linear on $[1/n,1/(n+1)]$ and we consider weights defined on U by

$$v_n(z_1,z_2) = e^{-\frac{|z_1|}{n}} \psi_n(|z_2|).$$

$HV(U)$ is a Fréchet algebra and the sequence $V = (v_n)_n$ has good local control.

The previous example can be seen as a particular case of the following:

Example 2.9. Let X_1, X_2 be two Banach spaces, $X = X_1 \oplus_p X_2$ and $U = X_1 \oplus B_{X_2}(0,R)$. We choose a strictly increasing sequence $(b_n)_n$ such that $b_n > 0$ for all n and $\lim_n b_n = R$. We consider ψ_n such that $\psi_n \equiv 1$ on $[0, b_n]$, $\psi_n \equiv 0$ on $[b_{n+1}, R]$ and ψ_n is linear on $[b_n, b_{n+1}]$ and take φ satisfying (4). Then we define weights by

$$v_n(x_1, x_2) = \varphi(\|x_1\|)^{1/n} \psi_n(\|x_2\|).$$

It is not difficult to check that the family $V = (v_n)_n$ has good local control and $HV(U)$ is a Fréchet algebra.

We consider now a symmetrically regular complex Banach space X. In particular, this covers the case $X = \mathbb{C}^n$ for $n = 1, 2 \ldots$ (in other words, the one or several complex variables cases). Let $U \subseteq X$ be open and unbounded, and V be a countable family of weights defined on U with good local control and satisfying (3). Our aim is to define an analytic structure on $\mathfrak{M}V(U)$.

In the particular case that U is an open convex and balanced subset of X, we denote by $\overset{\circ}{\overline{U}}{}^{w^*}$ the norm-interior in X^{**} of the weak-star closure of U. Given a holomorphic function of bounded type $f : U \subseteq X \longrightarrow \mathbb{C}$ we denote the Aron-Berner extension of f by $\bar{f} : \overset{\circ}{\overline{U}}{}^{w^*} \subseteq X^{**} \longrightarrow \mathbb{C}$. Then for each $z \in \overset{\circ}{\overline{U}}{}^{w^*}$, the mapping δ_z defined by $\delta_z(f) = \bar{f}(z)$ is in $\mathfrak{M}V(U)$. In this way, we can do $\overset{\circ}{\overline{U}}{}^{w^*} \hookrightarrow \mathfrak{M}V(U)$. For details see e.g. [13, page 620] and use that the family of weights V satisfies condition I.

For a general open set U, we do not have a canonical set in the bidual to which any holomorphic function can be extended. But we can fruitfully use a kind of 'local Aron-Berner extensions' as the following method shows. Our next step is, for a given $\phi \in \mathfrak{M}V(U)$, to find $r = r(\phi) > 0$ such that for any $z \in X^{**}$ with $\|z\| < r$, we can define a new $\phi^z \in \mathfrak{M}V(U)$ which is, in some sense, close to ϕ. To do that we need some preliminary work. Given any $f \in HV(U)$ and $x \in U$ we take the Taylor series expansion $\sum_{k=0}^{\infty} P_n(x)$ (each $P_n(x) \in \mathscr{P}(^n X)$) of f at x and, for each polynomial, we consider its Aron-Berner extension $\overline{P_n(x)} \in \mathscr{P}(^n X^{**})$. Hence, for each fixed $z \in X^{**}$ the mapping $U \to \mathbb{C}$ defined by $x \rightsquigarrow \overline{P_n(x)}(z)$ is well defined and holomorphic. Let us see that it also belongs to $HV(U)$; this follows from the next result.

Lemma 2.10. *Suppose V has good local control. Given $v \in V$, there exist $C, s > 0$ and $w \in V$ so that $\|x \rightsquigarrow \overline{P_n(x)}(z)\|_v \leq C\|f\|_w$ for every $f \in HV(U)$, $\|z\| < s$ and $n = 0, 1, 2, \ldots$*

Proof. Given $v \in V$, by the good local control of V there exist $C, s > 0$ and $w \in V$ so that $\operatorname{supp} v + \overline{B}_X(0,s) \subseteq U$ and $v(x) \leq Cw(x+y)$ for all $x \in \operatorname{supp} v$ and all $\|y\| \leq s$. Fixed $x \in \operatorname{supp} v$, $f \in HV(U)$ and $0 < r < s$, we have that $f \in H_b(B_X(x,r))$, and hence we can use a 'local Aron-Berner extension' $\bar{f}_x : B_{X^{**}}(x,s) \longrightarrow \mathbb{C}$ given by

$$\bar{f}_x(u) = \sum_{n=0}^{\infty} \overline{P_n(x)}(u-x),$$

for $u \in X^{**}$ such that $\|u - x\| < s$. Now, by Cauchy's integral formula we have for each n

$$\overline{P_n(x)}(z) = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{\bar{f}_x(x+\lambda z)}{\lambda^{n+1}} d\lambda.$$

Then

$$v(x)|\overline{P_n(x)}(z)| \leq \frac{1}{2\pi} \int_{|\lambda|=1} v(x)|\bar{f}_x(x+\lambda z)| d|\lambda|.$$

Now, for every fixed $x \in U$ and $z \in X^{**}$ with $\|z\| < s$ we get, by [15, Lemma page 355], some $(x_\alpha)_\alpha \subseteq X$ with $\|x_\alpha\| \leq \|z\| < s$ for all α so that $|\bar{f}_x(x+\lambda z)| = \lim_\alpha |f(x+\lambda x_\alpha)|$. Then, using (1) we have

$$\begin{aligned} v(x)|\bar{f}_x(x+\lambda z)| = \lim_\alpha v(x)|f(x+\lambda x_\alpha)| &\leq \sup_\alpha v(x)|f(x+\lambda x_\alpha)| \\ &\leq \sup_\alpha Cw(x+\lambda x_\alpha)|f(x+\lambda x_\alpha)| \leq C\|f\|_w. \end{aligned}$$

This altogether gives

$$v(x)|\overline{P_n(x)}(z)| \leq C\|f\|_w,$$

for all $x \in U$ and all $z \in X^{**}$, $\|z\| < s$. □

Now, for each given $\phi \in \mathfrak{M}V(U)$ there exists some $v \in V$ such that $|\phi(f)| \leq \|f\|_v$. With this v we have s, C, w satisfying (2). Now, if $z \in X^{**}$ and $\|z\| < s$ then

$$\begin{aligned} \sum_{n=0}^{\infty} |\phi(x \rightsquigarrow \overline{P_n(x)}(z))| \leq \sum_{n=0}^{\infty} \|x \rightsquigarrow \overline{P_n(x)}(z)\|_v &= \sum_{n=0}^{\infty} \sup_{x\in U} v(x) \Big| \overline{P_n(x)}\big(\frac{s}{\|z\|}z\big)\big(\frac{\|z\|}{s}\big)^n \Big| \\ &= \sum_{n=0}^{\infty} \Big\| x \rightsquigarrow \overline{P_n(x)}\big(\frac{s}{\|z\|}z\big)\Big\|_v \big(\frac{\|z\|}{s}\big)^n \leq C\|f\|_w \sum_{n=0}^{\infty} \big(\frac{\|z\|}{s}\big)^n. \end{aligned}$$

This implies that for each fixed $\phi \in \mathfrak{M}V(U)$ and $z \in X^{**}$ with $\|z\| < s$, the mapping $\phi^z : HV(U) \to \mathbb{C}$ given by

$$\phi^z(f) = \sum_{n=0}^{\infty} \phi(x \rightsquigarrow \overline{P_n(x)}(z)) \tag{6}$$

is well defined, linear and continuous. Moreover, proceeding as in [2, page 551], using the fact that ϕ is multiplicative, we have that $\phi^z(fg) = \phi^z(f)\phi^z(g)$ for all $f, g \in HV(U)$. Then $\phi^z \in \mathfrak{M}V(U)$.

If $X^* \hookrightarrow HV(U)$, we can follow [1] and [2], and consider the mapping $\pi : \mathfrak{M}V(U) \to X^{**}$ given by $\pi(\phi) = \phi|_{X^*}$.

Now, if $x^* \in X^*$, we have $x^*(y) = x^*(x) + x^*(y - x)$; this means that the Taylor series expansion of $x^* \in HV(U)$ around x consists of $P_0(x) = x^*(x)$ and $P_1(x)(y - x) = x^*(y - x)$; considering the extensions to the bidual we get $\overline{P_0(x)}(z) = x^*(x)$ and $\overline{P_1(x)}(z) = z(x^*)$. Then

$$\phi^z(x^*) = \sum_n \phi(x \rightsquigarrow \overline{P_n(x)}(z)) = \phi(x \rightsquigarrow x^*(x)) + \phi(x \rightsquigarrow z(x^*)) = \phi(x^*) + z(x^*).$$

Since this holds for every $x^* \in X^*$, this means $\pi(\phi^z) = \pi(\phi) + z$.

Our aim is to show that the sets

$$V_{\phi,\varepsilon} = \{\phi^z \colon z \in X^{**}\,,\ \|z\| < \varepsilon\} \tag{7}$$

where $\varepsilon < s$ for some $s > 0$ depending on ϕ, form a basis of neighbourhoods of a Hausdorff topology on $\mathfrak{M}V(U)$. We need first the following technical lemma. We denote by $A_n(x)$ and $\overline{A_n(x)}$ the symmetric n-linear forms associated to $P_n(x)$ and $\overline{P_n(x)}$, respectively.

Lemma 2.11. *For $f \in HV(U)$, let $\overline{P_n(x)}$ and $\overline{A_n(x)}$ be as before. If V has good local control, then given $v \in V$, there exist $C, s > 0$ and $w \in V$ so that, for every $z_1, \dots, z_n \in X^{**}$, the following holds*

$$\|x \rightsquigarrow \overline{A_n(x)}(z_1, \dots, z_n)\|_v \leq C \frac{n^n}{n! s^n} \|f\|_w \|z_1\| \dots \|z_n\|.$$

Proof. First of all we have

$$\begin{aligned}\overline{A_n(x)}(z_1, \dots, z_n) &= \frac{1}{2^n n!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n \overline{P_n(x)}(\varepsilon_1 z_1 + \ldots + \varepsilon_n z_n) \\ &= \frac{n^n}{2^n n!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n \overline{P_n(x)}\Big(\frac{\varepsilon_1 z_1 + \ldots + \varepsilon_n z_n}{n}\Big).\end{aligned}$$

If $\|z_1\|, \dots, \|z_n\| < s$ then $\left\|\frac{\varepsilon_1 z_1 + \ldots + \varepsilon_n z_n}{n}\right\| < s$. Now, given $v \in V$ we have C, s, w from (2) and, applying Lemma 2.10,

$$\begin{aligned}v(x)|\overline{A_n(x)}(z_1, \dots, z_n)| &\leq v(x) \frac{n^n}{2^n n!} \sum_{\varepsilon_j = \pm 1} \Big|\overline{P_n(x)}\Big(\frac{\varepsilon_1 z_1 + \cdots + \varepsilon_n z_n}{n}\Big)\Big| \\ &\leq \frac{n^n}{2^n n!} \sum_{\varepsilon_j = \pm 1} C \|f\|_w = \frac{n^n}{n!} C \|f\|_w.\end{aligned}$$

For general $z_1, \dots z_n \in X^{**}$ the conclusion follows easily. $\square$

Now we are ready to show that for symmetrically regular Banach spaces, we have an analytic structure on $\mathfrak{M}V(U)$, where the neighbourhoods are given by (7). In the proof of the following theorem, by (y^k, z^{n-k}) we mean that y is repeated k times and z is repeated $n-k$ times.

Theorem 2.12. *Suppose U is an open subset of a symmetrically regular Banach space X and V is a countable family of weights which has good local control such that $HV(U)$ is a Fréchet algebra. Then, $\pi : \mathfrak{M}V(U) \to X^{**}$ gives a structure of Riemann analytic manifold on $\mathfrak{M}V(U)$, where π is defined by $\pi(\phi) = \phi|_{X^*}$ for $\phi \in \mathfrak{M}V(U)$.*

Proof. We are going to see that the system defined in (7) for $\varepsilon < \frac{s}{2e}$ $(s = s(\phi))$ is a basis of neighbourhoods of a Hausdorff topology on $\mathfrak{M}V(U)$. The main step is to check that, for any given $\phi \in \mathfrak{M}V(U)$ and $y, z \in X^{**}$, we have $(\phi^y)^z = \phi^{y+z}$. To begin with, we fix $\phi \in \mathfrak{M}V(U)$ and we choose $v \in V$ such that $|\phi(f)| \leq \|f\|_v$ for every $f \in HV(U)$. For that $v \in V$ and all $y, z \in X^{**}$ (C, s, w are taken satisfying (2)), we have

$$\sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\binom{m}{k}\|x \rightsquigarrow \overline{A_m(x)}(y^{m-k}, z^k)\|_v = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\|x \rightsquigarrow \overline{A_m(x)}(y^{m-k}, z^k)\|_v$$
$$\leq \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k} C\frac{m^m}{m!s^m}\|f\|_w\|y\|^{m-k}\|z\|^k$$
$$\leq \sum_{m=0}^{\infty}\Big(\sum_{k=0}^{m}\binom{m}{k}\Big(\frac{e}{s}\Big)^m\|y\|^{m-k}\|z\|^k\Big)C\|f\|_w$$
$$= C\|f\|_w\sum_{m=0}^{\infty}\Big(\frac{\|y\| + \|z\|}{s/e}\Big)^m,$$

and the last sum is finite whenever $\|y\|, \|z\| < \frac{s}{2e}$. This means that the series $\sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\binom{m}{k}[x \rightsquigarrow \overline{A_m(x)}(y^{m-k}, z^k)]$ converges (absolutely) in $HV(U)$ and then

$$\phi\Big(x \rightsquigarrow \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\binom{m}{k}\overline{A_m(x)}(y^{m-k}, z^k)\Big) = \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\binom{m}{k}\phi(x \rightsquigarrow \overline{A_m(x)}(y^{m-k}, z^k)),$$

and this last series converges absolutely too.

Now we take $y, z \in X^{**}$ with $\|y\|, \|z\| < \frac{s}{2e}$ and $f \in HV(U)$. In order to compute $(\phi^y)^z(f)$ we need the Taylor series expansion of the mapping $x \rightsquigarrow \overline{P_m(x)}(z)$

(since we will have to apply ϕ^y to this mapping). This is done in [2]. We have

$$(\phi^y)^z(f) = \sum_{k=0}^{\infty} \phi^y(x \rightsquigarrow \overline{P_k(x)}(z)) = \sum_{k=0}^{\infty}\sum_{m=k}^{\infty} \binom{m}{k} \phi(x \rightsquigarrow \overline{A_m(x)}(y^{m-k}, z^k))$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{m} \binom{m}{k} \phi(x \rightsquigarrow \overline{A_m(x)}(y^{m-k}, z^k)) = \sum_{m=0}^{\infty} \phi\Big(x \rightsquigarrow \sum_{k=0}^{m} \binom{m}{k} \overline{A_m(x)}(y^{m-k}, z^k)\Big)$$

$$= \sum_{m=0}^{\infty} \phi(x \rightsquigarrow \overline{P_m(x)}(z+y)) = \phi^{z+y}(f).$$

From this, it easily follows that the sets $V_{\phi,\varepsilon}$ define a topology. The fact that this topology is Hausdorff follows as in [2].

The composition of the inverse of any chart with another chart given by the suitable restrictions of π, gives always the identity on certain open subsets of X. This is obviously a holomorphic mapping, so we get that $\pi : HV(U) \to X^{**}$ produces an analytic manifold structure in $\mathfrak{M}V(U)$. For details see [2, Corollary 2.4]. $\square$

Corollary 2.13. *Suppose U is an open subset of $\mathbb{C}^n$ and V is a countable family of weights on U such that it has good local control and $HV(U)$ is a Fréchet algebra. Then the spectrum $\mathfrak{M}V(U)$ of the algebra $HV(U)$ has structure of Riemann analytic manifold given by $\pi : \mathfrak{M}V(U) \to \mathbb{C}^n$.*

This corollary extends to weighted spaces on open subsets of $\mathbb{C}^n$ the classical result for $H(U)$ (see [21, H.7.Lemma]).

3. Extensions to the bidual and the spectrum of $HV(X)$

One of the aims of this section is to show that the spectrum of $HV(X)$ can be seen as a disjoint union of analytic copies of X^{**}, just as in the case of the spectrum of $H_b(X)$ [2, 16]. For this, we must study the extensions of a weight on X to its bidual X^{**}. Given a weight v on X, we define, in the spirit of the associated weight, the associated extension

$$\hat{v}(z) = \frac{1}{\sup\{|\bar{f}(z)| : f \in Hv(X),\ \|f\|_v \le 1\}}. \tag{8}$$

Note that $\hat{v}(x) = \tilde{v}(x)$ whenever x belongs to X.

This extension will be seen to have many good properties. It can be hard to compute, but we will show that this weight is somehow equivalent to more natural and simple extensions. As a consequence, the simple extensions will share the good properties of the associated extension.

First, let us see that the Aron-Berner extension is an isometry from $Hv(X)$ into $H\hat{v}(X^{**})$. Indeed, since $\|f\|_v = \|f\|_{\tilde{v}}$ for all f in $Hv(X)$, we have

$$\begin{aligned} \|f\|_v &= \sup_{x\in X} |f(x)|\tilde{v}(x) \le \sup_{x\in X} |f(x)|\tilde{v}(x) \\ &\le \sup_{z\in X^{**}} |\bar{f}(z)|\hat{v}(z) = \|\bar{f}\|_{\hat{v}}. \end{aligned}$$

On the other hand,

$$|\bar{f}(z)| = \|f\|_v \left|\overline{\left(\frac{f}{\|f\|_v}\right)}(z)\right| \le \|f\|_v \ \sup\{|\bar{g}(z)| : g \in Hv(X),\ \|g\|_v \le 1\} = \frac{\|f\|_v}{\hat{v}(z)},$$

from which the reverse inequality follows. We have shown:

Proposition 3.1. *Let v be a weight on X and $\hat{v}$ be its associated extension to X^{**}. For each $f \in Hv(X)$ we have that $\bar{f}$ belongs to $H\hat{v}(X^{**})$ and $\|\bar{f}\|_{\hat{v}} = \|f\|_v$.*

Now, suppose we have a family V of weights on X such condition (2) is fulfilled. Let us see that the family of associated extensions also verify (2). First, note that if we have $z \in X^{**}$ and a net (x_α) with $\|x_\alpha\| \le \|z\|$ for all α such that x_α converges polynomially to z (in the sense of Davie-Gamelin [15]), then for any x in X and any $f \in H_b(X)$ we have that $f(x + x_\alpha) \to \bar{f}(x + z)$. Also, given z in X we can define $\tau_z f : X \to \mathbb{C}$ as $\tau_z f(x) = \bar{f}(x + z)$. Recall that if X is symmetrically regular, then the Aron-Berner extension of $\tau_z f$ is simply $\overline{\tau_z f}(z') = \bar{f}(z' + z)$ [1]. Therefore, for (x_α) as before, we also have $\bar{f}(z' + x_\alpha) \to \bar{f}(z' + z)$

Now we can state the following.

Theorem 3.2. *If the family V of weights on the symmetrically regular Banach space X satisfies condition (2), then so does the family $\hat{V} = \{\hat{v} : v \in V\}$.*

Proof. Given $v \in V$, let C, s and w be as in (2). Fix z, z' in X^{**} with $\|z\| < s$ and consider a net (x_α) converging polynomially to z such that $\|x_\alpha\| < s$ for all α. Note that

$$\begin{aligned} v(x)|\tau_z f(x)| &= \lim_\alpha v(x)|f(x + x_\alpha)| \le C \limsup w(x + x_\alpha)|f(x + x_\alpha)| \\ &\le C\|f\|_w, \end{aligned}$$

and therefore $\|\tau_z f\|_v \le C\|f\|_w$ for all $z \in X^{**}$ with $\|z\| < s$. Now we have

$$\begin{aligned} \frac{1}{\hat{w}(z' + z)} &= \sup\{|\bar{f}(z' + z)| : \|f\|_w \le 1\} = \sup\{|\overline{\tau_z f}(z')| : \|f\|_w \le 1\} \\ &\le \sup\{|\overline{\tau_z f}(z')| : \|\tau_z f\|_v \le C\} \le C \sup\{|\bar{g}(z')| : \|g\|_v \le 1\} \\ &= C\frac{1}{\hat{v}(z')}, \end{aligned}$$

which completes the proof. $\square$

As we have already mentioned, condition (1) is automatic when the domain under consideration is X. Note also that the dependence of s on v in condition (2) is forced precisely by condition (1), so for entire functions we feel that the following variant of condition (2) is more appropriate.

Definition 3.3. A family V of weights on X is said to satisfy condition ($\star$) if there exists $s > 0$ such that, for any $v \in V$, we can find $C > 0$ and $w \in V$ for which $v(x) \leq Cw(x+y)$ for all $x, y \in X$, $\|y\| < s$. We will say that the family V has excellent local control when it satisfies all the conditions involved in the good local control, but changing condition (2) to condition ($\star$).

All examples in Section 2 of weighted spaces of entire functions satisfying good local control actually also satisfy excellent local control condition. In fact, for any family of the form $\{v^{1/m} : m \in \mathbb{N}\}$, conditions (4) and ($\star$) are equivalent, since the value of s that works for v, works for $v^{1/m}$ for all m. It is clear that with the same proof as Theorem 3.2 we have the following.

Theorem 3.4. *If the family V of weights on the symmetrically regular Banach space X satisfies condition ($\star$), then so does the family $\hat{V} = \{\hat{v} : v \in V\}$.*

So we have:

Corollary 3.5. *Let V be a family of weights on X satisfying condition ($\star$) and suppose X is symmetrically regular. For each $z \in X^{**}$, the mapping $\tau_z : HV(X) \to HV(X)$ given by $\tau_z f(x) = \bar{f}(x+z)$ is continuous.*

Proof. Choose $N \in \mathbb{N}$ such that $\|z\| < Ns$. For $v \in V$, we can take C, s and w be as in Definition 3.3. Proceeding as in Theorem 3.2, we can see that $\|\tau_{z/N} f\|_v \leq C\|f\|_w$. This means that $\tau_{z/N} : HV(X) \to HV(X)$ is a continuous operator. As a consequence, $\tau_z = (\tau_{z/N})^N$ is continuous.

□

Now we are ready to simplify the description of $\mathfrak{M}V(X)$, for X symmetrically regular and V with excellent local control. Indeed, given $\phi \in \mathfrak{M}V(X)$, we can give an alternative definition of ϕ^z that works for all $z \in X^{**}$. First, let us define $J_\phi(z) : HV(X) \to \mathbb{C}$ by $J_\phi(z)(f) = \phi(\tau_z f)$. Since τ_z is multiplicative (because the Aron-Berner extension is multiplicative) and, as we have shown, is continuous, we have that $J_\phi(z)$ belongs to $\mathfrak{M}V(X)$.

Since X is symmetrically regular, we have that $\tau_{z'+z}(g) = \tau_{z'} \circ \tau_z(g)$ for all $z, z' \in X^{**}$ and all $g \in H_b(X)$ [1, Theorem 8.3.(vii)] or [2, Lemma 2.1]. Therefore, since $HV(X)$ is contained in $H_b(X)$ we have that $J_\phi(z'+z)(f) = J_\phi(z')(\tau_z f)$ for all $z, z' \in X^{**}$ and all $f \in HV(X)$. This fact will allow us to show the following.

Lemma 3.6. *If X is a symmetrically regular Banach space and V is a countable family of weights with excellent local control such that $HV(X)$ is a Fréchet algebra, then the mapping*

$$J_\phi : X^{**} \to \mathfrak{M}V(X)$$

is bicontinuous into its image (in fact, J_ϕ is bianalytic), when $\mathfrak{M}V(X)$ is endowed with the analytic structure defined in Section 2.

Proof. Fix $z' \in X^{**}$ and let us show that J_ϕ is continuous at z'. For this, let us take the constant $s > 0$ provided by condition $(\star)$. If we consider $\|z\| < s$, then $J_\phi(z'+z)(f) = J_\phi(z')(\tau_z f) = (J_\phi(z'))^z(f)$ (the last equality follows expanding $\tau_z f$ and comparing this with the definition of Ψ^z for $\Psi \in \mathfrak{M}V(X)$). Since the analytic structure of $\mathfrak{M}V(X)$ was defined so that $z \mapsto \Psi^z$ is analytic in $\|z\| < s$ for any $\Psi \in \mathfrak{M}V(X)$, we have that $J_\phi(z'+z)$ is an analytic function of z in $\|z\| < s$, so in particular J_ϕ is analytic at z'.

On the other hand, since $\pi(J_\phi(z)) = \pi(\phi) + z$, we have that $J_\phi^{-1}(\Psi) = \pi(\Psi) - \pi(\phi)$ for $\Psi \in J_\phi(X^{**})$ and the inverse of J_ϕ is also continuous (and analytic). $\square$

If we look at the previous proof for the particular case $z' = 0$, we note that for $\|z\| < s$, $J_\phi(z)$ and ϕ^z coincide. Therefore, we could define ϕ^z for arbitrary z by setting $\phi^z := J_\phi(z) = \phi \circ \tau_z$, and this will extend our previous definition of ϕ^z in Section 2.

As a consequence of Lemma 3.6 we have an analytic copy of X^{**} in the connected component of $\mathfrak{M}V(X)$ containing ϕ. Since this analytic copy of X^{**} is necessarily open and closed, it must coincide with the connected component. Then we have:

Theorem 3.7. *Let X be a symmetrically regular Banach space and V a countable family of weights on X with excellent local control such that $HV(X)$ is a Fréchet algebra. Then, $\mathfrak{M}V(X)$ is a disjoint union of analytic copies of X^{**}. Each copy is given by $\{\phi \circ \tau_z : z \in X^{**}\}$ for some $\phi \in \mathfrak{M}V(X)$, where $\tau_z f(x) = \bar{f}(x+z)$ for all $x \in X$, $z \in X^{**}$ and $f \in HV(X)$.*

We have seen that the associated extension defined in (8) has some good properties: it makes the Aron-Berner extension an isometry and preserves conditions (2) or $(\star)$ for families of weights. However, it may be hard to compute. Many weights have natural (and simple) extensions to the bidual. For example, if we have $v(x) = \varphi(\|x\|)$ for some appropriate function φ, then $\bar{v}(z) = \varphi(\|z\|_{X^{**}})$ for $z \in X^{**}$ is clearly the most natural extension of v to X^{**}. More generally, suppose we have a continuous non-negative function $\varphi : \mathbb{R}^n_+ \to \mathbb{R}_+$ and for $j = 1, \ldots n$ we have Banach spaces Y_j and continuous linear operators $T_j : X \to Y_j$. We can define a weight on X by

$$v(x) = \varphi(\|T_1(x)\|_{Y_1}, \ldots, \|T_n(x)\|_{Y_n}). \tag{9}$$

Then a natural extension to X^{**} is

$$\bar{v}(z) = \varphi(\|T_1^{**}(z)\|_{Y_1^{**}}, \ldots, \|T_n^{**}(z)\|_{Y_n^{**}}). \tag{10}$$

Particular cases of weights as in (9) are obtained by decomposing X as a direct sum of closed subspaces and defining the weight as a function of the norms of the projections (like in Examples 2.8 and 2.9). Also, note that any φ as above can be considered as a weight on $\mathbb{C}^n$, extending it as a function of the modulus of the coordinates.

We will see that these natural extensions share some good properties with our associated extension. First we have the following general result:

Lemma 3.8. *Let v be a weight on X and $\bar{v}$ any extension of v to X^{**}. Then, for each $f \in HV(X)$ we have $\|\bar{f}\|_{\hat{v}} \leq \|\bar{f}\|_{\bar{v}}$.*

Proof. Suppose $\|\bar{f}\|_{\bar{v}} \leq 1$. Clearly, $\|f\|_v \leq 1$ and then

$$|\bar{f}(z)| \leq \sup\{|\bar{g}(z) : \|g\|_v \leq 1\} \leq \frac{1}{\hat{v}}.$$

This means that $\|\bar{f}\|_{\hat{v}} \leq 1$, which completes the proof. □

In order to show that the extension defined by (10) makes the Aron-Berner extension an isometry, we need the following version of Goldstine's theorem.

Lemma 3.9. *Let X, $Y_1, \ldots, Y_n$ be Banach spaces and $T_j : X \to Y_j$, $j = 1, \ldots, n$, be continuous linear operators. For each $z_0 \in X^{**}$ and each $\delta > 0$, there exists a net $(x_\alpha) \subseteq X$ such that $x_\alpha \xrightarrow{w*} z_0$ and for all α, we have*

(a) $\|x_\alpha\| \leq \|z_0\|$,
(b) $\|T_j x_\alpha\| \leq \|T_j^{**} z_0\|$ *if* $T_j^{**} z_0 \neq 0$,
(c) $\|T_j x_\alpha\| < \delta$ *if* $T_j^{**} z_0 = 0$.

Proof. For simplicity, we will assume that $T_j^{**} z_0 = 0$ only for $j = n$. Let us define on X the following equivalent norm:

$$|||x||| = \max\Big\{\frac{\|x\|}{\|z_0\|}, \frac{\|T_1 x\|}{\|T_1^{**} z_0\|}, \ldots, \frac{\|T_{n-1} x\|}{\|T_{n-1}^{**} z_0\|}, \frac{1}{\delta}\|T_n x\|\Big\}.$$

This norm makes the mapping $T : X \to \ell_\infty(X, Y_1, \ldots, Y_n)$ given by $T(x) = (\frac{x}{\|z_0\|}, \frac{T_1 x}{\|T_1^{**} z_0\|}, \ldots, \frac{T_{n-1}x}{\|T_{n-1}^{**} z_0\|}, \frac{1}{\delta} T_n x)$ an isometry. By [17, Ex. 3.22], the bitranspose T^{**} is also an isometry, which means that the norm in $(X, |||\cdot|||)^{**}$ is given by

$$|||z||| = \max\Big\{\frac{\|z\|}{\|z_0\|}, \frac{\|T_1^{**}z\|}{\|T_1^{**}z_0\|}, \ldots, \frac{\|T_{n-1}^{**}z\|}{\|T_{n-1}^{**}z_0\|}, \frac{1}{\delta}\|T_n^{**}z\|\Big\}.$$

By Goldstine's theorem applied to X and X^{**} with the norm $|||\cdot|||$, we obtain a net (x_α) converging weak-star to z_0 with $|||x_\alpha||| \le |||z_0|||$. This inequality implies the three inequalities in the statement and, by the equivalence of the norms, the obtained weak-star convergence is the desired one. □

Theorem 3.10. *Let v be a weight on X of the form (9), where φ is non-increasing in each coordinate. If $\bar{v}$ is the extension defined in (10), then for each $f \in HV(X)$ we have $\|\bar{f}\|_{\bar{v}} = \|\bar{f}\|_{\hat{v}} = \|f\|_v$.*

Proof. We have already seen the second equality and, for the first one, one inequality follows from Lemma 3.8. For the reverse one, given $z \in X^{**}$ fixed and $\varepsilon > 0$, let $\delta > 0$ be such that for any $\lambda \in \mathbb{C}^n$ with $\|\lambda - (\|T_1^{**}z\|, \ldots, \|T_n^{**}z\|)\|_\infty < \delta$, we have $|h(\lambda) - h(\|T_1^{**}z\|, \ldots, \|T_n^{**}z\|)| < \varepsilon$.

For this δ and for the given z, let $(x_\alpha)_\alpha$ be the net obtained in Lemma 3.9. Applying Davie and Gamelin's procedure to the net (x_α), we can find another net (y_β) polynomially converging to z. Thus we can find β_0 such that $|f(y_{\beta_0}) - \bar{f}(z)| < \varepsilon$. Moreover, every y_β is a convex combination of elements of the net (x_α), hence (y_β) satisfy the inequalities (a), (b) and (c) too. Since φ is non-increasing in each coordinate and by the choice of δ, we have:

$$\begin{aligned}
|\bar{f}(z)|\bar{v}(z) &= |\bar{f}(z)|\varphi(\|T_1^{**}(z)\|, \ldots, \|T_n^{**}(z)\|) \\
&\le |\bar{f}(z)|(\varphi(\|T_1(y_{\beta_0})\|, \ldots, \|T_n(y_{\beta_0})\|) + \varepsilon) \\
&\le (|f(y_{\beta_0})| + \varepsilon)\varphi(\|T_1(y_{\beta_0})\|, \ldots, \|T_n(y_{\beta_0})\|) + \varepsilon|\bar{f}(z)| \\
&\le \|f\|_v + \varepsilon\varphi(0, \ldots, 0) + \varepsilon|\bar{f}(z)|.
\end{aligned}$$

Consequently, $\|\bar{f}\|_{\bar{v}} \le \|f\|_v = \|\bar{f}\|_{\hat{v}}$. □

Our next step is to obtain an analogous result, but for **every** function g defined on X^{**} (not only for Aron-Berner extensions). For this, we must impose some orthogonality-like conditions on the operators $T_1, \ldots, T_n$. First, denoting by $\tilde{\bar{v}}$ the associated weight of $\bar{v}$, we have the following.

Corollary 3.11. *With the hypotheses of Theorem 3.10, we have $\bar{v} \le \tilde{\bar{v}} \le \hat{v}$ and then $H\hat{v}(X^{**})$ is contained in $H\bar{v}(X^{**})$ with $\|g\|_{\bar{v}} \le \|g\|_{\hat{v}}$ for all g in $H\hat{v}(X^{**})$.*

Proof. Since $\|\bar{f}\|_{\hat{v}} = \|\bar{f}\|_{\bar{v}} = \|f\|_v$ for all $f \in Hv(X)$, we have for $z \in X^{**}$,

$$\sup\{|g(z)| : \|g\|_{\bar{v}} \leq 1\} \geq \sup\{|\bar{f}(z)| : \|\bar{f}\|_{\bar{v}} \leq 1\} = \sup\{|\bar{f}(z)| : \|f\|_v \leq 1\}.$$

Therefore, we have $\bar{v} \leq \tilde{\bar{v}} \leq \hat{v}$ (the first inequality always holds). In particular, $\|g\|_{\bar{v}} = \|g\|_{\tilde{\bar{v}}} \leq \|g\|_{\hat{v}}$ for all g in $H\hat{v}(X^{**})$. □

Definition 3.12. We say that the operators $T_j : X \to Y_j$ satisfy the orthogonality condition if each $x \in X$ can be written as $x = x_1 + \cdots + x_n$ in such a way that $T_j(x_i) = 0$ if $i \neq j$ and $T_j(x_j) = 0$ only if $x_j = 0$.

The orthogonality condition is readily verified if, for example, we can decompose X in a direct sum of n closed subspaces and $T_1, \ldots, T_n$ are the composition of injective operators with the corresponding projections. Namely, if we write $X = X_1 \oplus \cdots \oplus X_k$, Π_j are the projection on X_j associated to this decomposition and $S_j : X_j \to Y_j$ are injective linear operators, then $T_j = S_j \circ \Pi_j$ $(j = 1, \ldots, n)$ satisfy the orthogonality condition. To illustrate this with an example, consider $X = \ell_1$ and the weight

$$v(x) = \varphi\left(\Big(\sum_i |x_{2i}|^2\Big)^{1/2}, \sup_i |x_{2i+1}| \right).$$

Here, $\ell_1 = X_1 \oplus X_2$, where X_1 (respectively X_2) is spanned by the even (respectively odd) elements of the canonical basis, and $S_1 : X_1 \to \ell_2$ and $S_2 : X_2 \to c_0$ are the formal inclusions.

As we have already mentioned, any continuous function $\varphi : \mathbb{R}_+^n \to \mathbb{R}_+$ can be considered as a weight on $\mathbb{C}^n$, by extending it as $\varphi(\lambda_1, \ldots, \lambda_n) = \varphi(|\lambda_1|, \ldots, |\lambda_n|)$. Therefore, we can consider its associated weight $\tilde{\varphi}$, which is also a weight on $\mathbb{C}^n$. This allows us to define the following weight on X:

$$w(x) = \tilde{\varphi}(\|T_1(x)\|_{Y_1}, \ldots, \|T_n(x)\|_{Y_n}). \tag{11}$$

Then a natural extension of w to X^{**} is

$$\bar{w}(z) = \tilde{\varphi}(\|T_1^{**}(z)\|_{Y_1^{**}}, \ldots, \|T_n^{**}(z)\|_{Y_n^{**}}). \tag{12}$$

Note that $\bar{w}$ is a weight on X^{**} related with v which is, in some sense, simpler than $\hat{v}$, since it involves 'taking associated weight' in a finite dimensional space. Comparing the different definitions, it is natural to wonder about the relationship between $\tilde{v}$ and w (which are associate-like weights on X) and between $\hat{v}$, $\tilde{\bar{v}}$ and $\bar{w}$ (associate-like weights on X^{**}). The following theorem presents some of these relationships. The most important is the third one, since it shows the isometry between $H\hat{v}(X^{**})$ and $H\bar{v}(X^{**})$, i.e., that the norms induced by the associate extension (the one with good properties) and the most simple extension coincide.

Theorem 3.13. *Let v be a weight as in (9), where φ is continuous and non-increasing in each coordinate. Then the following hold:*

(a) $\bar{w} \geq \hat{v}$.

(b) If $T_1 \dots, T_n$ satisfy the orthogonality condition, then $\tilde{v} = w$.

(c) If the bitransposes $T_1^{**} \dots, T_n^{**}$ satisfy the orthogonality condition, then $\hat{v} = \bar{w} = \tilde{\bar{v}}$. In particular, $\|g\|_{\hat{v}} = \|g\|_{\bar{v}}$ for all $g \in H\hat{v}(X^{**}) = H\bar{v}(X^{**})$.

Proof. (a) To establish the inequality it is enough to show the following: fixed $z_0 \in X^{**}$, for any $h \in H(\mathbb{C}^n)$ with $\|h\|_\varphi \leq 1$ and any $\varepsilon > 0$, there exists $f \in H(X)$ with $\|f\|_v \leq 1$ such that $|\bar{f}(z_0)| \geq |h(\|T_1^{**}z_0\|, \dots, \|T_n^{**}z_0\|)| - \varepsilon$.

Let $\delta > 0$ be such that for any $\lambda \in \mathbb{C}^n$ with $\|\lambda - (\|T_1^{**}z_0\|, \dots, \|T_n^{**}z_0\|)\|_\infty < \delta$, we have $|h(\lambda) - h(\|T_1^{**}z_0\|, \dots, \|T_n^{**}z_0\|)| < \varepsilon$. Choose $y_j^* \in Y_j^*$ such that $\|y_j^*\| = 1$ and $T_j^{**}z_0(y_j^*) > \|T_j^{**}z_0\| - \delta$ for $j = 1 \dots n$.

Now we define $f(x) = h(y_1^*(T_1x), \dots, y_n^*(T_nx))$, and observe that its Aron-Berner extension is given by $\bar{f}(z) = h(T_1^{**}z(y_1^*), \dots, T_n^{**}z(y_n^*))$. Also,

$$\begin{aligned} |f(x)|v(x) &= h(y_1^*(T_1x), \dots, y_n^*(T_nx))\varphi(\|T_1x\|, \dots, \|T_nx\|) \\ &\leq h(y_1^*(T_1x), \dots, y_n^*(T_nx))\varphi(y_1^*(T_1x), \dots, y_n^*(T_nx)) \leq \|h\|_\varphi, \end{aligned}$$

so $\|f\|_v \leq 1$.

Finally, we have $|\bar{f}(z_0) - h(\|T_1^{**}z_0\|, \dots, \|T_n^{**}z_0\|)| < \varepsilon$, which ends the proof.

(b) We adapt the proof of [14, Proposition 2], so we omit some details. Fix $x_0 \in X$ and choose $y_j^* \in Y_j^*$ such that $\|y_j^*\| = 1$ and $y_j^*(T_jx_0) = \|T_jx_0\|$ for $j = 1 \dots n$. For $h \in H(\mathbb{C}^n)$ with $\|h\|_\varphi \leq 1$, we can define $f(x) = h(y_1^*(T_1x), \dots, y_n^*(T_nx))$. We can easily check that $\|f\|_v \leq 1$ and, clearly, $f(x_0) = h(\|T_1x_0\|, \dots, \|T_nx_0\|)$. Therefore,

$$\sup\{|h(\|T_1x_0\|, \dots, \|T_nx_0)\|)| : \|h\|_\varphi \leq 1\} \leq \sup\{|f(x_0)| : \|f\|_v \leq 1\}. \tag{13}$$

As a consequence, $\tilde{v}(x_0) \leq w(x_0)$.

On the other hand, take $x_0 \in X$ and f with $\|f\|_v \leq 1$. Using the orthogonality condition, we can write $x_0 = x_1 + \dots + x_n$ with $T_jx_i = 0$ for $i \neq j$. If we define

$$h(\lambda_1, \dots, \lambda_n) = f\big(\lambda_1 \frac{x_1}{\|T_1x_1\|} + \dots + \lambda_n \frac{x_n}{\|T_nx_n\|}\big)$$

(we write 0 instead of $\frac{x_j}{\|T_j x_j\|}$ whenever $x_j = 0$), since $\|f\| \leq 1$ we have

$$\varphi(\lambda_1, \dots, \lambda_n)|h(\lambda_1, \dots, \lambda_n)| = \varphi(|\lambda_1|, \dots, |\lambda_n|)|h(\lambda_1, \dots, \lambda_n)| = \\ v\big(\lambda_1 \frac{x_1}{\|T_1 x_1\|} + \cdots + \lambda_n \frac{x_n}{\|T_n x_n\|}\big)|f\big(\lambda_1 \frac{x_1}{\|T_1 x_1\|} + \cdots + \lambda_n \frac{x_n}{\|T_n x_n\|}\big)| \leq 1.$$

Then, $\|h\|_\varphi \leq 1$ and, since $h(\|T_1 x_0\|, \dots, \|T_n x_0\|) = f(x_0)$, we obtain the reverse inequality in (13).

(c) Note that we can apply (b) to $\bar{v}$ to obtain $\tilde{\bar{v}} = \bar{w}$. Also, by Corollary 3.11 we have $\tilde{\bar{v}} \leq \hat{v}$. This, together with (a), gives the conclusion. $\square$

It is easy to see that $T_1^{**} \dots, T_n^{**}$ satisfy the orthogonality condition if $T_1, \dots T_n$ are projections associated to a decomposition of X as a direct sum.

Acknowledgement. We warmly give thanks to Antonio Galbis for providing us with Example 2.6. We also thank the referee for the useful comments which have improved the paper.

References

[1] R. M. Aron, B. Cole, and T. W. Gamelin. Spectra of algebras of analytic functions on a Banach space. *J. Reine Angew. Math.*, **415** (1991), 51–93.

[2] R. M. Aron, P. Galindo, D. García, and M. Maestre. Regularity and algebras of analytic functions in infinite dimensions. *Trans. Amer. Math. Soc.*, **348(2)** (1996), 543–559.

[3] K. D. Bierstedt, J. Bonet, and A. Galbis. Weighted spaces of holomorphic functions on balanced domains. *Michigan Math.*, **40** (1993), 271–297.

[4] K. D. Bierstedt, J. Bonet, and J. Taskinen. Associated weights and spaces of holomorphic functions. *Studia Math.*, **127(2)** (1998), 137–168.

[5] K. D. Bierstedt, R. G. Meise, and W. Summers. A projective description of weighted inductive limits. *Trans. Amer. Math Soc.*, **272(2)** (1982), 107–160.

[6] K. D. Bierstedt and W. Summers. Biduals of weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. (Series A)*, **54(2)** (1993), 70–79.

[7] C. A. Berenstein, B. Q. Li and A. Vidras. Geometric characterization of interpolating varieties for the (FN)-space A_p^0 of entire functions. *Canad. J. Math.* **47(1)** (1995), 28–43.

[8] J. Bonet, P. Domański, and M. Lindström. Essential norm and weak compactness of composition operators on weighted spaces of analytic functions. *Canad. Math. Bull.*, **42(2)** (1999), 139–148.

[9] J. Bonet, P. Domański, M. Lindström, and J. Taskinen. Composition operators between wighted Banach spaces of analytic functions. *J. Austral. Math. Soc. (Series A)*, **64** (1998), 101–118.

[10] J. Bonet and M. Friz. Weakly compact composition operators on locally convex spaces. *Math. Nachr.*, **24** (2002), 26–44.

[11] J. Bonet, M. Friz, and E. Jordá. Composition operators between weighted inductive limits of sapces of holomorphic functions. *Publ. Math. Debrecen.* **67(3-4)** (2005), 333–348.

[12] R. W. Braun, Weighted algebras of entire functions in which each closed ideal admits two algebraic generators. *Michigan Math. J.* **34(3)** (1987), 441–450.

[13] D. Carando, G. García, M. Maestre, Homomorphisms and composition operators on algebras of analytic functions of bounded type, *Advances in Mathematics* **197** (2005), 607-629.

[14] D. Carando and P. Sevilla-Peris. Spectra of weighted algebras of holomorphic functions. *Math. Zeit.*, DOI 10.1007/s00209-008-0444-0, (2009).

[15] A. Davie and T. Gamelin. A theorem on polynomial-star approximation. *Proc. Am. Math. Soc.*, **106(2)** (1989), 351–356.

[16] S. Dineen. *Complex analysis on infinite-dimensional spaces.* Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 1999.

[17] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, V Zizler. *Functional analysis and infinite-dimensional geometry.*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. 8. New York, NY: Springer, 2001.

[18] D. García, M. Maestre, and P. Rueda. Weighted spaces of holomorphic functions on Banach spaces. *Studia Math.*, **138(1)** (2000), 1–24.

[19] D. García, M. Maestre, and P. Sevilla-Peris. Composition operators between weighted spaces of holomorphic functions on Banach spaces. *Ann. Acad. Sci. Fenn. Math.*, **29(1)** (2004), 81–98.

[20] D. García, M. Maestre, and P. Sevilla-Peris. Weakly compact composition operators between weighted spaces. *Note Mat.*, **25(1)** (2005/06), 205–220.

[21] R. C. Gunning, *Introduction to holomorphic functions of several variables. Vol. I. Function theory.* The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990.

[22] R. Meise, and B. A. Taylor, Sequence space representations for (FN)-algebras of entire functions modulo closed ideals. *Studia Math.* **85(3)** (1987), 203–227.

Dep. Matemática, Fac. C. Exactas y Naturales, Universidad de Buenos Aires, Pab I, Ciudad Universitaria, 1428, Buenos Aires, Argentina

dcarando@dm.uba.ar

Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain

domingo.garcia@uv.es

Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain

manuel.maestre@uv.es

Instituto Universitario de Matemática Pura y Aplicada and DMA, ETSMRE, Universidad Politécnica de Valencia, Av. Blasco Ibáñez, 21, E–46010 Valencia, Spain

psevilla@mat.upv.es