On large orbits of supersoluble subgroups of linear groups

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Abstract
We prove that if $G$ is a finite soluble group, $V$ is a finite faithful completely reducible $G$-module, and $H$ is a supersoluble subgroup of $G$, then $H$ has at least one regular orbit on $V \oplus V$.

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1 Introduction
Let $G$ be a finite group acting on a finite set $\Omega$. An element $\omega$ of $\Omega$ is in a regular orbit if $C_G(\omega) = \{g \in G \mid \omega g = \omega\} = 1$, i.e., the orbit of $\omega$ is as large as possible and it has size $|G|$. Regular orbits of actions of linear groups acting on finite vector spaces arise in a variety of contexts, including the study of soluble groups, representation theory of finite groups and finite permutation groups, and it is a lively area of current research.

One of the most important questions in this context is to determine conditions which force a given subgroup of a finite linear group to have a regular orbit. This problem has been extensively investigated with a lot of results available (see [3, 4, 5, 15, 16]). In [11, Theorem A], a common extension of the main results of these papers has been showed.

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**Theorem 1** ([11] Theorem A). *If \( G \) is a finite soluble group, \( V \) is a faithful completely reducible \( G \)-module (possibly of mixed characteristic) and \( H \) is a subgroup of \( G \) such that the semidirect product \( VH \) is \( S_4 \)-free, then \( H \) has at least two regular orbits on \( V \oplus V \). Furthermore, if \( H \) is \( \Gamma(2^3) \)-free and \( \text{SL}(2,3) \)-free, then \( H \) has at least three regular orbits on \( V \oplus V \).*

Halasi and Maróti also proved in [7] that if \( V \) is a finite vector space over a finite field of order \( q \geq 5 \) and of characteristic \( p \) and \( G \leq \text{GL}(V) \) is a \( p \)-soluble completely reducible linear group, then there exists a base for \( G \) on \( V \) of size at most \( 2 \). As a consequence, under this hypothesis \( G \) possesses a regular orbit over \( V \oplus V \). On the other hand, Wolf [12, Theorem A] showed that a finite supersoluble and completely reducible subgroup \( G \) of \( \text{GL}(V) \), for a finite vector space \( 0 \neq V \), has at least one regular orbit on \( V \oplus V \).

The results just mentioned suggest that the answer to the question of whether or not Wolf’s theorem holds for every supersoluble subgroup of a finite completely reducible soluble subgroup \( G \) of \( \text{GL}(V) \), even if the supersoluble subgroup is not completely reducible, is a natural next objective.

The main aim of this paper is to give a complete answer to this question.

**Theorem A.** *Let \( G \) be a finite soluble group and \( V \) be a finite faithful completely reducible \( G \)-module (possibly of mixed characteristic). Suppose that \( H \) is a supersoluble subgroup of \( G \). Then \( H \) has at least one regular orbit on \( V \oplus V \).*

By [11] Corollary 3, the answer is affirmative if \( V \) is of odd order. Therefore it will be enough to prove Theorem A for a module \( V \) over a field of characteristic 2.

The following two examples will show that in Theorem A the subgroup \( H \) is not completely reducible on \( V \) in general.

**Example 1.** Let \( G = \text{GL}(2,3) \) and \( V = \text{GF}(3) \oplus \text{GF}(3) \) the natural faithful module of \( G \) over \( \text{GF}(3) \). Let \( H = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \). Observe that \( H \cong S_3 \) is supersoluble and \( V \) is a non completely reducible \( H \)-module. In fact, \( V_1 = \{(0, x) \mid x \in \text{GF}(3)\} \) is an \( H \)-submodule of \( V \) and no complement of \( V_1 \) in \( V \) is \( H \)-invariant.

**Example 2.** Let \( K = \text{GL}(2,2) \) and \( W = \text{GF}(2) \oplus \text{GF}(2) \) the natural faithful module of \( K \) over \( \text{GF}(2) \). Let \( S \cong S_2 \) be the symmetric group on \( \Omega = \{1, 2\} \). Write \( G = K \wr S \) and \( V = W^\Omega \). Then \( V \) is a faithful, irreducible \( G \)-module (see Section 2). Set

\[
H = \{(f, \sigma) \in G \mid f \in K, \sigma \in S, f(1) = f(2)\} \cong S_3 \times C_2.
\]
Then $H$ is a supersoluble subgroup of $G$. Let $V_1 = \{v \in V = W^\Omega \mid v(1) = v(2)\}$. Then $V_1$ is an $H$-submodule of $V$. Suppose that there exists an $H$-submodule $V_2$ such that $V = V_1 \oplus V_2$ and take $0 \neq u \in V_2$. We have that $u(1) \neq u(2)$, and $u(1) + u(2) \neq 0$. Then $u^{(1,\sigma)} + u \in V_1 \cap V_2 = 0$. This contradiction shows that $V$ is not a completely reducible $H$-module. Note that $G$ does not have regular orbits on $V \oplus V$ by Wolf’s formula, but $H$ does.

We bring the introduction to a close with an application of Theorem A. Denote by $\mathfrak{U}$ the class of all finite supersoluble groups, which is a subgroup-closed saturated formation. Denote by $G^\mathfrak{U}$ the $\mathfrak{U}$-residual of group $G$. Clearly, $G^\mathfrak{U} \subseteq G^{\mathfrak{N}}$, where $G^{\mathfrak{N}}$ denotes the nilpotent residual. The following corollary generalises a result of Keller and Yang [9, Theorem 1.2] by replacing the nilpotent residual by the supersoluble residual.

**Corollary 2.** Let $G$ be a finite soluble group and $V$ a finite faithful completely reducible $G$-module, possibly of mixed characteristic. Let $M$ be the largest orbit size in the action of $G$ on $V$. Then

$$|G : G^\mathfrak{U}| \leq M^2.$$ 

**Proof.** Since $\mathfrak{U}$ is a saturated formation, by [8, Theorem 3.9] we can take a subgroup $H$ such that $G^\mathfrak{U}H = G$ and $H \in \mathfrak{U}$. Then $H$ is supersoluble. By Theorem A, $H$ has a regular orbit on $V \oplus V$. It implies that $|C_H(v)| \leq |H|^{1/2}$ for some $v \in V$. Let $M_H$ be the largest orbit size of $H$ on $V$. Then it follows that $|H| \leq |H : C_H(v)|^2 \leq M_H^2$. Hence clearly $|G/G^\mathfrak{U}| \leq |H| \leq M_H^2 \leq M^2$, as desired.

## 2 Background results

All groups considered in the sequel will be finite.

The following elementary lemma appears in [11, Lemma 3].

**Lemma 3.** Suppose that a group $G$ acts on a non-empty finite set $\Omega$. Then:

1. If $|\Omega| - \left| \bigcup_{1 \neq g \in G} C_G(g) \right| > k|G|$ for some non-negative integer $k$, then $G$ has at least $k + 1$ regular orbits on $\Omega$. In particular, if $k = 0$, then $G$ has at least one regular orbit on $\Omega$.

2. If $G$ has $k$ regular orbits on $\Omega$, then a subgroup $H$ of $G$ has at least $|G : H|k$ regular orbits on $\Omega$. 


The following notation and arguments appear in [11, Section 3]. We summarise them here for the benefit of the reader.

Recall that an irreducible $G$-module $V$ is called *imprimitive* if there is a non-trivial decomposition of $V$ as a direct sum of subspaces $V = V_1 \oplus \cdots \oplus V_n$ ($n > 1$) such that $G$ permutes the set $\{V_1, \ldots, V_n\}$. The irreducible $G$-module $V$ is *primitive* if $V$ is not imprimitive. A linear group $G \leq GL(d, p^k)$, $p$ a prime, is said to be *primitive* if the natural $G$-module is primitive.

Let $G$ be a group and let $V$ be a faithful $G$-module. Let $V = \hat{W}_1 \oplus \cdots \oplus \hat{W}_m$, with $m \geq 2$, be a decomposition of $V$ as a direct sum of subspaces such that $\hat{\Omega} = \{\hat{W}_1, \ldots, \hat{W}_m\}$ is permuted transitively by $G$. The action of $G$ on $\hat{\Omega}$ induces a homomorphism $\sigma : G \to S_{\hat{\Omega}}$, where $\hat{\Omega} = \{1, \ldots, m\}$. Write $W = \hat{W}_1$ and $H = N_G(W)/C_G(W)$ and $S = \sigma(G)$. Let

$$\hat{G} = H \wr S = \{(f, \sigma) \mid f : \Omega \to H, \sigma \in S\}$$

with the product $(f_1, \sigma_1)(f_2, \sigma_2) = (g, \sigma_1 \sigma_2)$, where $g(\omega) = f_1(\omega)f_2(\omega^{\sigma_1})$ for all $\omega \in \Omega$ be the permutational wreath product of $K$ with $S$ (see [8, Kapitel I, Satz 15.3]). Let

$$W^\Omega = \{f \mid f : \Omega \to W \text{ is a map}\}. \quad (1)$$

If $Y$ is a subgroup of $H$, we set $Y^\Omega = \{(f, 1) \in H \wr S \mid f(\omega) \in Y \text{ for all } \omega \in \Omega\}$. In particular, $B = H^\Omega$ is called the *base group* of $H \wr S$. If $W$ is a $H$-module, then $W^\Omega$, considered as a subgroup of $([W]H) \wr S$, becomes a $H \wr S$-module with the action given by $g(f, \sigma)(\omega) = g(\omega^{\sigma^{-1}})f(\omega^{\sigma^{-1}})$.

**Lemma 4** ([11, Lemma 9]). There exists a monomorphism $\tau : G \to \hat{G}$ such that:

1. The actions of $G$ on $V$ and $\tau(G)$ on $W^\Omega$ are equivalent.
2. $\hat{G} = H^\Omega \tau(G)$.
3. Write $W_i = \{f \in W^\Omega \mid f(j) = 0, \forall j \neq i\}$ for each $i \in \Omega$. Then

$$N_{\tau(G)}(W_i)/C_{\tau(G)}(W_i) \cong H, \forall i \in \Omega.$$

Therefore if we are interested in regular orbits of the action of $G$ on $V$ and $V$ is not primitive, we may assume, by Lemma 4, that $G$ is a supersoluble subgroup of a wreath product $\hat{G} = K \wr S$, where $K$ is a group, $W$ is a faithful $K$-module, and $S$ is a non-trivial primitive permutation group on a finite set $\Omega$ such that $\hat{G} = K^\Omega$ and $V = W^\Omega$. Since this situation will appear several times in our arguments, we will use some abbreviations to refer to it.
Notation 5. We say that \((\hat{G}, G, H, S, \Omega)\) satisfies Condition A if

- \(H\) is a group;
- \(S\) is a primitive group on the finite set \(\Omega\);
- \(\hat{G} = H \wr S\);
- \(G\) is a supersoluble subgroup of \(\hat{G}\) such that \(H \nabla G = \hat{G}\).

Notation 6. We say that \((\hat{G}, G, H, S, \Omega, V, W)\) satisfies Condition B if

- \((\hat{G}, G, H, S, \Omega)\) satisfies Condition A;
- \(W\) is a faithful \(H\)-module over \(\mathbb{GF}(2)\);
- \(V = W^\Omega\) (see Equation (1)), naturally is a faithful \(\hat{G}\)-module;
- Write \(W_i = \{f \in V \mid f(j) = 0, \forall j \neq i\}\) for each \(i \in \Omega\);
- \(N_G(W_i)/C_G(W_i) \cong H\) for each \(i \in \Omega\).

As in [11, Section 3], we are interested here in regular orbits of a group \(G\) on completely reducible \(G\)-modules \(V\) over finite fields and so, in looking for regular orbits of \(G\) on \(V\), we can assume without loss of generality that the field is a prime field.

In this context, a result of Wolf [13] that provides a formula to count the exact number of regular orbits \(\hat{G}\) on \(W^\Omega\) is extremely useful. Let \(S\) be a transitive permutation group on a finite set \(\Omega\) and denote by \(\Pi_l(\Omega, S)\) the set of all partitions \(\{\Delta_1, \ldots, \Delta_l\}\) of length \(l\) of \(\Omega\) having the property that the subgroup \(\{s \in S \mid \Delta^s_i = \Delta_i \text{ for all } i\}\) of \(S\) is trivial.

Theorem 7 (Wolf’s formula, [13]). Suppose that \((\hat{G}, G, H, S, \Omega, V, W)\) satisfies Condition B. Let \(k\) be the number of regular orbits of \(H\) on \(W\). Then the number of regular orbits of \(\hat{G}\) (also \(G\)) on \(V = W^\Omega\) is at least

\[
\frac{1}{|S|} \sum_{2 \leq l \leq m} P(k, l)|\Pi_l(\Omega, S)|,
\]

where \(P(k, l) = k!/(k - l)!\) if \(k \geq l\) and \(P(k, l) = 0\) otherwise.

The following result is useful to obtain regular orbits in a direct sum of \(G\)-modules starting from regular orbits of its terms.

Lemma 8. Let \(G\) be a group and \(V\) be a faithful \(G\)-module such that \(V = W_1 \oplus \cdots \oplus W_s\), where \(W_i\) is a \(G\)-module, \(1 \leq i \leq s\). If \(G/C_G(W_i)\) has \(t_i\) regular orbits on \(W_i \oplus W_i\), then \(G\) has at least \(\prod_{i=1}^s t_i\) regular orbits on \(V \oplus V\).
The following result about supersoluble primitive permutation groups is crucial in our inductive arguments.

**Lemma 9.** Let $S$ be a supersoluble primitive permutation group on a finite set $\Omega = \{1, \ldots, n\}$ with $n \geq 2$. Then $\text{Stab}_S(1) \cap \text{Stab}_S(2) = 1$.

**Proof.** Since $S$ is supersoluble and primitive, we have that $|\Omega|$ is a prime. Hence $S$ is a transitive permutation group of prime degree. The conclusion follows from [3, Theorem 3.6 (d)].

**Lemma 10.** Assume that $(\hat{G}, G, H, S, \Omega)$ satisfies Condition A. Write $N = H^2 \cap G$ and assume that $O_p(N) = 1$ for some prime $p$. If $f$ is a $p$-element of $H^2$ such that $(f, 1) \in N$ and $f(i_0) = 1$ for some $i_0 \in \Omega$, then $f = 1$.

**Proof.** Observe that $S \cong \hat{G}/H^2 \cong G/N$ is supersoluble. Since $S$ is a primitive permutation group, we conclude that $S$ has a unique minimal normal subgroup $X$ such that $|X| = |\Omega| = q$ for some prime $q$.

Let $P \in \text{Syl}_p(N)$ such that $(f, 1) \in P$. By the Frattini Argument, $G = NN_G(P)$ and, consequently, $\hat{G} = H^2N_G(P)$. Let $\rho \in X$, $\rho \neq 1$. Then $\rho^q = 1$. Since $\hat{G} = H^2N_G(P)$, there exists $u \in H^2$ such that $(u, \rho) \in N_G(P)$ whose projection onto $S$ is $\rho$. Assume that $o((u, \rho)) = qm$ with $\gcd(q, m) = 1$ and $\alpha \in \mathbb{N}$. Then there exist $\lambda, \mu \in \mathbb{Z}$ such that $\lambda q + \mu m = 1$, and so $(u, \rho)^{1-\lambda q} = (u, \rho)^{\mu m}$ is a $q$-element of the form $(g, \rho^{1-\lambda q}) = (g, \rho) \in N_G(P)$.

Let $T = P((g, \rho))$. Note that $T' \leq P$ is a $p$-group and observe that $T' \leq G' \leq F(G)$ since $G$ is supersoluble. Thus $T' \leq O_p(G)$. Then $[(f, 1), (g, \rho)] \in T' \cap N \leq O_p(G) \cap N = O_p(N) = 1$. Thus we have $(f, 1)(g, \rho) = (g, \rho)(f, 1)$, that is, $f(i)g(i) = g(i)f(i^\rho)$ for all $i \in \Omega$. Therefore $f(i) = 1$ if and only if $f(i^\rho) = 1$.

Recall that $X$ acts transitively on $\Omega$. For each $i \in \Omega$, there exists $\rho_i$ (depending on $i$) in $S$ such that $i_0^\rho_i = i$. Since $f(i_0) = 1$, we have that $f(i) = f(i_0^\rho_i) = 1$. Thus $f(i) = 1$ for each $i \in \Omega$ and the statement is proved.

## 3 Lemmas

In order to prove Theorem [A], we will argue by induction by decomposing $V$ as a direct sum of subspaces permuted transitively by $G$. Therefore our first step will be the study of the case in which there is no such a proper decomposition, that is, $V$ is primitive. In attaining this aim, the following two lemmas are crucial. The first one concerns primitive soluble linear groups over a field of characteristic two.
Let $V$ be the Galois field $\text{GF}(p^n)$ for some prime $p$ and integer $n$. Then $V$ can be regarded as a vector space over $\text{GF}(p)$ of dimension $n$. Recall that the semi-linear group of $V$ is

$$\Gamma(V) = \Gamma(p^n) = \{ x \mapsto ax^\tau \mid a \in \text{GF}(p^n)^*, \tau \in \text{Gal}(\text{GF}(p^n)/\text{GF}(p))\}.$$  

**Lemma 11.** Let $G$ be a supersoluble group and $V$ be a faithful primitive $G$-module over $\text{GF}(2)$. Then $G$ has at least four regular orbits on $V \oplus V$ unless $G = \Gamma(V)$ and $|V| = 2^n$, $2 \leq n \leq 4$. In these cases, $G$ has exactly $n - 1$ regular orbits on $V \oplus V$.

**Proof.** Let $A$ be a maximal abelian normal subgroup of $G$. Clearly $A \leq C_G(A) \leq G$. Suppose that $A < C_G(A)$. Then we can take a chief factor $T/A$ of $G$ such that $T \leq C_G(A)$. Since $G$ is supersoluble, $T/A$ is cyclic and $T = \langle A, x \rangle$ for some $x \in C_G(A)$. Then $T$ is an abelian subgroup of $G$, contrary to the choice of $A$. Thus $A = C_G(A)$. Since $V$ is a primitive $G$-module, $V_A$ is homogeneous by Clifford’s theorem [2, Chapter B, Theorem 7.3]. By [10] Lemma 2.2, $V_A$ is irreducible. It follows from [10] Theorem 2.1 that $G \leq \Gamma(V)$. Write $|V| = 2^n$ where $n \geq 1$ is an integer.

First we assume that $G = \Gamma(V)$. Equivalently, if suffices to consider the regular orbits of $\Gamma(2^n)$ acting on the additive group of the field $\text{GF}(2^n)$. Take the field automorphism $\sigma: \text{GF}(2^n) \rightarrow \text{GF}(2^n)$ given by $u \mapsto u^2$. The Galois group $C = \text{Gal}(\text{GF}(2^n)/\text{GF}(2)) = \langle \sigma \rangle$ is of order $n$.

For each prime $p$ dividing $n$, $\langle \sigma^{n/p} \rangle$ is the unique subgroup of $C$ with order $p$ since $C$ is cyclic. Then we have that

$$C_{\text{GF}(2^n)}(\sigma^{n/p}) = \{ u \in \text{GF}(2^n) \mid u^{2^{n/p}} = u \}$$

is a subfield of $\text{GF}(2^n)$ isomorphic to $\text{GF}(2^{n/p})$. Thus $|C_{\text{GF}(2^n)}(\sigma^{n/p})| = 2^{n/p}$.

In order to prove that $C$ has at least four regular orbits on $\text{GF}(2^n)$ when $n \geq 5$, by Lemma [3] it suffices to show that

$$2^n - \sum_{p|n} 2^{n/p} > 3n$$

holds for $n \geq 5$. Observe that $\sum_{p|n} 2^{n/p} \leq \log_2 n \cdot 2^{n/2}$. It is not difficult to check that $2^n - \sum_{p|n} 2^{n/p} \geq 2^n - \log_2 n \cdot 2^{n/2} > 3n$ for $n \geq 8$ and it is easy to find that the inequality also holds for $n = 5, 6, 7$.

Thus we have proved that $G \leq \Gamma(V)$ has at least four regular orbits on $V \oplus V$ when $n \geq 5$.

Assume that $n = 1$. Then $|V| = 2$ and $G = 1$. Hence $G$ has exactly four regular orbits on $V \oplus V$.
Assume that $n = 2$. Then $|V| = 2^2$ and $G \leq \Gamma(V) \cong S_3$. If $G < \Gamma(V)$, then $G$ has a regular orbit on $V$. In this case, $G$ has at least $|V| = 4$ regular orbits on $V \oplus V$. If $G = \Gamma(V)$, we can check that $G$ has exactly one regular orbit on $V \oplus V$.

Assume that $n = 3$. Then $|V| = 2^3$ and $G \leq \Gamma(V) \cong [C_7]C_3$. If $G = \Gamma(V)$, then $G$ has exactly two regular orbits on $V \oplus V$. Thus, if $G < \Gamma(V)$, $G$ has at least four regular orbits on $V \oplus V$.

Assume that $n = 4$. Then $|V| = 2^4$ and $G \leq \Gamma(V) \cong [C_{15}]C_4$. If $G = \Gamma(V)$, then $G$ has exactly three regular orbits on $V \oplus V$. Thus, if $G < \Gamma(V)$, $G$ has at least six regular orbits on $V \oplus V$.

Thus the lemma is completely proved.

\textbf{Lemma 12.} Let $G$ be a soluble primitive group of $GL(d, 2)$, and let $V$ be the natural $G$-module. Assume that $H$ is a supersoluble subgroup of $G$. Then $H$ has at least three regular orbits on $V \oplus V$ unless one of the following two cases occurs:

1. $d = 2$ and $H = \Gamma(V) \cong S_3$, then $H$ has just one regular orbit on $V \oplus V$.

2. $d = 3$ and $H = \Gamma(V) \cong \Gamma(2^3)$, then $H$ has just two regular orbits on $V \oplus V$.

Furthermore if $H$ is of odd order, then $H$ has four regular orbits on $V \oplus V$ unless the case 2 occurs.

\textbf{Proof.} Assume first that $H = G$. Then $G$ is supersoluble. It follows from Lemma 11 that the hypothesis of the lemma is satisfied. Now we may assume that $H < G$. By [3, Theorem 3.4], $H$ has at least four regular orbits on $V \oplus V$ provided that $G$ is not isomorphic to $GL(2, 2), 3^{1+2}.SL(2, 3)$ or $3^{1+2}.GL(2, 3)$.

If $H$ is a proper subgroup of $G = GL(2, 2) \cong S_3$, then $H$ is of prime order and there exists $v \in V$ such that $C_H(v) = 1$. Hence $H$ has at least $|V| = 4$ regular orbits on $V \oplus V$.

Suppose that $G$ is isomorphic to $3^{1+2}.SL(2, 3)$ or $3^{1+2}.GL(2, 3)$ (as a subgroup of $GL(6, 2)$). In this case, one checks with GAP [6] that $H$ has at least three (four if $|H|$ is odd) regular orbits on $V \oplus V$.

The next definitions reflect what happens in the exceptional cases of Lemma 12.

\textbf{Definition 13.} Let $G$ be a group and let $V$ be a faithful $G$-module. We say that the $G$-module $V$ satisfies Property I if the following conditions hold:

1. $G$ is an odd order group and $O_3(G) = 1$. 

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2. There exists \( 0 \neq x \in V \) such that \( C_G(x) \) has at least four different orbits on \( V \) with representatives \( y_1, y_2, z_1, z_2 \) satisfying that \( C_G(x) \cap C_G(y_i) = 1 \) and \( C_G(x) \cap C_G(z_i) \) is a 3-group for each \( i \).

**Definition 14.** Let \( G \) be a group and let \( V \) be a faithful \( G \)-module. We say that the \( G \)-module \( V \) satisfies **Property II** if the following conditions hold:

1. \( G \) is an even order group with \( O_2(G) = 1 \).

2. There exists \( 0 \neq x \in V \) such that \( C_G(x) \) at least three different orbits on \( V \) with representatives \( y, z_1, z_2 \) satisfying that \( C_G(x) \cap C_G(y) = 1 \) and \( C_G(x) \cap C_G(z_i) \) is a 2-group for each \( 1 \leq i \leq 2 \).

Note that if the faithful \( G \)-module \( V \) satisfies either **Property I** or **Property II**, then \( G \) has at least one regular orbit on \( V \oplus V \). Our strategy will consist in showing by induction that \( G \) has at least three regular orbits on \( V \oplus V \) or \( G \) satisfies either **Property I** or **Property II**. As we will see in Lemmas 15 and 16 below, the existence of regular orbits on \( V \oplus V \) in the situation of **Condition B** will depend on the existence of some special orbits of \( H \) on \( W_1 \oplus W_1 \) allowing us to apply Lemma 10. This situation is guaranteed when **Property I** or **Property II** holds.

Let \( G \) be a group and \( \Omega \) be a transitive \( G \)-set. Recall that a subset \( \Delta \subseteq \Omega \) is said to be a **block** if for every \( g \in G \), either \( \Delta^g = \Delta \) or \( \Delta^g \cap \Delta = \emptyset \). Clearly every transitive \( G \)-set \( \Omega \) has a block \( \Delta \) such that \( 1 \leq |\Delta| < |\Omega| \) if \( |\Omega| \geq 2 \). If we take such a block \( \Delta \) of maximal size, then \( \text{Stab}_G(\Delta) \) is maximal in \( G \). (see II Definition 1.1.1 and Proposition 1.1.2).

**Lemma 15.** Assume that \((\bar{G}, G, H, S, \Omega, V, W)\) satisfies **Condition B**.

1. If \( O_p(H) = 1 \) for some prime \( p \) and write \( N = H^2 \cap G \), then \( O_p(N) = 1 \).

Let \( x \in W \). Suppose that \( v \in V = W^\Omega \) is defined by \( v(\omega) = x \) for all \( \omega \in \Omega \).

2. If \((f, \sigma) \in C_G(v)\), then \( f(\omega) \in C_H(x) \) for all \( \omega \in \Omega \).

3. Assume that \( \{\Delta_1, \ldots, \Delta_s\} \) is a partition of \( \Omega \) such that \( \bigcap_i \text{Stab}_S(\Delta_i) = 1 \). Assume also that \( C_H(x) \) has different orbits on \( W \) with representatives \( y_1, \ldots, y_s \) such that \( C_H(x) \cap C_H(y_i) \) is a \( p \)-group for \( 1 \leq i \leq s \).

Construct the elements \( v, u \in V = W^\Omega \) as \( v(\omega) = x \) for \( \omega \in \Omega \) and \( u(\omega) = y_i \) if \( \omega \in \Delta_i \), where \( 1 \leq i \leq s \), for \( \omega \in \Omega \).

Then \( C_G(v) \cap C_G(u) \) is a \( p \)-group. Furthermore, if \( O_p(H) = 1 \) and \( C_H(x) \cap C_H(y_k) = 1 \) for some \( 1 \leq k \leq s \), then \( C_G(v) \cap C_G(u) = 1 \), in particular, \((v, u)\) generates a regular orbit in \( V \oplus V \).
4. If $\Omega = \{1, 2, 3\}$, $y, z \in W$ belong to different orbits of $C_H(x)$ on $W$, $C_H(x) \cap C_H(y) = 1$, $C_H(x) \cap C_H(z)$ is a 2-group, and $u \in V$ is defined by $u(1) = u(2) = y$, $u(3) = z$, then $C_G(u) \cap C_G(v)$ is a 2-group.

5. If $\Omega = \{1, 2\}$, $y \in W$ satisfies that $C_H(x) \cap C_H(y) = 1$, and $u \in V$ is defined by $u(1) = u(2) = y$, then $C_G(u) \cap C_G(v)$ is a 2-group.

Assume that $\Omega = \{1, 2\}$ and $0 \neq x \in W$. Suppose that $O_2(H) = 1$ and that $v' \in V = W^\Omega$ is defined by $v'(1) = 0$, $v'(2) = x$.

6. If $O_2(G) \neq 1$, $y_1$ and $y_2$ lie in different orbits of $C_H(x)$ on $W$, $C_H(x) \cap C_H(y_2) = 1$ and $u \in V = W^\Omega$ is defined by $u(1) = y_1$, $u(2) = y_2$, then $C_G(v') \cap C_G(u) = 1$.

Proof. 1. Write $W_i = \{f \in V \mid f(j) = 0, \forall j \neq i\}$ for each $i \in \Omega$ and note that $N = \bigcap_i N_G(W_j) \trianglelefteq G$. Consequently, $N$ is a normal subgroup of $N_G(W_j)$ for each $j$. $N/(N \cap C_G(W_j)) \cong NC_G(W_j)/C_G(W_j) \trianglelefteq N_G(W_j)/C_G(W_j)$, which is isomorphic to $H$. Since $O_p(H) = 1$, we conclude that $O_p(N) \leq C_G(W_j)$ for each $j$. Therefore

$$O_p(N) \leq \bigcap_j C_G(W_j) = C_G(V) = 1,$$

because $G$ acts faithfully on $V$.

2. Suppose that $(f, \sigma) \in C_G(v)$. Given $\omega \in \Omega$, $v(\omega)^{f(\omega)} = v(\omega\sigma)$, which implies that $x^{f(\omega)} = x$ and so $f(\omega) \in C_H(x)$ for all $\omega \in \Omega$.

3. Let $(f, \sigma) \in C_G(v) \cap C_G(u)$. Given $\omega \in \Omega$, $v(\omega)^{f(\omega)} = v(\omega\sigma)$, which implies that $x^{f(\omega)} = x$ and so $f(\omega) \in C_H(x)$ for $\omega \in \Omega$. Moreover, $u(\omega)^{f(\omega)} = u(\omega\sigma)$ for $\omega \in \Omega$. If $\omega \in \Delta_i$, since the $y_i$ belong to different orbits under the action of $C_H(x)$, we conclude that $\omega\sigma \in \Delta_i$. It follows that $\sigma \in \bigcap_i \text{Stab}_S(\Delta_i) = 1$ and, if $\omega \in \Delta_i$, $y_i^{f(\omega)} = y_i$, that is, $f(\omega) \in C_H(x) \cap C_H(y_i)$, which is a $p$-group for all $i$. Therefore $(f, \sigma) = (f, 1)$ is a $p$-element. It follows that $C_G(v) \cap C_G(u)$ is a $p$-group.

Suppose that, in addition, $O_p(H) = 1$ and that $C_H(x) \cap C_H(y_1) = 1$. In this case, for $\omega \in \Delta_i$, we obtain that $y_1^{f(\omega)} = y_1$, and hence $f(\omega) \in C_H(x) \cap C_H(y_1) = 1$. Consequently $f(\omega) = 1$ for $\omega \in \Delta_1$. Furthermore, $(f, \sigma) = (f, 1) \in H^1 \cap G = N$ is a $p$-element and $f(\omega) = 1$ for $\omega \in \Delta_1$. Since $O_p(H) = 1$, we obtain that $O_p(N) = 1$ by the statement $\square$ By Lemma $\square$ we conclude that $f = 1$.

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4. Let \((f, \sigma) \in C_G(v) \cap C_G(u)\). Then \(v(i)^f = \sigma(i)^v\). It follows that 
\(x^f = x\), that is, \(f(i) \in C_H(x)\) for all \(i \in \Omega\). Moreover, \(u(i)^f = u(i)^\sigma\). Since \(y\) and \(z\) belong to different orbits of \(C_G(x)\) in \(W_1\), we conclude that \(\sigma \in \langle (12) \rangle\). Moreover, \(u(i)^f = u(i)^\sigma\) for \(i \in \{1, 2\}\) implies that \(y^f = y\), that is, \(f(1), f(2) \in C_H(x) \cap C_H(y) = 1\), and \(u(3)^f = u(3)^\sigma = u(3)\) implies that \(z^f = z\), that is, \(f(3) \in C_H(x) \cap C_H(z)\), a 2-group. Therefore \((f, \sigma)\) is a 2-element.

5. The proof of this statement is similar to the proof of the previous statement.

6. Since \(O_2(H) = 1\), we have that \(O_2(N) = 1\) by Statement 1. Since \(G/N \cong S \cong S_2\), we have that \(N = \text{a maximal subgroup of } G\). Moreover, \(N \cap O_2(G) \leq O_2(N) = 1\). As \(O_2(G) \neq 1\), consequently, \(G = N O_2(G) = H^2 O_2(G)\) and \([N, O_2(G)] = 1\).

Let \((f, \sigma) \in C_G(v') \cap C_G(u)\), with \(f \in H^3\), \(\sigma \in S\). Since \(v'(2)^{\sigma} = v'(2)^f = x^{f(2)} \neq 0\), we conclude that \(\sigma = 1\). Furthermore, \(u(2)^{\sigma} = u(2)^f\), which implies that \(f(2) \in C_H(x) \cap C_H(y_2) = 1\). Note that \((f, \sigma) = (f, 1) \in H^3 \cap G = N\).

Let \(\rho = (12) \in S\). Since \(G = H^2 O_2(G)\), there exists \(g \in H^3\) such that \((g, \rho) \in O_2(G)\). Since \([N, O_2(G)] = 1\), \(f(g, \rho) = (g, \rho)(f, 1)\). It follows that \(f(1) = f(2^\rho) = f(2)^{\rho(2)} = 1\), and so \(f(1) = 1\). Consequently, \((f, \sigma) = (1, 1)\). We conclude that \(C_G(v') \cap C_G(u) = 1\). \(\square\)

The arguments needed for the induction step are collected in the following lemma.

**Lemma 16.** Let \(G\) be a supersolvable group and \(V\) be a faithful \(G\)-module over \(GF(2)\). Assume that there is a decomposition \(V = V_1 \oplus \cdots \oplus V_m\) \((m \geq 1)\) as a direct sum of subspaces which are permuted transitively by \(G\). Let \(K = N_G(V_1)/C_G(V_1)\), then \(V_1\) can be regarded as a faithful \(K\)-module. Then we have:

1. If \(K\) has at least four regular orbits on \(V_1 \oplus V_1\), then \(G\) has at least four regular orbits on \(V \oplus V\).

2. If \(K\) is of even order and \(K\) has at least three regular orbits on \(V_1 \oplus V_1\), then \(G\) has at least three regular orbits on \(V \oplus V\).

3. If the \(K\)-module \(V_1\) satisfies Property I and \(G\) is of odd order, then \(G\) has at least four regular orbits on \(V \oplus V\) or the \(G\)-module \(V\) satisfies Property I.

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4. If the $K$-module $V_1$ satisfies Property II, then either $G$ has three regular orbits on $V \oplus V$ or the $G$-module $V$ satisfies Property II.

5. If the $K$-module $V_1$ satisfies Property I, then either $G$ has three regular orbits on $V \oplus V$ or the $G$-module $V$ satisfies Property I or Property II.

Proof. We argue by induction on $m$. Clearly Statements 1, 5 hold when $m = 1$. Now we assume that $m \geq 2$. Since $G$ acts transitively on $\{V_1, \ldots, V_m\}$, we can take a block $\Delta$ of $\{V_1, \ldots, V_m\}$ such that $\text{Stab}_G(\Delta)$ is maximal in $G$. Without loss of generality, we may assume that $\Delta = \{V_1, \ldots, V_t\}$ with $t \geq 1$.

Let $W = \sum_{i=1}^t V_i$ and $L = N_G(W)$. Then $L = \text{Stab}_G(\Delta)$ is maximal in $G$. Assume that $\{g_1, g_2, \ldots, g_t\}$, where $g_1 = 1$, is a right transversal of $L$ in $G$ with $t = |G : L| \geq 2$. Note that $V = W_{g_1} \oplus \cdots \oplus W_{g_t}$ and the action of $G$ on $\{W_{g_1}, \ldots, W_{g_t}\}$ induces a homomorphism $\sigma : G \rightarrow S_\Omega$ such that $W_{g_i}g = W_{g_i\sigma(g)}$, where $\Omega = \{1, \ldots, m\}$. Write $S = \sigma(G)$ and $S$ acts faithfully and primitively on $\Omega$.

Let $H = L/C_G(W), G = H \wr S$. By Lemma 4 there exists a monomorphism $\tau : G \rightarrow \hat{G}$ such that:

1. The actions of $G$ on $V$ and $\tau(G)$ on $W^\Omega$ are equivalent.

2. $\hat{G} = H^2\tau(G)$.

3. Write $W_i = \{f \in W^\Omega \mid f(j) = 0, \forall j \neq i\}$ for each $i \in \Omega$. Then

$$N_{\tau(G)}(W_i)/C_{\tau(G)}(W_i) \cong H, \forall i \in \Omega.$$

It is easy to check that $(\hat{G}, \tau(G), H, S, \Omega, W^\Omega, W)$ satisfies Condition B. Since the action of $G$ on $V$ and the action of $\tau(G)$ on $W^\Omega$ are equivalent, without loss of generality, we may assume that $G = \tau(G)$, $V = W^\Omega$ and $(\hat{G}, G, H, S, \Omega, V, W)$ satisfies Condition B.

Write $N = H^2 \cap G$ and $W_i = \{f \in V \mid f(j) = 0, \forall j \neq i\}$ for each $i \in \Omega$. It is easy to see that $N = \bigcap_i N_G(W_i)$, moreover, $S \cong \hat{G}/H^2 \cong G/N$ is supersoluble. Thus $t$ is a prime.

Recall that $W = V_1 \oplus \cdots \oplus V_s$ is a faithful $H$-module and $\Delta = \{V_1, \ldots, V_s\}$ is a block of the action of $G$ on $\{V_1, \ldots, V_m\}$. It follows from Theorem 1.13 that $L$ (and also $H$) acts transitively on $\Delta = \{V_1, \ldots, V_s\}$. Write $J = N_H(V_1)/C_H(V_1)$ and $J_0 = N_L(V_1)C_G(V_1)/C_G(V_1) \leq K$. It is not difficult to see that the action of $J$ on $V_1$ is equivalent to the action of $J_0$ on $V_1$.

Now we will prove Statements 1, 5. Our strategy is first to apply induction on $(W, H, V_1, J)$ and then to calculate the number of regular orbits by Theorem 7.
1. By hypothesis, \( J_0 \leq K \) has at least four regular orbits on \( V_1 \oplus V_1 \). Thus \( J \) has at least four regular orbits on \( V_1 \oplus V_1 \). Since \( s = m/t < m \), by induction, \( H \) has at least four regular orbits on \( W \oplus W \).

Suppose that \( S \) has a regular orbit on the power set of \( \Omega \). Then \( |\Pi_2(\Omega, S)| \geq |S|/2 \). Consequently, in this case, \( \widehat{G} = H \wr S \) has at least four regular orbits on \( V \oplus V \) by Theorem 7, and so does \( G \). Therefore we may assume that \( S \) has no regular orbit on \( \mathcal{P}(\Omega) \) and so \( S \) is one of the exceptional cases of \([10, \text{Theorem } 5.6]\) and \( 3 \leq t \leq 9 \). By \([13, \text{Theorem } 3.1 \text{ (iii)}]\), we have that \( |\Pi_3(\Omega, S)| \geq |S| \) for \( 5 \leq t \leq 9 \), which implies that \( G \leq H \wr S \) has at least four regular orbits on \( V \oplus V \) by Theorem 7. Thus we may assume that \( t = 3 \) since \( t \) is a prime. In this case, \( S \cong S_3 \). It is not difficult to calculate that \( |\Pi_2(\Omega, S)| = 0 \) and \( |\Pi_3(\Omega, S)| = 1 \). Thus \( G \), as a subgroup of \( \widehat{G} \), has at least four regular orbits on \( V \oplus V \). Thus Statement 1 is proved.

2. If \( J \) is of odd order, then so is \( J_0 \). Since \( K \) is of even order, \( |K : J_0| \geq 2 \). Thus \( J_0 \) (and also \( J \)) has at least six regular orbits on \( V_1 \oplus V_1 \). Applying Statement 1 on \( (W, H, V_1, J) \), we conclude that \( H \) has at least four regular orbits on \( W_1 \oplus W_1 \). Applying Statement 1 on \((V, G, W, H)\) again, we obtain that \( G \) has at least four regular orbits on \( V \oplus V \), as desired.

Now we assume that \( J \) is of even order. By induction, \( H \) has at least three regular orbits on \( W \oplus W \). By \([13, \text{Proposition } 3.2 \text{ (2)}]\) and Theorem 7, we may assume that \( t \leq 4 \) and \( S \) has no regular orbit on \( \mathcal{P}(\Omega) \). Note that \( t \) is a prime. Thus, by \([10, \text{Theorem } 5.6]\), we conclude that \( |\Omega| = 3 \) and \( S \cong S_3 \). In this case, \( |\Pi_2(\Omega, S)| = 0 \) and \( |\Pi_3(\Omega, S)| = 1 \). In particular, \( \widehat{G} \) has at least one regular orbit on \( V \oplus V \).

Observe that \( H \) is of even order since \( J \) is of even order. Then \( \widehat{G} \) has a subgroup isomorphic to \( C_2 \wr S_3 \) and so \( \widehat{G} \) is not supersoluble. Thus we have that \( G \) is a proper subgroup of \( \widehat{G} \). Suppose that \( |\widehat{G} : G| = 2 \). Then \( G \triangleleft \widehat{G} \) and \( B = H^2 \) is not contained in \( G \). Recall that \( N = B \cap G \). Then \( N \) is normal in \( \widehat{G} \) and \( |B : N| = 2 \). In particular, there exists a direct factor \( H_1 \cong H \) of \( B \) which is not contained in \( N \). Then \( B = H_1 N \) and \( |H_1 : H_1 \cap N| = 2 \). Note that \( C = (H_1 \cap N)^2 \) is a normal subgroup of \( \widehat{G} \) contained in \( B \) such that \( \widehat{G}/C \cong C_2 \wr S_3 \). Thus there exists a normal subgroup \( X \) of \( \widehat{G} \) contained in \( B \) such that \( \widehat{G}/X \cong S_4 \) and clearly \( |B : X| = 2^2 \). If \( X \leq G \), we have that \( X \leq N \) and \( |N : X| = 2 \). It implies that \( N/X \) is a normal subgroup with order \( 2 \) of \( G/X \cong S_4 \), which is impossible. Therefore \( \widehat{G} = XG \) and \( G/G \cap X \cong \widehat{G}/X \cong S_4 \),
contrary to assumption. Consequently, \(|\hat{G} : G| \geq 3\) and so \(G\) has at least three regular orbits on \(V \oplus V\). Thus the conclusion is proved.

3. Since the \(K\)-module \(V_1\) satisfies Property I, \(K\) has at least two regular orbits on \(V_1 \oplus V_1\). If \(J_0\) is a proper subgroup of \(K\), then \(J_0\) has at least four regular orbits on \(V_1 \oplus V_1\) and so does \(J\). Applying Statement 1 twice, we obtain that \(G\) has at least four regular orbits on \(V \oplus V\).

Then we may assume \(J_0 = K\). Consequently the \(J_0\)-module \(V_1\) (and also \(V_1\) as a \(J\)-module) satisfies Property I. By induction, \(H\) has at least four regular orbits on \(W \oplus W\) or the \(H\)-module \(W\) satisfies Property I. If \(H\) has at least four regular orbits on \(W \oplus W\), then, by Statement 1, \(G\) has at least four regular orbits on \(V \oplus V\), as desired.

Now we assume that the \(H\)-module \(W\) satisfies Property I. By hypothesis, we have that \(O_3(H) = 1\). Moreover, there exists \(0 \neq x \in W\) such that \(C_H(x)\) has at least four different orbits on \(W\) with representatives \(y_1, y_2, z_1, z_2\) satisfying that \(C_H(x) \cap C_H(y_i) = 1\) and \(C_H(x) \cap C_H(z_i)\) is a 3-group for each \(i\).

Since \(G\) is of odd order, we have that \(S\) is of odd order. Consequently \(t\) is an odd prime and \(t \geq 3\). By \([10]\) Theorem 5.6, \(S\) has a strongly regular orbit on \(P(\Omega)\). We may assume that \(\Delta \subseteq \Omega\) satisfies that \(\text{Stab}_S(\Delta) = 1\) and \(|\Delta| \neq |\Omega \setminus \Delta|\). Take \(v \in V = W_1^\Omega\) such that \(v(i) = x\) for each \(i \in \Omega\) and define \(u_j, 1 \leq j \leq 4\), as follows:

\[
\begin{align*}
  u_1(i) &= y_1, \quad i \in \Delta; \\
  u_2(i) &= y_2, \quad i \in \Delta; \\
  u_3(i) &= y_1, \quad i \in \Delta; \\
  u_4(i) &= y_2, \quad i \in \Delta; \\
  u_1(i) &= y_2, \quad i \in \Omega \setminus \Delta; \\
  u_2(i) &= y_1, \quad i \in \Omega \setminus \Delta; \\
  u_3(i) &= z_1, \quad i \in \Omega \setminus \Delta; \\
  u_4(i) &= z_2, \quad i \in \Omega \setminus \Delta.
\end{align*}
\]

It is not difficult to find that \(u_j, 1 \leq j \leq 4\), lie in different orbits of \(C_G(v)\) on \(V\). By Lemma \([15] (3)\), \((v, u_j), 1 \leq j \leq 4\), generate four different regular orbits of \(G\) on \(V \oplus V\). Thus the conclusion is proved.

4. Since the \(K\)-module \(V_1\) satisfies Property II, we may assume that

(a) \(K\) is an even order group with \(O_2(K) = 1\), and

(b) there exist \(0 \neq x' \in V_1\) and three different \(C_K(x')\)-orbits with representatives \(y', z'_1, z'_2\) satisfying that \(C_K(x') \cap C_K(y') = 1\) and \(C_K(x') \cap C_K(z'_i)\) is a 2-group for each \(i\).

If \(J_0\) is of odd order, then \(J_0\) is proper in \(K\). Then \(J_0\) has at least two regular orbits on \(V \oplus V\) and \(C_{J_0}(x') \cap C_{J_0}(z'_i)\) is a 2-group for each \(i\),
which implies that \( J_0 \) has at least four regular orbits on \( V_1 \oplus V_1 \) and so does \( J \). Applying Statement 3 twice, we see that \( G \) has at least four regular orbits on \( V \oplus V \).

Thus we may assume that \( J_0 \) is of even order. Suppose that \( |K : J_0| \geq 3 \). Then \( J_0 \) (also \( J \)) has at least three regular orbits on \( V_1 \oplus V_1 \). It follows from Statement 2 that \( H \) has at least three regular orbits on \( W \oplus W \). Observe that \( |H| \) is even since \( |J| \) is even. Applying Statement 2 again, we conclude that \( G \) has at least three regular orbits on \( V \oplus V \).

Now we may assume that \( |K : J_0| \leq 2 \). Consequently \( J_0 \triangleleft K \) and \( O_2(J_0) \leq O_2(K) = 1 \). Then \( V_1 \), as a \( J \)-module (and so as a \( J_0 \)-module), satisfies **Property II**.

By induction, \( H \) has at least three regular orbits on \( W \oplus W \) or the \( H \)-module \( W \) satisfies **Property II**. Suppose that \( H \) has at least three regular orbits on \( W \oplus W \). Since \( |H| \) is even, \( G \) has at least three regular orbits on \( V \oplus V \) by Statement 2 as desired.

Now we assume that the \( H \)-module \( W \) satisfies **Property II**, that is:

(a) \( H \) is an even order group with \( O_2(H) = 1 \).

(b) There exist \( 0 \neq x \in W \) and three different \( C_H(x) \)-orbits with representatives \( y, z_1, z_2 \) satisfying that \( C_H(x) \cap C_H(y) = 1 \) and \( C_H(x) \cap C_H(z_i) \) is a 2-group for each \( i \).

First we consider the case \( |\Omega| = t \geq 5 \). By Lemma 9, \( \text{Stab}_S(1) \cap \text{Stab}_S(2) = 1 \). Let us take \( v \in V = W^\Omega \) such that \( v(i) = x \) for each \( i \in \Omega \). Consider the elements \( u_j \in V \), with \( 1 \leq j \leq 3 \), defined by

\[
\begin{align*}
    u_1(1) &= y; & u_1(2) &= z_2; & u_1(i) &= z_1, & i \in \Omega \setminus \{1, 2\}; \\
    u_2(1) &= z_1; & u_2(2) &= y; & u_2(i) &= z_2, & i \in \Omega \setminus \{1, 2\}; \\
    u_3(1) &= z_2; & u_3(2) &= z_1; & u_3(i) &= y, & i \in \Omega \setminus \{1, 2\}.
\end{align*}
\]

Since \( y, z_1, z_2 \) lie in different orbits of \( C_H(x) \) on \( W_1 \), it is not difficult to conclude that \( u_1, u_2 \) and \( u_3 \) lie in different orbits of \( C_G(v) \) on \( V \). By Lemma 9, we have that \( (v, u_j), 1 \leq j \leq 3 \), generate three different regular orbits of \( G \) on \( V \oplus V \), as desired.

Recall that \( |\Omega| = t \) is a prime. Thus we only have to consider the cases \( t = 2 \) or \( t = 3 \).

Assume that \( t = 3 \). In this case, \( S = S_3 \) or \( S = \langle (123) \rangle \). Take \( v \in V = W^\Omega \) such that \( v(i) = x \) for each \( i \in \Omega \). Consider the elements
\(u_j \in V\), where \(1 \leq j \leq 3\), defined by

\[
\begin{align*}
    u_1(1) &= y, & u_1(2) &= z_1, & u_1(3) &= z_2; \\
    u_2(1) &= y, & u_2(2) &= y, & u_2(3) &= z_1; \\
    u_3(1) &= y, & u_3(2) &= y, & u_3(3) &= z_2.
\end{align*}
\]

It is clear that \(u_1, u_2\) and \(u_3\) belong to different orbits of \(C_G(v)\) on \(V\).

By Lemma 15 (3), \(C_G(v) \cap C_G(u_1) = 1\). By Lemma 15 (4), we have that \(C_G(v) \cap C_G(u_j)\) is 2-group for \(j \in \{2, 3\}\).

As \(O_2(H) = 1\), by Lemma 15 (1), \(O_v(N) = 1\). Observe that \(O_2(G/N) \cong O_2(S) = 1\) and consequently \(O_2(G) \leq O_2(N) = 1\). Furthermore, \(G\) is of even order since \(H\) is of even order. Thus the \(G\)-module \(V\) satisfies \textbf{Property II}, as desired.

Finally we assume that \(|\Omega| = 2\) and \(S \cong S_2\). Take \(v \in V\) such that \(v(i) = x\) for each \(i \in \Omega\) and consider the elements \(u_1, u_2, u_3 \in V\) defined by

\[
\begin{align*}
    u_1(1) &= z_1, & u_1(2) &= y; \\
    u_2(1) &= z_2, & u_2(2) &= y; \\
    u_3(1) &= z_1, & u_3(2) &= z_2.
\end{align*}
\]

We have that \(u_1, u_2\) and \(u_3\) belong to different orbits of \(C_G(v)\) on \(V\) and, by Lemma 15 (3), \(C_G(v) \cap C_G(u_j) = 1\) for \(j \in \{1, 2\}\) and \(C_G(v) \cap C_G(u_3)\) is 2-group.

Assume first that \(O_2(G) = 1\). Then, since \(G\) is of even order, we can conclude that the \(G\)-module \(V\) satisfies \textbf{Property II}, as desired. Now we assume that \(O_2(G) \neq 1\). By Lemma 15 (6), if we take \(v' \in V\) such that \(v'(1) = 0\) and \(v'(2) = x\), then \(C_G(v') \cap C_G(u_1) = 1\). We observe that \((v, u_1), (v, u_2)\) and \((v', u_1)\) lie in different regular orbits of \(G\) on \(V \oplus V\), as desired. Thus the conclusion 4 is completely proved.

5. Since the \(K\)-module \(V_1\) satisfies \textbf{Property I}, \(K\) has at least two regular orbits on \(V_1 \oplus V_1\). If \(J_0\) is proper in \(K\), then \(J_0\) has at least four regular orbits on \(V_1 \oplus V_1\) and so does \(J\). By Statement 4, \(H\) has at least four regular orbits on \(W_1 \oplus W_1\). Applying Statement 4 again, we obtain that \(G\) has at least four regular orbits on \(V \oplus V\). Thus we may assume \(J_0 = K\). Consequently \(V_1\) as a \(J\)-module, and so as a \(J_0\)-module, satisfies \textbf{Property I}.

When \(H\) is of even order, by induction, \(H\) has at least three regular orbits on \(W \oplus W\) or the \(H\)-module \(W_1\) satisfies \textbf{Property I} or \textbf{Property II}. Since \(H\) is of even order, clearly the \(H\)-module \(W\) does not
satisfy Property I. If $H$ has at least three regular orbits on $W \oplus W$, then it follows from Statement 2 that $G$ has at least three regular orbits on $V \oplus V$, as desired. If the $H$-module $W$ satisfies Property II, then we can conclude by Statement 4 that $G$ has at least three regular orbits on $V \oplus V$ or the $G$-module $V$ satisfies Property II, as desired. When $H$ is of odd order, applying Statement 3 on $(W, H, V_1, J)$, we can conclude that the $H$-module $W$ satisfies Property I or $H$ has at least four regular orbits on $W \oplus W$. If the latter case holds, then it follows from Statement 1 that $G$ has at least four regular orbits on $V \oplus V$, as desired.

Thus we can suppose that the $H$-module $W$ satisfies Property I. Then we have:

(a) $H$ is an odd order group and $O_3(H) = 1$.

(b) There exist $0 \neq x \in W$ and four different $C_H(x)$-orbits with representatives $y_1, y_2, z_1, z_2$ satisfying that $C_H(x) \cap C_H(y_i) = 1$ and $C_H(x) \cap C_H(z_i)$ is a 3-group for each $i$.

First we consider the case $|\Omega| = t \geq 3$. By Lemma 9, $\text{Stab}_S(1) \cap \text{Stab}_S(2) = 1$.

Take $v \in V = W^\Omega$ such that $v(i) = x$ for each $i \in \Omega$. Consider the elements $u_j \in V$, where $1 \leq j \leq 3$, defined by

$$
\begin{align*}
  u_1(1) &= y_1; & u_1(2) &= y_2; & u_1(i) &= z_1, & i \in \Omega \setminus \{1, 2\}; \\
  u_2(1) &= y_1; & u_2(2) &= y_2; & u_2(i) &= z_2, & i \in \Omega \setminus \{1, 2\}; \\
  u_3(1) &= y_1; & u_3(2) &= z_1; & u_3(i) &= z_2, & i \in \Omega \setminus \{1, 2\}.
\end{align*}
$$

Since $y_1, y_2, z_1$ and $z_2$ lie in different orbits of $C_H(x)$ on $W$, it follows that $u_1, u_2$ and $u_3$ lie in different orbits of $C_G(v)$ on $V$. By Lemma 15 (3), we have that $C_G(v) \cap C_G(u_j) = 1$ for $1 \leq j \leq 3$. Thus $G$ has at least three regular orbits on $V \oplus V$, as desired.

Now we assume that $|\Omega| = 2$ and $S \cong S_2$. Let $v \in V$ such that $v(i) = x$ for each $i \in \Omega$ and consider the elements $u_1, u_2, u_3 \in V$ defined by

$$
\begin{align*}
  u_1(1) &= y_1, & u_1(2) &= y_2; \\
  u_2(1) &= y_1, & u_2(2) &= y_1; \\
  u_3(1) &= y_2, & u_3(2) &= y_2.
\end{align*}
$$

Clearly $u_j, 1 \leq j \leq 3$ lie in different orbits of $C_G(v)$ on $V$. By Lemma 15 (3), $C_G(v) \cap C_G(u_1) = 1$. By Lemma 15 (5), $C_G(v) \cap C_G(u_j)$ are 2-groups for $j \in \{2, 3\}$. 

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Assume first that $O_2(G) = 1$. Then, since $G/N \cong S_2$, $G$ has even order and we conclude that the $G$-module $V$ satisfies Property II, as desired.

Now we assume that $O_2(G) \neq 1$. By Lemma [15][16], if we take $v' \in V$ such that $v'(1) = x$ and $v'(2) = 0$ and define $u'_j \in V$, $1 \leq j \leq 2$ as follows:

$$u'_1(1) = y_1, \quad u'_1(2) = z_1; \quad u'_2(1) = y_1, \quad u'_2(2) = z_2.$$ 

We have that $C_G(v') \cap C_G(u'_j) = 1$, $1 \leq j \leq 2$. We also observe that $(v, u_1), (v', u'_1)$ and $(v', u'_2)$ lie in different regular orbits of $G$ on $V \oplus V$, as desired. Thus the conclusion [5] is completely proved. □

4 Proof of the main theorems

**Theorem 17.** Let $G$ be a soluble group and let $V$ be an irreducible and faithful $G$-module over $GF(2)$. If $H$ is an odd order supersoluble subgroup of $G$, then $H$ has at least four regular orbits on $V \oplus V$ or the $H$-module $V$ satisfies Property I.

**Proof.** We argue by induction on $|G|$. By Lemma [12] if $V$ is primitive, then $H$ has four regular orbits on $V \oplus V$ or $|V| = 2^2, H = \Gamma(V) \cong [C_7]C_3$. In the latter case, Property I holds, as desired. Now we may assume that $V$ is an imprimitive $G$-module. Assume that $V = V_1 \oplus \cdots \oplus V_m$ ($m \geq 2$) is a direct sum of subspaces which are permuted transitively by $G$. If we do this so that $m$ is as small as possible, then we can assume that $L = N_G(V_1)$ is maximal in $G$, and we observe also that $L$ acts irreducibly on $V_1$. Write $U = L/C_G(V_1)$ and $V_1$ is a faithful and irreducible $U$-module.

Assume that $\Omega_1, \ldots, \Omega_s$ ($s \geq 1$) are all the $H$-orbits in $\{V_1, \ldots, V_m\}$. Set $W_j = \sum_{W \in \Omega_j} W$. First we claim that $H/C_H(W_j)$ has at least four regular orbits on $W_j \oplus W_j$ or the $H/C_H(W_j)$-module $W_j$ satisfies Property I for each $j$.

We can assume without loss of generality $j = 1$ and $\Omega_1 = \{V_1, \ldots, V_t\}$, where $t = |H : L \cap H|$. Write $W = W_1$, $K = H/C_H(W_1)$ and $J = N_K(V_1)/C_K(V_1)$.

Now we claim that $K$ has at least four regular orbits on $W \oplus W$ or the $K$-module $W$ satisfies Property I. Observe that the action of $J$ on $V_1$ is equivalent to the action of $A := (L \cap H)C_G(V_1)/C_G(V_1) \leq U$ on $V_1$. Then the triple $(U, A, V_1)$ satisfies the hypotheses of the theorem. By induction, $A$
(and so $J$) has at least four regular orbits on $V_1 \oplus V_1$ or the $A$-module $V_1$ (and so the $J$-module $V_1$) satisfies Property I. If $J$ has at least four regular orbits on $V_1 \oplus V_1$, then it follows from Lemma 16(1) that $K$ has at least four regular orbits on $W \oplus W$, as claimed. If the $J$-module $V_1$ satisfies Property I, since $|H|$ is odd, then it follows from Lemma 16(3) that $K$ has at least four regular orbits on $W \oplus W$ or the $K$-module $W$ satisfies Property I, as claimed.

Thus $H/C_H(W_j)$ has at least two regular orbits on $W_j \oplus W_j$ for each $1 \leq j \leq s$. If $s \geq 2$, then $H$ has at least four regular orbits on $V \oplus V$ by Lemma 8 as desired. Now we may assume that $s = 1$, that is, $H$ acts transitively on $\{V_1, \ldots, V_m\}$. Thus $H = K$ and $W = V$, and consequently $H$ has at least four regular orbits on $V \oplus V$ or the $H$-module $V$ satisfies Property I. The theorem is proved.

**Theorem 18.** Let $G$ be a soluble group and $V$ be an irreducible and faithful $G$-module over $GF(2)$. If $H$ is a supersoluble subgroup of $G$, then either $H$ has at least three regular orbits on $V \oplus V$ or $V$, as an $H$-module, satisfies Property I or Property II.

**Proof.** Work by induction on $|GV|$. If $V$ is a primitive $G$-module, it follows from Lemma 11 that either $H$ has at least three regular orbits on $V \oplus V$ or the $H$-module $V$ satisfies:

1. $|V| = 2^2$ and $H = \Gamma(V) \cong S_3$, or
2. $|V| = 2^3$ and $H = \Gamma(V) \cong [C_7]C_3$.

It is not difficult to find that, in the first case, $V$ satisfies Property II and in the second case, $V$ satisfies Property I, as desired. Consequently, we assume that $V$ is an imprimitive $G$-module. Then there $V = V_1 \oplus \cdots \oplus V_m$, $(m \geq 2)$ is a direct sum of subspaces which are permuted transitively by $G$. If we do this so that $m$ is as small as possible, then we can assume that $L = N_G(V_1)$ is maximal in $G$, and we observe also that $L$ acts irreducibly on $V_1$. Write $U = L/C_G(V_1)$ and $V_1$ is a faithful, irreducible $U$-module.

Assume that $\Omega_1, \ldots, \Omega_s$ ($s \geq 1$) are all the $H$-orbits in $\{V_1, \ldots, V_m\}$. Set $W_j = \sum_{w \in \Omega_j} W$.

First we claim that $H/C_H(W_j)$ has at least three regular orbits on $W_j \oplus W_j$ or the $H/C_H(W_j)$-module $W_j$ satisfies Property I or Property II for each $j$.

Without loss of generality, we may suppose $j = 1$ and $\Omega_1 = \{V_1, \ldots, V_t\}$, where $t = |H : L \cap H|$. Write $W = W_1$, $K = H/C_H(W_1)$ and $J = N_K(V_1)/C_K(V_1)$. Then $W$ is a faithful $H$-module. Observe that the action of $J$ on $V_1$ is equivalent to the action of $A := (L \cap H) C_G(V_1)/C_G(V_1) \leq U$ on $V_1$. Then the triple $(U, A, V_1)$ satisfies the hypotheses of the theorem. By
induction, either \( A \) (and also \( J \)) has at least three regular orbits on \( V_1 \oplus V_1 \) or \( V_1 \) regarded as an \( A \)-module (and also as a \( J \)-module) satisfies Property I or Property II.

If the \( J \)-module \( V_1 \) satisfies Property I, then our claim follows from Lemma 16 (5). If the \( J \)-module \( V_1 \) satisfies Property II, then our claim follows from Lemma 16 (4). Now we assume that \( J \) has at least three regular orbits on \( V_1 \oplus V_1 \). If \( J \) is of even order, then \( K \) has at least three regular orbits on \( W \oplus W \) by Lemma 16 (2). If \( J \) is of odd order, then \( A \) is of odd order and the triple \( (U, A, V_1) \) satisfies the hypotheses of Theorem 17. Thus \( A \) (and also \( J \)) has at least four regular orbits on \( V_1 \oplus V_1 \) or \( V_1 \), regarded as an \( A \)-module (also as a \( J \)-module) satisfies Property I. If \( J \) has at least four regular orbits on \( V_1 \oplus V_1 \), then \( K \) has at least four regular orbits on \( W \oplus W \) by Lemma 16 (1), as claimed. If the \( J \)-module \( V_1 \) satisfies Property I, then, by Lemma 16 (5) again, our claim holds.

Now we have proven that \( H/C_H(W_j) \) has at least three regular orbits on \( W_j \oplus W_j \) or the \( H/C_H(W_j) \)-module \( W_j \) satisfies Property I or Property II for each \( 1 \leq j \leq s \). In particular, \( H/C_H(W_j) \) has at least one regular orbit on \( W_j \oplus W_j \) for each \( 1 \leq j \leq s \). If there exists some \( j \in \{1, \ldots, s\} \) such that \( H/C_H(W_j) \) has at least three regular orbits on \( W_j \oplus W_j \), then we can conclude that \( H \) has at least three regular orbits on \( V \oplus V \) by Lemma 8, as desired.

Now we can assume that the \( H/C_H(W_j) \)-module \( W_j \) satisfies Property I or Property II for each \( 1 \leq j \leq s \). Thus if \( s = 1 \), then \( V \), as an \( H \)-module, satisfies Property I or Property II, as desired. Consequently, we can assume \( s \geq 2 \).

Take

\[ C = \{1 \leq j \leq s \mid \text{the } H/C_H(W_j) \text{ module } W_j \text{ satisfies Property II}\} \]

First we assume that \( C = \emptyset \). Then the \( H/C_H(W_j) \)-module \( W_j \) satisfies Property I for each \( 1 \leq j \leq s \). It implies that \( H/C_H(W_j) \) has at least two regular orbits on \( W_j \oplus W_j \). Since \( s \geq 2 \), then we can conclude that \( H \) has at least four regular orbits on \( V \oplus V \) by Lemma 8, as desired.

Now we assume that \( C \neq \emptyset \), then, without loss of generality, we may assume that \( C = \{1, \ldots, l\} \) for some \( 1 \leq l \leq s \).

Write \( K_j = H/C_H(W_j) \). For \( j = 1 \), we have

1. \( K_1 \) is an even order group and \( O_2(K_1) = 1 \).
2. There exists \( 0 \neq x_1 \in W_1 \) such that \( C_{K_1}(x_1) \) has three different orbits on \( V_1 \) with representatives \( y_1, z_1, z_2 \) such that \( C_{K_1}(x_1) \cap C_{K_1}(y_1) = 1 \) and \( C_{K_1}(x_1) \cap C_{K_1}(z_i) \) is a 2-group for \( i = 1, 2 \).
Recall that $K_j$ has at least one regular orbit on $V_j \oplus V_j$ for each $2 \leq j \leq s$. We can assume that $C_{K_j}(x_j) \cap C_{K_j}(y_j) = 1$ for some $x_j, y_j \in V_j$.

Thus we can conclude that $C_H(x_j) \cap C_H(y_j) \subseteq C_H(W_j)$ for each $1 \leq j \leq s$ and $X_i/C_H(W_1)$ is a 2-group, where $X_i = C_H(x_i) \cap C_H(z_i)$ for $i = 1, 2$.

Write $v = \sum_{i=1}^s x_i$, $u = \sum_{i=1}^s y_i$, $w_1 = z_1 + \sum_{i=2}^s y_i$, and $w_2 = z_1 + \sum_{i=2}^s y_i$. It is not difficult to find that $u, w_1, w_2$ lie in different orbits of $C_H(v)$ on $V$.

Moreover, we have

$$C_H(v) \cap C_H(u) = \bigcap_{j=1}^s (C_H(x_j) \cap C_H(y_j)) \subseteq \bigcap_{j=1}^s C_H(W_j) = 1$$

and

$$C_H(v) \cap C_H(w_i) \subseteq X_i \cap \bigcap_{j=2}^s C_H(W_j) \cong (X_i \cap \bigcap_{j=2}^s C_H(W_j))C_H(W_1)/C_H(W_1)$$

is a 2-group for $i = 1, 2$.

On the other hand, $H$ is of even order since $H/C_H(W_j)$ is of even order. Moreover, for each $j \in \mathcal{C}$, we have that $H/C_H(W_j)$ is an even order group and $O_2(H/C_H(W_j)) = 1$, and for each $j \in \{1, \ldots, s\} \setminus \mathcal{C}$, we have that $H/C_H(W_j)$ is an odd order group. Thus $O_2(H) \leq \bigcap_{i=1}^s C_H(W_j) = 1$. Hence the $H$-module $V$ satisfies Property II, as desired. Thus the theorem is completely proved.

**Proof of Theorem A.** Assume that the theorem is false and let $(G, H, V)$ be the counterexample such that $|G| + |H| + |V|$ minimal. First we claim that $V$ is an irreducible $G$-module. Assume that this is false. Let $V = V_1 \oplus V_2$, where $0 \neq V_i$ is a $G$-module for $i \in \{1, 2\}$. Then $V_i$ is a faithful, completely reducible $G/C_G(V_i)$-module for $i \in \{1, 2\}$. Observe that $HC_G(V_i)/C_G(V_i)$ satisfies the hypotheses for $i \in \{1, 2\}$. Hence, by the choice of $(G, H, V)$, $HC_G(V_i)/C_G(V_i)$ has at least one regular orbit on $V_i \oplus V_i$ for $i \in \{1, 2\}$. Thus $H$ has at least one regular orbit on $V \oplus V$, against the choice of $(G, H, V)$. This contradiction shows that $V$ is an irreducible $G$-module over a field of characteristic $p$ for some prime $p$. Then $V$ is a completely reducible $G$-module over the field $GF(p)$ of $p$ elements.

Arguing as in the previous paragraph, we may assume that $V$ is an irreducible, faithful $G$-module over $GF(p)$. If $p$ is odd, then it follows from Lemma [12] Corollary 3] that $H$ has at least two regular orbits on $V \oplus V$. Thus we may assume that $p = 2$. It follows from Theorem [15] that $H$ has at least three regular orbits on $V \oplus V$, or the $H$-module $V$ satisfies Property I or Property II. In all these cases, we can conclude that $H$ has at least one regular orbit on $V \oplus V$ and the main theorem is completely proved.  

\[\square\]
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