Whitney Extension Operators with arbitrary loss of differentiability.

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Abstract

For a compact set $K \subset \mathbb{R}^d$ we characterize the existence of a linear extension operator $E : \mathscr{E}(K) \to \mathscr{C}^\infty(\mathbb{R}^d)$ for the space of Whitney jets $\mathscr{E}(K)$ with a certain loss of derivatives σ, that is, the operator satisfies the following continuity estimates for all $n \in \mathbb{N}_0$ and all $F \in \mathscr{E}(K)$

$$\sup\{|\partial^\alpha E(F)(x) : |\alpha| \leq n, x \in \mathbb{R}^d|\} \leq C_n \|F\|_{\sigma(n)},$$

where $\|\cdot\|_{\sigma(n)}$ denotes the Whitney norm and the map $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ is monotonically increasing with $\sigma(n) \geq n$ and $\sigma(0) = 0$. From our main result it follows directly that if a compact set K admits an extension operator, then it is always possible to construct a second extension operator resembling the original Whitney operators $E_n : \mathscr{E}^n(K) \to \mathscr{C}^n(\mathbb{R}^d)$ where the evaluations of the jet occuring in the Taylor polynomials are approximated by measures.

1 Introduction

The question of describing the spaces of continuously partially differentiable functions on $\mathbb{R}^d$ which are restricted to some closed set M has been fully answered by Whitney [Whi34]. He established the spaces $\mathscr{E}^n(M)$ respectively the projective limit of them $\mathscr{E}(M)$ which contain families $(f^{(\alpha)})_{|\alpha|\leq n}$ respectively $(f^{(\alpha)})_{|\alpha|\in\mathbb{N}_0^d}$ of continuous functions defined on M. For a compact set K he equipped the spaces $\mathscr{E}^n(K)$ with a norm $\|\cdot\|_n$. This norm establishes a connection between the functions $f^{(\alpha)}$ in a way that it is always possible to find a $g \in \mathscr{C}^n(\mathbb{R}^d)$ such that $\partial^\alpha g = f^{(\alpha)}$ on K for all $\alpha \in \mathbb{N}_0^d$. In other words the mapping

$$r_n : \mathscr{C}^n(\mathbb{R}^d) \to \mathscr{E}^n(K), f \mapsto (\partial^\alpha f\big|_K)$$

is surjective. The same holds for the spaces $\mathscr{E}(K)$, $\mathscr{E}^n(M)$ and $\mathscr{E}(M)$ which are all constructed as projective limits. He overcame the problem of not having a differentiability structure on K with

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the aid of the formal Taylor polynomial and the remainder which are defined for $f := (f^{(\alpha)})_{|\alpha|\leq n}$ and $x \in K$ as follows

$$T_x^n(f)(y) := \sum_{|\alpha|\leq n} \frac{f^{(\alpha)}(x)}{\alpha!}(y-x)^\alpha,$$
$$R_x^n(f)^{(\alpha)} := f^{(\alpha)} - \partial^\alpha(T_x^n(f))\Big|_K.$$

And with

$$q_n(K, f, t) := \sup\{|R_x^n(f)^{(\alpha)}(y)||y-x|^{|\alpha|-n} : x, y \in K, 0 < |x-y| \leq t, |\alpha| \leq n\}$$

the Whitney norm $\|\cdot\|_n$ is defined as

$$\|\cdot\|_n := |f|_n + \sup\{q_n(K, f, t) : t > 0\},$$

where

$$|f|_n := \sup\{|f^{(\alpha)}(x)| : x \in K, |\alpha| \leq n\}.$$

The space $\mathscr{E}^n(K)$ is then defined as the space of all families $(f^{(\alpha)})_{|\alpha|\leq n}$ for which $\lim_{t\to 0} q_n(K, f, t) = 0$, equipped with this norm. The elements of spaces $\mathscr{E}_n(K), \mathscr{E}_n(F), \mathscr{E}(K)$ and $\mathscr{E}(F)$ are referred to as Whitney jets of n-th order and of infinite order respectively. For $K \subset \mathbb{R}^d$ compact, the space $\mathscr{E}(K)$ is dense in $\mathscr{E}^m(K)$ (via canonical identification) for al $m \in \mathbb{N}$. Furthermore, Whitney proved that the restriction operator r_n has always a linear and continuous right inverse E_n, called extension operator. In contrast to that, the question if the restriction $r : \mathscr{E}(K) \to \mathscr{C}^\infty(\mathbb{R}^d)$ has also a linear and continuous right inverse turned out to be much more complicated. In 1961 B. S. Mityagin in his paper [Mit61] observed that a singleton does not possesses the extension property and posed as a problem to give a characterization of the extension property in geometric terms. Tidten showed in [Tid79] that there are compact sets which are the closure of its interior and do not have the extension property. In [GU17] it is shown that there is no such complete description in terms of densities of Hausdorff contents or related characteristics. Also the extension property cannot be characterized in terms of growth of Markov's factors for the set. This Markov's inequalities however provide a complete description of the extension property for compact sets with tame linear o no loss of differentiability [FJW11, FJW16]. In [Gon96] it is shown that there is a compact set $K \subset \mathbb{R}$ with the extension property but not satisfying any Markov inequality. Hence this compact K possess the extension property but does not admit a tame linear extension operator. In this note we show that compacts with the extension property can be completely characterized as those admitting certain supported measures satisfying certain conditions of uniform approach of the derivatives of any order for smooth functions. This is far from being a geometric characterization. However, as a consequence, we obtain that any compact K which has the extension property satisfies that there is an extension operator on $\mathscr{E}(K)$ which can be obtained by a perturbation of the original Whitney's extension operators for finitely order jets. The results that we present form part of the PhD thesis of the second named author [Jak18], which was advised by the other authors.

Given $B \subset \mathbb{R}^d$ open, the space $\mathscr{D}(B)$ is defined as the subspace of $\mathscr{C}^\infty(\mathbb{R}^d)$ formed by the functions supported in B. The space $\mathscr{D}$ of test functions is the formed by all $\mathscr{C}^\infty$-functions with compact support. Our notation for functional analysis is standard. We refer to [MV97]. If E is a nonnormable locally convex space we denote its dual by E'. If X is a Banach space X^* stands for the dual.

Before we start with the proof of the main result, we recall the special partition of unity which Whitney used in his original work and which we also apply for our construction. Throughout this paper we make use of the following notations. For a set $M \subset \mathbb{R}^d$ we denote with $\mathrm{diam}\,(M) = \sup\{|x-y| : x, y \in M\}$ the diameter of M and for $z \in \mathbb{R}^d$ arbitrary we denote with $\mathrm{d}\,(z, M) = \inf\{|z-x| : x \in M\}$ the distance of z to M and for N being another set we also denote with $d(M, N) = \inf\{|z-x| : x \in M, z \in N\}$ the distance of the two sets.

1 Lemma. *Whitney's decomposition Let $K \subset \mathbb{R}^d$ be compact. Then there exists an open set $\Omega \supset K$ and a family $(\varphi_i)_{i\in\mathbb{N}_0} \in \mathscr{D}(\Omega\setminus K)^{\mathbb{N}_0}$ of posititve test functions with the following properties.*

1. *$\sum_{i=0}^{\infty} \varphi_i = 1$ for all $x \in \Omega \setminus K$ and each point belongs to at most N supports supp (φ_i) for some constant $N \in \mathbb{N}$.*
2. *supp $(\varphi_i) \to K$ for $i \to \infty$, that is, for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that supp $(\varphi_i) \subset \{x \in \mathbb{R}^d : d\,(x, K) < \varepsilon\}$ for all $i \geq k$.*
3. *diam(supp $(\varphi_i)) \leq 2d\,($supp $(\varphi_i), K)$.*
4. *There are constants c_β such that $|\partial^\beta \varphi_i(x)| \leq c_\beta d\,(x, K)^{-|\beta|}$ for all $i \in \mathbb{N}, \beta \in \mathbb{N}_0^d$, and $x \in \mathbb{R}^d$.*

Proof. For a proof see for instance [Ste70]. □

We will now list some properties of Whitneys decomposition which will get important in the construction of an extension operator in the main theorem. The proof of this corollary is contained in the proof of Theorem 3 in [FJW16]. , where x_i denotes the element of K in which the distance is attained.

2 Corollary. *Let $K \subset \mathbb{R}^d$ be compact. Let $(\varphi_i)_{i\in\mathbb{N}_0} \in \mathscr{D}(\Omega \setminus K)^{\mathbb{N}_0}$ be the partition of unity corresponding to K in Lemma 1 has the following properties. Set $\gamma_i := d\,(K, \text{supp}\,(\varphi_i)) = d\,(x_i, \text{supp}\,(\varphi_i))$, where x_i is a point in K minimizing the distance of K to supp φ_i.*

1. *Given $|\alpha| \leq n, |\beta| \leq n$ there exist constants $C_n > 0$ independent of i such that for all $x \in$ supp (φ_i)*
$$\left|\partial^\beta((x-x_i)^\alpha \varphi_i(x_i))\right| \leq C_n \gamma_i^{|\alpha|-|\beta|}.$$
2. *For $|\beta| \leq n$ and $|\alpha| > n$ we find constants C_n not depending on i such that for all $x \in$ supp (φ_i)*
$$\left|\partial^\beta((x-x_i)^\alpha \varphi_i(x))\right| \leq C_n 3^{|\alpha|} \sup_{\gamma\leq\beta,\alpha} \frac{\alpha!}{(\alpha-\gamma)!}\gamma_i^{|\alpha|-|\beta|}.$$
3. *For $|\beta| \leq n$ we have*
$$\sum_{|\alpha|>n} \sup_{\gamma\leq\beta,\alpha} \frac{1}{(\alpha-\gamma)!} 3^{|\alpha|} \leq e^{3d}(n+1)^d 3^n.$$

2 Main Result

For the formulation of our main result we introduce the following notation to describe the continuity of an extension operator $E : \mathscr{E}(K) \to \mathscr{C}^\infty(\mathbb{R}^d)$. Given a map $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ which is monotonically increasing with $\sigma(0) = 0$ and $\sigma(n) \geq n$ for all $n \in \mathbb{N}_0$, we say that E has loss of derivatives σ if and only if for each $n \in \mathbb{N}_0$ there are positive constants C_n such that the inequality $|E(f)|_n \leq C_n \|f\|_{\sigma(n)}$ holds for all $f \in \mathscr{E}(K)$. Since $\mathscr{E}(K)$ is dense in $\mathscr{E}^m$ for each $m \in \mathbb{N}$, this is equivalent to say that E maps $\mathscr{E}^{\sigma(n)}(K)$ continuously into $\mathscr{C}^n(\mathbb{R}^d)$ for all $n \in \mathbb{N}_0$, by identifying E with its unique extension on the corresponding Banach spaces of jets of finite order.

3 Theorem. *For $K \subset \mathbb{R}^d$ compact and $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ monotonically increasing with $\sigma(0) = 0$ and $\sigma(n) \geq n$, the following are equivalent:*

1. *$\mathscr{E}(K)$ admits a continuous linear extension operator with loss of derivatives σ.*
2. *For all $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ and $\varepsilon > 0$ there are measures $\nu_{\alpha,x,\varepsilon}$ on K such that, for each $f \in \mathscr{E}^{\sigma(n)}(K)$ and $n \in \mathbb{N}_0$,*

$$\lim_{\varepsilon\to 0} \sup_{|\alpha|\leq n, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)}) - \varepsilon^{|\alpha|} f^{(\alpha)}(x)|}{\varepsilon^n} = 0 \quad \textit{and}$$
$$\lim_{\varepsilon\to 0} \sup_{|\alpha|> n, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^n} = 0.$$

We prove our main result in the theorems 9 and 12.

2.1 Construction of the measures

Our first goal is to construct local measures $\mu_{\alpha,\varepsilon}$, supported on an arbitrary ball $B_0 \subset \mathbb{R}^d$ containing 0 in its interior, which approximate for $|\alpha| \leq n$ the α-th partial derivatives of the function $f \in \mathscr{C}^n(\mathbb{R}^d)$ at the origin. For a given compact set $K \subset \mathbb{R}^d$ we construct in a next step measures $\mu_{\alpha,x_0,\varepsilon}$, supported on a ball B containing K in its interior, simply by shifting the aforementioned measures, having similar approximation properties not only at the origin but at each point x_0 of the boundary of K. In a last step we apply the extension operator, which exists by assumption, to simply yield the final measures $\nu_{\alpha,x_0,\varepsilon}$ by taking he composition $\nu_{\alpha,x_0,\varepsilon} = \mu_{\alpha,x_0,\varepsilon} \circ E$.

The main tool in the construction of the measures is Eidelheit's Theorem. It provides a characterization of the solvability of an infinite system of equations. Let E be a Fréchet space. If U is a 0-neighbourhood E, E'_{U° denotes the *local Banach space* $\mathrm{span}(U^\circ) \subseteq E'$ endowed with the norm $p_{U^\circ}(x) := \inf\{\lambda > 0 : x \in \lambda U^\circ\}$, $x \in U^\circ$.

4 Theorem. *Eidelheit's Theorem*
Let E be a Fréchet space, $(U_k)_{k\in\mathbb{N}}$ be a fundamental system of zero neighbourhoods in E and let $(T_j)_{j\in\mathbb{N}}$ be linearly independent, continuous linear forms on E. Then the infinite system of equations

$$T_j x = y_j \quad \textit{for all } j \in \mathbb{N}$$

is solvable for each sequence $y \in \omega$ if, and only if, the following holds:

$$dim((E')_{U_k^\circ} \cap span\{T_j : j \in \mathbb{N}\}) < \infty \quad \textit{for all } k \in \mathbb{N}.$$

In the following we deal with sequence spaces. We will now shortly recall the definitions. In contrast to the "classical" definitions of sequence spaces, we w define them over the index set $\mathbb{N}_0^d$. Basically nothing changes.

5 Definition. *For a given dimension $d \in \mathbb{N}$ we define the sequence spaces of sequences over the index set $\mathbb{N}_0^d$:*

- $\omega(\mathbb{N}_0^d) := \mathbb{C}^{\mathbb{N}_0^d}$.
- $\varphi(\mathbb{N}_0^d) := \left\{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \omega(\mathbb{N}_0^d) : \sharp\left\{\alpha \in \mathbb{N}_0^d : x_\alpha \neq 0\right\} < \infty\right\}$.
- $s(\mathbb{N}_0^d) := \left\{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \omega(\mathbb{N}_0^d) : \lim_{|\alpha|\to\infty} |x_\alpha||\alpha|^k = 0 \textit{ for all } k \in \mathbb{N}\right\}$. *Since this space is by Example 29.4 in [MV97] nuclear, the topology is generated by the fundamental system of seminorms* $p_k((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) = \sum_{\alpha\in\mathbb{N}_0^d} |x_\alpha||\alpha|^k$ *as well as by the system* $\tilde{p}_k((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) = \sup_{\alpha\in\mathbb{N}_0^d} |x_\alpha||\alpha|^k$.
- $s'(\mathbb{N}_0^d) := \left\{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \omega(\mathbb{N}_0^d) : \sup_{\alpha\in\mathbb{N}_0^d} |x_\alpha||\alpha|^{-k} < \infty \textit{ for one } k \in \mathbb{N}\right\}$ *is the dual space of* $s(\mathbb{N}_0^d)$.

We are now able to prove the following representation theorem.

6 Lemma. *The mapping $T : s(\mathbb{N}_0^d) \to \omega(\mathbb{N}_0^d), (\lambda_\beta)_{\beta\in\mathbb{N}_0^d} \mapsto \left(\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta \beta^\gamma\right)_{\gamma\in\mathbb{N}_0^d}$ is surjective.*

Proof. The mapping T can be decomposed in countably infinite many linear forms T_γ where for each $\gamma \in \mathbb{N}_0^d$

$$T_\gamma\left((\lambda_\beta)_{\beta\in\mathbb{N}_0^d}\right) = \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta \beta^\gamma,$$

and all these linear forms are obviously elements of $s'(\mathbb{N}_0^d)$. A fundamental system $(U_k)_{k\in\mathbb{N}}$ for $s(\mathbb{N}_0^d)$ is given by

$$U_k = \left\{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in s(\mathbb{N}_0^d) : p_k((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) < 1\right\},$$

and therefore

$$U_k^\circ = \left\{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in s'(\mathbb{N}_0^d) : p'_k((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) \leq 1\right\}.$$

Let $p'_k((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) := \sup_{\alpha\in\mathbb{N}_0^d} |x_\alpha||\alpha|^{-k}$. The local Banach space $s'(\mathbb{N}_0^d)_{U_k^\circ}$ can be described as follows

$$s'(\mathbb{N}_0^d)_{U_k^\circ} = \left\{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in s'(\mathbb{N}_0^d) : p'_k((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) < \infty\right\}.$$

In order to apply now Eidelheit's Theorem, we thus have to check for each $k \in \mathbb{N}$ that

$$\dim\left\{\left(\varrho_\gamma\right)_{\gamma\in\mathbb{N}_0^d} \in \varphi(\mathbb{N}_0^d) : \sup_{\beta\in\mathbb{N}_0^d}\left|\sum_{\gamma\in\mathbb{N}_0^d}\varrho_\gamma\beta^\gamma\right||\beta|^{-k}\right\} < \infty.$$

We show by contradiction that for each $(\varrho_\gamma)_{\gamma\in\mathbb{N}_0^d} \in \varphi(\mathbb{N}_0^d)$ with

$$\sup_{\beta\in\mathbb{N}_0^d}\left|\sum_{\gamma\in\mathbb{N}_0^d}\varrho_\gamma\beta^\gamma\right||\beta|^{-k} < \infty \tag{1}$$

it is true that $\varrho_\gamma = 0$ for $|\gamma| > k$. So we assume that $m := \max\{|\gamma| : \varrho_\gamma \neq 0\} > k$ and let $P_m(x) = \sum_{|\gamma|=m}\varrho_\gamma x^\gamma$. Following the assumption, there is $x = (x_1, ..., x_d) \in \mathbb{R}^d$ with $x_1, ..., x_d > 0$ such that $P_m(x) \neq 0$. Since the set $\{t\beta : \beta \in \mathbb{N}_0^d, t > 0\}$ is dense in $\{x \in \mathbb{R}^d : x \geq 0\}$, there is $\beta_0 \in \mathbb{N}_0^d$ with $P_m(\beta_0) \neq 0$. It is easy to see that for all $\beta_l := l\beta_0$ it is for $\gamma \in \mathbb{N}_0^d$ true that $\beta_l^\gamma = l^{|\gamma|}\beta_0^\gamma$ and thus we get

$$\left|\sum_{|\gamma|=m}\varrho_\gamma\beta_l^\gamma\right| = |P_m(\beta_0)|l^m$$

and

$$\left|\sum_{|\gamma|<m}\varrho_\gamma\beta_l^\gamma\right| \leq l^{m-1}\sum_{|\gamma|<m}|\varrho_\gamma||\beta_0^\gamma|.$$

Since $m > k$ we have

$$\lim_{l\to\infty}\left|\sum_{|\gamma|\leq m}\varrho_\gamma\beta_l^\gamma\right||\beta_l|^{-k} = \infty,$$

which is a contradition to (1). □

The following lemma gives the solution to the central moment problem. This result can be compared to Proposition 5 in [FJW16].

7 Lemma. *Let $(k_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \omega(\mathbb{N}_0^d)$ be arbitrary. Then there is $(\varrho_\beta)_{\beta\in\mathbb{N}_0^d} \in s(\mathbb{N}_0^d)$ with $\varrho_\beta \geq 0$ for all $\beta \in \mathbb{N}_0^d$ such that for all $\alpha \in \mathbb{N}_0^d$ there is a sequence $(\lambda_\beta^{(\alpha)}) \in s(\mathbb{N}_0^d)$ with $|\lambda_\beta^{(\alpha)}| \leq \varrho_\beta$ for all $\beta \in \mathbb{N}_0^d$, satisfying*

$$\left(\sum_{\beta\in\mathbb{N}_0^d}\lambda_\beta^{(\alpha)}\beta^\gamma\right)_{\gamma\in\mathbb{N}_0^d} = (k_\alpha\delta_{\alpha\gamma})_{\gamma\in\mathbb{N}_0^d}.$$

Proof. We first observe that the set $\{(k_\alpha\delta_{\alpha,\gamma})_{\gamma\in\mathbb{N}_0^d} : \alpha \in \mathbb{N}_0^d\}$ is compact in the Fréchet space of all sequences $\omega(\mathbb{N}_0^d)$, which follows directly by Tychonov's Theorem. By [MV97, Corollary 26.22] surjective maps between Fréchet spaces lift compact sets. This gives a compact set $K \subset s(\mathbb{N}_0^d)$ such that we have with the surjective mapping T of the previous lemma

$$T(K) \supset \{(k_\alpha\delta_{\alpha,\gamma})_{\gamma\in\mathbb{N}_0^d} : \alpha \in \mathbb{N}_0^d\}.$$

By [MV97, Corollary 27.6] we can choose a sequence $(\varrho_\beta)_{\beta\in\mathbb{N}_0^d} \in s(\mathbb{N}_0^d)$ fullfilling $|\lambda_\beta| \le \varrho_\beta$ for all $\beta \in \mathbb{N}_0^d$ and all $(\lambda_\beta)_{\lambda\in\mathbb{N}_0^d} \in K$. □

In the following theorem we construct measures approximating the derivatives of functions in $\mathscr{E}$ on the boundary of some compact set $K \subset \mathbb{R}^d$. Given a functional $x^* \in C(K)^*$, by Riesz representation theorem there is a measure μ supported in K such that $x^*(f) = \int f d\mu$. If $u \in \mathscr{E}^n(K)'$ or $u \in \mathscr{E}(K)'$ satisfies that $\|u(f)\|_0 \le C\|f\|_0$, the density of $\mathscr{E}(K)$ in $C(K)$ permits to conclude that there is a (unique) measure μ supported in K such that $u(f) = \int f^0 d\mu$. For such a functional we identify u with μ and we write $\mu(f) := \int g d\mu$ for every $g \in C(K)$.

8 Theorem. *Let $n \in \mathbb{N}_0$ and $K \subset \mathbb{R}^d$ be compact. Then for all $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ and $\varepsilon > 0$ there are measures $\mu_{\alpha,x,\varepsilon}$ supported on a ball $\mathcal{B}$ with $\mathcal{B}^\circ \supset K$ such that for all $f \in \mathscr{E}^n(\mathbb{R}^d)$*

1. $$\lim_{\varepsilon\to 0} \sup_{|\alpha|\le n, x\in\partial K} \frac{|\mu_{\alpha,x,\varepsilon}(f) - \varepsilon^{|\alpha|}\partial^\alpha f(x)|}{\varepsilon^n} = 0,$$
2. $$\lim_{\varepsilon\to 0} \sup_{|\alpha|> n, x\in\partial K} \frac{|\mu_{\alpha,x,\varepsilon}(f)|}{\varepsilon^n} = 0.$$

Proof. By cutting off it is enough to show the assertion for $f \in \mathcal{D}^n(\mathcal{B}) := \{g \in \mathscr{E}^n(\mathbb{R}^d) : \operatorname{supp} g \subset \mathcal{B}\}$ where $\mathcal{B}$ is a fixed ball containing K in its interior. We apply the previous lemma choosing $k_\alpha = \alpha!$ for each $\alpha \in \mathbb{N}_0^d$. Then there exist sequences $(\lambda_\beta^{(\alpha)})_{\beta\in\mathbb{N}_0^d}$ and $(\varrho_\beta)_{\beta\in\mathbb{N}_0^d}$, both in $s(\mathbb{N}_0^d)$, such that for all $\alpha \in \mathbb{N}_0^d$

$$\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \beta^\gamma = \begin{cases} \alpha! & \text{for } \gamma = \alpha, \\ 0 & \text{else,} \end{cases} \tag{2}$$

and such that $|\lambda_\beta^{(\alpha)}| \le \varrho_\beta$ for all $\alpha, \beta \in \mathbb{N}_0^d$. With this we define for each $g \in \mathcal{D}^0(\mathcal{B})$ and $x \in \partial K$

$$\mu_{\alpha,x,\varepsilon}(g) := \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} g(x + \varepsilon\beta).$$

We start with the proof of 1., so let $|\alpha| \le n$. With Taylor's theorem we get $\xi \in [x, x+\varepsilon\beta]$ such that

$$\left|\mu_{\alpha,x,\varepsilon}(f) - \varepsilon^{|\alpha|}\partial^\alpha f(x)\right| = \left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} f(\varepsilon\beta) - \varepsilon^{|\alpha|}\partial^\alpha f(x)\right|$$
$$= \left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \left(\sum_{|\gamma|<n} \frac{\partial^\gamma f(x)}{\gamma!}\varepsilon^{|\gamma|}\beta^\gamma + \varepsilon^n \sum_{|\gamma|=n} \frac{\partial^\gamma f(\xi)}{\gamma!}\beta^\gamma\right) - \varepsilon^{|\alpha|}\partial^\alpha f(x)\right|.$$

Using (2) we get

$$\varepsilon^{|\alpha|}\partial^\alpha f(x) = \alpha!\varepsilon^{|\alpha|}\frac{\partial^\alpha f(x)}{\alpha!} = \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \beta^\alpha \varepsilon^{|\alpha|} \frac{\partial^\alpha f(x)}{\alpha!},$$

and

$$0 = \sum_{|\gamma|\leq n, \gamma\neq\alpha} \left(\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \beta^\gamma \right) \varepsilon^{|\gamma|} \frac{\partial^\gamma f(x)}{\gamma!} = \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \sum_{|\gamma|\leq n, \gamma\neq\alpha} \beta^\gamma \varepsilon^{|\gamma|} \frac{\partial^\gamma f(x)}{\gamma!},$$

which results in

$$\varepsilon^{|\alpha|} \partial^\alpha f(x) = \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \sum_{\gamma\leq n} \beta^\gamma \varepsilon^{|\gamma|} \frac{\partial^\gamma f(x)}{\gamma!}.$$

Therefore we get

$$\begin{aligned} \left| \mu_{\alpha,x,\varepsilon}(f) - \varepsilon^{|\alpha|} \partial^\alpha f(x) \right| &= \varepsilon^n \left| \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \sum_{|\gamma|=n} \beta^\gamma \frac{1}{\gamma!} |\partial^\gamma f(\xi) - \partial^\gamma f(x)| \right| \\ &\leq \varepsilon^n \sum_{|\gamma|=n} \sum_{\beta\in\mathbb{N}_0^d} \varrho_\beta |\beta|^n \sup_{t\in(0,\varepsilon)} |\partial^\gamma f(x+t\beta) - \partial^\gamma f(x)|, \end{aligned}$$

where the last term is independent of α. We show now that the supremum over all boundary points of K of the last sum converges to 0 for each $|\gamma| = n$. To do this we split the sum for some index $m \in \mathbb{N}$ into

$$\begin{aligned} \sup_{x\in\partial K} \sum_{\beta\in\mathbb{N}_0^d} \varrho_\beta |\beta|^n \sup_{t\in(0,\varepsilon)} |\partial^\gamma f(t\beta) - \partial^\gamma f(x)| &\leq \sup_{x\in\partial K} \sum_{|\beta|\leq m} \varrho_\beta |\beta|^n \sup_{t\in(0,\varepsilon)} |\partial^\gamma f(x+t\beta) - \partial^\gamma f(x)| \\ &+ \sup_{x\in\partial K} \sum_{|\beta|>m} \varrho_\beta |\beta|^n \sup_{t\in(0,\varepsilon)} |\partial^\gamma f(x+t\beta) - \partial^\gamma f(x)| \end{aligned}$$

Since $f \in \mathcal{D}^n(B)$ and therefore has a compact support, there exists a constant $C > 0$ such that $\sup_{x,y\in\mathbb{R}^d} |\partial^\gamma f(x) - \partial^\gamma f(y)| < C$ for all $|\gamma| = n$. And since $(\varrho_\beta)_{\beta\in\mathbb{N}_0^d} \in s(\mathbb{N}_0^d)$ with $\varrho_\beta \geq 0$ for all β it follows that

$$\lim_{m\to\infty} \sum_{|\beta|>m} \varrho_\beta |\beta|^n = 0.$$

Thus the second sum converges to 0 for $m \to \infty$ and is independent of the choice of ε. It is left to show that the first sum converges also to 0 but for $\varepsilon \to 0$. For this we use again the fact that since $f \in \mathcal{D}^n(B)$, all the $\partial^\gamma f$ are uniformly continuous, so we have that $\sup_{t\in(0,\varepsilon)} |\partial^\gamma f(x+t\beta) - \partial^\gamma f(x)| \to 0$ uniformly for $\varepsilon \to 0$. Using then

$$\sum_{\beta\leq m} \varrho_\beta |\beta|^n \leq \sum_{\beta\in\mathbb{N}_0^d} \varrho_\beta |\beta|^n,$$

for all $m \in \mathbb{N}$, finishes the proof for the case $|\alpha| \leq n$.

For the case $|\alpha| > n$ we argue now similarly. With Taylor's theorem we find again a $\xi \in [x, x+\varepsilon\beta]$ such that

$$\left| \mu_{\alpha,x,\varepsilon}(f) \right| = \left| \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} f(x+\varepsilon\beta) \right| = \left| \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \left(\sum_{|\gamma|<n} \frac{\partial^\gamma f(x)}{\gamma!} \varepsilon^{|\gamma|} \beta^\gamma + \varepsilon^n \sum_{|\gamma|=n} \frac{\partial^\gamma f(\xi)}{\gamma!} \beta^\gamma \right) \right|.$$

Since $|\alpha| > n$, we have that
$$\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \beta^\gamma = 0$$
for all $|\gamma| \leq n$, and therefore we get
$$\left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \sum_{|\gamma|<n} \frac{\partial^\gamma f(x)}{\gamma!} \varepsilon^{|\gamma|} \beta^\gamma\right| = 0 = \left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \sum_{|\gamma|=n} \frac{\partial^\gamma f(x)}{\gamma!} \varepsilon^{n} \beta^\gamma\right|.$$
This results in
$$\begin{aligned}
\left|\mu_{\alpha,x,\varepsilon}(f)\right| &= \left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \left(\sum_{|\gamma|<n} \frac{\partial^\gamma f(x)}{\gamma!} \varepsilon^{|\gamma|} \beta^\gamma + \varepsilon^n \sum_{|\gamma|=n} \frac{\partial^\gamma f(\xi)}{\gamma!} \beta^\gamma\right)\right| \\
&= \left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \left(-\sum_{|\gamma|=n} \frac{\partial^\gamma f(x)}{\gamma!} \varepsilon^{|\gamma|} \beta^\gamma + \varepsilon^n \sum_{|\gamma|=n} \frac{\partial^\gamma f(\xi)}{\gamma!} \beta^\gamma\right)\right| \\
&\leq \varepsilon^n \sum_{|\gamma|=n} \sum_{\beta\in\mathbb{N}_0^d} \varrho_\beta |\beta|^n \sup_{t\in(0,\varepsilon)} |\partial^\gamma f(x+t\beta) - \partial^\gamma f(x)|,
\end{aligned}$$
where again the last term is independent of α which ensures that the supremum over all $\alpha > n$ exists. The same argument as in the first case shows also here the desired convergence property.

□

9 Theorem. *Let $K \subset \mathbb{R}^d$ such that $\mathscr{E}(K)$ admits an extension operator with loss σ. Then for all $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ and $\varepsilon > 0$ there are measures $\nu_{\alpha,x,\varepsilon}$ on K such that for each $f \in \mathscr{E}^{\sigma(n)}(K)$ and $n \in \mathbb{N}_0$*

1. $$\lim_{\varepsilon\to 0} \sup_{|\alpha|\leq n, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)}) - \varepsilon^{|\alpha|} f^{(\alpha)}(x)|}{\varepsilon^n} = 0$$
2. $$\lim_{\varepsilon\to 0} \sup_{|\alpha|> n, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^n} = 0.$$

Proof. Let E denote the extension operator with loss σ. Since $\mathscr{E}(K)$ is dense in $\mathscr{E}^{\sigma(n)}(K)$ for all $n \in \mathbb{N}_0$, there exists a uniquely determined continuous and linear extension $E_n : \mathscr{E}^{\sigma(n)}(K) \to \mathscr{E}^n$ of E. Then we get that for all $f = \left(f^{(\alpha)}\right)_{|\alpha|\leq\sigma(n)} \in \mathscr{E}^{\sigma(n)}(K)$ and all $x \in K$
$$f^{(\alpha)}(x) = \partial^\alpha E_n(f)(x).$$
Since $E_n(f) \in \mathscr{E}^n$, we can by taking the measures $\mu_{\alpha,x,\varepsilon}$ from the previous theorem define
$$\nu_{\alpha,x,\varepsilon}(f^{(0)}) := E_n^*(\mu)(f) := \mu_{\alpha,x,\varepsilon}(E_n(f)).$$
The proof is complete by applying the properties of the measures $\mu_{\alpha,x,\varepsilon}$. □

2.2 Construction of the operator

The following proposition gives a sufficient condition to the existence of partial derivatives. We will use it to show that the extension operator which we construct takes its values in the space of continuously partially differentiable functions.

10 Proposition. *Let $K \subset \mathbb{R}^d$ be compact, $n \in \mathbb{N}_0$, $f : \mathbb{R}^d \to \mathbb{R}$ such that $f \in \mathscr{C}^\infty(K^c)$, $f \equiv 0$ on K and for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq n$, $x_0 \in \partial K$ and $\varepsilon > 0$ exists a $\delta > 0$ with $\left|\frac{\partial^\beta f(x)}{d(x,K)^{n-|\beta|}}\right| < \varepsilon$ for all $x \in U_\delta(x_0) \cap K^c$,or shorter*

$$|\partial^\beta f(x)| = o(d(x,K)^{n-|\beta|}) \text{ for all } \beta \in \mathbb{N}_0^d \text{ with } |\beta| \leq n \text{ and } x \to \partial K. \tag{3}$$

Then f admits continuous partial derivatives up to order n in $\mathbb{R}^d$.

Proof. We will prove the existence of the partial derivatives only for those of first order, so we show that

$$\lim_{h\to 0} \frac{|f(x_0 + he) - f(x_0)|}{h} = 0 \tag{4}$$

for an arbitrary $x_0 \in \partial K$ and some unit vector e. One can use the same argument for all the other partial derivatives. First we remark that f is continuous in $\mathbb{R}^d \backslash \partial K$ by assumption and the continuity in ∂K follows immediately from (3). In order to show (4) let $\varepsilon > 0$. With (3) we find a $\delta > 0$ such that

$$\left|\frac{\partial f}{\partial x_1}(x)\right| / |x - x_0|^{n-1} < \varepsilon \text{ for all } x \in U_\delta(x_0) \cap K^c. \tag{5}$$

We choose $h \in U_\delta(0)$ such that $x_0 + he \in K^c \cap U_\delta(x_0)$ (if such h does not exists we have $\frac{\partial f}{\partial x_1}(x) = 0$). Because K is compact we find $\tilde{x} \in [x_0 + he, x_0] \cap K$ which minimizes the distance of $x_0 + he$ to K. According to $\tilde{x}$ we find a $0 < |\tilde{h}| < |h|$ which fulfills $x_0 + he = \tilde{x} + \tilde{h}e$ and we get

$$\begin{aligned} \frac{|f(x_0 + he) - f(x_0)|}{h} &= \frac{|f(x_0 + he) - f(\tilde{x})|}{h} \\ &= \frac{|f(\tilde{x} + \tilde{h}e) - f(\tilde{x})|}{\tilde{h}} \frac{\tilde{h}}{h} \\ &\leq \frac{|f(\tilde{x} + \tilde{h}e) - f(\tilde{x})|}{\tilde{h}}. \end{aligned} \tag{6}$$

Because $[x_0 + he, \tilde{x}) \subset K^c$, f is continuous on $[x_0 + he, \tilde{x}]$ and $\frac{\partial f}{\partial x_1}$ exists on $(x_0 + he, \tilde{x})$ we can apply the mean value theorem to get $\hat{x} \in (x_0 + he, \tilde{x})$, such that

$$\frac{\partial f}{\partial x_1}(\hat{x}) = \frac{f(\tilde{x} + \tilde{h}e) - f(\tilde{x})}{\tilde{h}}.$$

Together with (5) and (6) and observing $h > |\hat{x} - x_0|$, we have for $h \leq 1$

$$\frac{|f(x_0 + he) - f(x_0)|}{h} \leq \frac{|f(\tilde{x} + \tilde{h}e) - f(\tilde{x})|}{\tilde{h}} \frac{1}{h^{n-1}} \leq \left|\frac{\partial f}{\partial x_1}(\hat{x})\right| \frac{1}{|\hat{x} - x_0|^{n-1}} < \varepsilon,$$

which gives the desired result. Since $f \equiv 0$ on K we already know that $\partial^\beta f \equiv 0$ on $\mathring{K}$ for all $\beta \leq n$. The calculation above shows that this is true on the whole set K. □

11 Remark. *The proof of Proposition 10 shows that the conclusion is true if one only assume* $|\partial^\beta f(x)| = o(1)$ *for all* $\beta \in \mathbb{N}_0^d$ *with* $|\beta| \leq n$ *and* $x \to \partial K$.

Now we show that 2 implies 1 in our main result.

12 Theorem. *Let* $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ *be an increasing function. If for all* $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ *and* $\varepsilon > 0$ *there are measures* $\nu_{\alpha,x,\varepsilon}$ *on K such that, for each* $f \in \mathscr{E}^{\sigma(n)}(K)$ *and* $n \in \mathbb{N}_0$,

$$1. \qquad \lim_{\varepsilon\to 0} \sup_{|\alpha|\leq n, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)}) - \varepsilon^{|\alpha|} f^{(\alpha)}(x)|}{\varepsilon^n} = 0 \quad \text{and}$$

$$2. \qquad \lim_{\varepsilon\to 0} \sup_{|\alpha|> n, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^n} = 0,$$

then K admits an extension operator with loss σ.

Proof. We set $\mu_{\alpha,i} := \nu_{\alpha,x_i,\gamma_i}/\gamma_i^{|\alpha|}$, and for $f \in \mathscr{E}^0(K)$ we define the operator

$$E(f) = \begin{cases} f^{(0)}(x), x \in K \\ \sum_{i\in\mathbb{N}} \varphi_i(x) \sum_{|\alpha|\leq i} \frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})(x-x_i)^\alpha, x \notin K \end{cases} \tag{7}$$

In order to show that E is an extension operator, we will compare it with Whitney's extension operator E_n which is defined as follows. For $f \in \mathscr{E}^n(K)$,

$$E_n(f)(x) = \begin{cases} f^{(0)}(x), x \in K \\ \sum_{i\in\mathbb{N}} \varphi_i(x) T^n_{x_i}(f)(x), x \notin K \end{cases} .$$

In what follows we show that for all $|\beta| \leq n$ and $f \in \mathscr{E}^{\sigma(n)}(K)(\hookrightarrow \mathscr{E}^n(K))$ we have

$$\left|\partial^\beta E(f)(x) - \partial^\beta E_n(f)(x)\right| = o(dist(x,K)^{n-|\beta|}) \text{ for } x \to \partial K. \tag{8}$$

This shows together with Proposition 10 that $(E - E_n)(f)$ admits derivatives up to order n and that they all vanish on K. Since the Whitney partition $(\varphi_i)_{i\in\mathbb{N}}$ is locally finite, $E(f)$ is clearly $\mathscr{C}^\infty$ on $\mathbb{R}^d \setminus K$.

Let us now prove (8). For $x \in \mathbb{R}^d \setminus K$ we define $i(x) := \min\{i \in I : x \in \operatorname{supp} \varphi_i\}$. Because of the property 2 of the Whitney decomposition in Lemma 1 we then have $i(x) \to \infty$ for $x \to \partial K$. For $|\beta| \leq n$, $f \in \mathscr{E}^{\sigma(n)}(K)$, and $i(x) > n$ we have

$$\begin{aligned} &\partial^\beta(E(f) - E_n(f))(x) = \\ &\sum_{i\geq i(x)} \sum_{|\alpha|\leq n} \frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta((x-x_i)^\alpha \varphi_i(x)) + \\ &\sum_{i\geq i(x)} \sum_{n<|\alpha|\leq i} \frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\partial^\beta((x-x_i)^\alpha \varphi_i(x)). \end{aligned}$$

We will now estimate both summands. Starting with the first, we use requirement 1. and the fact that $\lim_{i\to\infty} \gamma_i = 0$ to get

$$\lim_{i\to\infty} \frac{|\mu_{\alpha,i}(f^{(0)})\gamma_i^{|\alpha|} - \gamma_i^{|\alpha|} f^{(\alpha)}(x_i)|}{\gamma_i^n} = 0,$$

and therefore

$$|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)| = o(\gamma_i^{n-|\alpha|}) \text{ as } i \to \infty.$$

From this and the first part of Corollary 2 we obtain

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d} \left| \sum_{|\alpha|\le n} \frac{1}{\alpha!} (\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)) \partial^\beta((x-x_i)^\alpha \varphi_i(x)) \right| / \gamma_i^{n-|\beta|} \\
&\le \sup_{x\in \operatorname{supp}\varphi_i} \sum_{|\alpha|\le n} \frac{1}{\alpha!} \frac{|(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))|}{\gamma_i^{n-|\alpha|}} \frac{|\partial^\beta((x-x_i)^\alpha \varphi_i(x))|}{\gamma_i^{|\alpha|-|\beta|}} \\
&\le \sup_{x\in \operatorname{supp}\varphi_i} \sum_{|\alpha|\le n} \frac{C_n}{\alpha!} \frac{|(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))|}{\gamma_i^{n-|\alpha|}} \longrightarrow 0 \quad (i\to\infty).
\end{aligned}$$

Hence using the facts that $(\varphi_i)_{i\in\mathbb{N}}$ is locally finite and $\mathrm{d}(x,K) \ge \gamma_i$ for all $x \in \operatorname{supp}\varphi_i$, we get

$$\lim_{x\to\partial K} \sum_{i\ge i(x)} \frac{1}{d(x,K)^{n-|\beta|}} \left| \sum_{|\alpha|\le n} \frac{1}{\alpha!} (\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)) \partial^\beta((x-x_i)^\alpha \varphi_i(x)) \right| = 0. \tag{9}$$

Now we estimate the second summand. By assumption we have

$$\lim_{i\to\infty} \sup_{|\alpha|>n} \frac{|\mu_{\alpha,i}(f^{(0)})| \gamma_i^{|\alpha|}}{\gamma_i^n} = 0.$$

The parts 2 and 3 of Corollary 2 imply then

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d} \left| \sum_{|\alpha|>n} \frac{1}{\alpha!} \mu_{\alpha,i}(f^{(0)}) \partial^\beta((x-x_i)^\alpha \varphi_i(x)) \right| / \gamma_i^{n-|\beta|} \\
&\le \sup_{x\in \operatorname{supp}\varphi_i} \sum_{|\alpha|>n} \frac{1}{\alpha!} |\mu_{\alpha,i}(f^{(0)})| |\partial^\beta((x-x_i)^\alpha \varphi_i(x))| / \gamma_i^{n-|\beta|} \\
&\le \sum_{|\alpha|>n} \frac{1}{\alpha!} |\mu_{\alpha,i}(f^{(0)})| C_n 3^{|\alpha|} \sup_{\gamma\le\beta,\alpha} \frac{\alpha!}{(\alpha-\gamma)!} \gamma_i^{|\alpha|-n} \\
&\le \sup_{\alpha>n} \left(\frac{|\mu_{\alpha,i}(f^{(0)})| \gamma_i^{|\alpha|}}{\gamma_i^n} \right) \sum_{|\alpha|>n} \frac{1}{\alpha!} C_n 3^{|\alpha|} \sup_{\gamma\le\beta,\alpha} \frac{\alpha!}{(\alpha-\gamma)!} \\
&\le \sup_{\alpha>n} \left(\frac{|\mu_{\alpha,i}(f^{(0)})| \gamma_i^{|\alpha|}}{\gamma_i^n} \right) C_n e^{3d} (n+1)^d 3^n \longrightarrow 0 \quad (i\to\infty).
\end{aligned}$$

From this we can conclude using again that $(\varphi_i)_{i\in\mathbb{N}}$ is locally finite and $\mathrm{d}(x,K) \ge \gamma_i$ for all $x \in \operatorname{supp}\varphi_i$

$$\lim_{x\to\partial K} \sum_{i\ge i(x)} \frac{1}{d(x,K)^{n-|\beta|}} \left| \sum_{n<|\alpha|\le i} \frac{1}{\alpha!} \mu_{\alpha,i}(f^{(0)}) \partial^\beta((x-x_i)^\alpha \varphi_i(x)) \right| = 0. \tag{10}$$

With (9) and (10) we have shown (8). Now we can apply proposition 10 on $E(f)-E_n(f)$ and get that this linear operator is well defined and takes its values in $\mathscr{I}^n(K)=\{g\in\mathscr{C}^n(\mathbb{R}^d):\partial^\alpha g(x)=0$ for all $x\in K, |\alpha|\leq n\}$. This operator is continuous if we equip $\mathscr{I}^n(K)$ with the topology of pointwise convergence, what follows from the locally finite property of the Whitney partition of unity. Hence we get the continuity of $E(f)-E_n(f):\mathscr{E}^{\sigma(n)}(K)\to\mathscr{C}^n(\mathbb{R}^d)$. The continuity of $E_n:\mathscr{E}^n(K)\to\mathscr{C}^n(\mathcal{B})$ permits to get positive constants C_n such that $|E(f)|_n\leq C_n\|f\|_{\sigma(n)}$ for $f\in\mathscr{E}^{\sigma(n)}(K)$. Hence the restriction to $\mathscr{E}(K)$ is an extension operator $0-0$ continuous. □

3 Consequences

Now that we have proved our main result, we can extract some conclusions.

13 Theorem. *The following are equivalent*

1. *$\mathscr{E}(K)$ admits a continuous linear extension operator.*
2. *$\mathscr{E}(K)$ admits a continuous linear extension operator which is $0-0$ continuous*
3. *For all $\alpha\in\mathbb{N}_0^d$, $x\in\partial K$ and $\varepsilon>0$ there are measures $\nu_{\alpha,x,\varepsilon}$ on K such that, for each $f\in\mathscr{E}(K)$ and $n\in\mathbb{N}_0$,*

$$\lim_{\varepsilon\to 0}\ \sup_{|\alpha|\leq n, x\in\partial K}\frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})-\varepsilon^{|\alpha|}f^{(\alpha)}(x)|}{\varepsilon^n}=0 \quad \textit{and}$$

$$\lim_{\varepsilon\to 0}\ \sup_{|\alpha|>n, x\in\partial K}\frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^n}=0.$$

Proof. The equivalence between 1 and 2 was proved by Frerick in [Fre07]. That 2 implies 3 is a consequence of of Theorem 3. 3 implies 2 by Theorem 12. □

As an outcome of our results we get that the Whitney operator can be seen as a standard construction plan for any extension operator on $\mathscr{E}(K)$.

14 Remark. From the proof of Theorem 13 we get that whenever $\mathscr{E}(K)$ admits a continuous linear extension operator then there exist measures $\mu_{\alpha,i}$ on K, for $\alpha\in\mathbb{N}_0^d$ and $i\in\mathbb{N}$ satisfying

$$\lim_{i\in\mathbb{N}}|\mu_{\alpha,i}(f)-f^{(\alpha)}(x_i)|=0$$

for each $f\in\mathscr{E}(K)$ such that

$$E(f)=\begin{cases} f^{(0)}(x), x\in K\\ \sum_{i\in\mathbb{N}}\varphi_i(x)\sum_{|\alpha|\leq i}\frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})(x-x_i)^\alpha, x\notin K\end{cases}$$

defines an extension operator.

The remark above tells us that whenever $\mathscr{E}(K)$ admits a continuous linear extension operator, it is always possible to construct another extension operator in a way that it is looking similar to the original Whitney operators for the spaces of jets of finite order.

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