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Interpolative Reich–Rus–Ćirić and Hardy–Rogers Contraction on Quasi-Partial b-Metric Space and Related Fixed Point Results

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Abstract: The aim of this paper was to obtain common fixed point results by using an interpolative contraction condition given by Karapinar in the setting of complete metric space. Here in this paper, we have redefined the Reich–Rus–Ćirić type contraction and Hardy–Rogers type contraction in the framework of quasi-partial b-metric space and proved the corresponding common fixed point theorem by adopting the notion of interpolation. The results are further validated with the application based on them.

Keywords: quasi-partial b-metric space; common fixed point; interpolation; Reich–Rus–Ćirić contraction; Hardy–Rogers contraction

MSC: 46T99; 47H10; 54H25

1. Introduction

In the year 1922, Banach [1] introduced one of the most prominent results called Banach contraction principle and its existence in metric fixed point theory i.e., Let J be a self map on a non-empty set X and d is a complete metric. If there exists a constant $\kappa \in [0,1)$ such that

$$d(J\mu, J\eta) \leq \kappa d(\mu, \eta) \quad \text{for all } \mu, \eta \in X,$$

then it possesses a unique fixed point in X. Due to the importance and application potential of the Banach contraction principle, this notion has been extended by several authors [2–4]. In 1994, Matthews [5] introduced the notion of partial-metric space as a part of the study of denotational semantics of dataflow networks. In 1968, the following contraction was proved by Kannan [6] i.e.,

$$d(J\mu, J\eta) \leq \rho[d(\mu, J\mu) + d(\eta, J\eta)] \quad \text{for all } \mu, \eta \in X,$$

where $\rho \in \left[0, \frac{1}{2}\right)$. In 2018, Karapinar [7] adopted the interpolative approach to define the generalized Kannan-type contraction on a complete metric space. We recall that a self-map $J : X \to X$ is said to be an interpolative Kannan type contraction for a metric space (X, d), if there are constants $\rho \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(J\mu, J\eta) \leq \rho[d(\mu, J\mu)]^{\alpha} \cdot [d(\eta, J\eta)]^{1-\alpha} \quad \text{for all } \mu, \eta \in X \setminus Fix(J)$$

where $Fix(J) = \{z \in X : Jz = z\}$.

In 1972, Reich [8] generalized the concepts of Kannan and Banach, e.g., a self map $J : X \to X$ is called a Reich-contraction mapping if there are $\alpha, \beta, \gamma \in [0, 1)$ and $\alpha + \beta + \gamma < 1$ such that

$$d(J\mu, J\eta) \leq \alpha d(\mu, J\mu) + \beta d(\eta, J\eta) + \gamma d(\mu, \eta) \quad \text{for all } \mu, \eta \in X.$$

Reich–Rus–Ćirić [9–15] independently proved the next theorem and its variants i.e., a self map $J : X \to X$ is said to be a Reich–Rus–Ćirić contraction map on a complete metric space (X, d) if there are $\rho \in \left[0, \frac{1}{3}\right)$ such that

$$d(J\mu, J\eta) \leq \rho[d(\mu, \eta) + d(\mu, J\mu) + d(\eta, J\eta)],$$

for all $\mu, \eta \in X$, then J possesses a unique fixed point. Very recently, Karapinar et al. [16,17] introduced the concept of interpolative Reich–Rus–Ćirić and Hardy–Rogers type contraction and proved the following fixed point results.

Theorem 1 ([16])**.** *In the notion of partial metric space (X, d), if a mapping $J : X \to X$ is an interpolative Reich–Rus–Ćirić type contraction, i.e., there are constants $\rho \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that $d(J\mu, J\eta) \leq \rho[d(\mu, \eta)]^{\beta}[d(\mu, J\mu)]^{\alpha} \cdot [d(\eta, J\eta)]^{1-\alpha-\beta}$ for all $\mu, \eta \in X \setminus Fix(J)$, then J owns a fixed point.*

Theorem 2 ([17])**.** *Let (X, d) be a metric space. If the self-mapping $J : X \to X$ is an interpolative Hardy–Rogers type contraction i.e., there exist $\rho \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that*

$$\begin{aligned} d(J\mu, J\eta) \leq\ & \rho[d(\mu, \eta)]^{\beta}[d(\mu, J\mu)]^{\alpha} \cdot d(\eta, J\eta)]^{\gamma} \\ & \cdot \left[\frac{1}{2}(d(\mu, J\eta) + d(\eta, J\mu))\right]^{1-\alpha-\beta-\gamma} \quad \textit{for all } \mu, \eta \in X \setminus Fix(J), \end{aligned}$$

then J possesses a fixed point of X.

In continuation, interesting work was done by many authors [18–27] which enriched this field.

The purpose of this paper was to revisit the approach of interpolative Reich–Rus–Ćirić and Hardy–Rogers type contractions to attain a common fixed point for quasi-partial b-metric spaces. Some examples are given to illustrate the new approach.

2. Preliminaries and Definitions

Definition 1 ([28])**.** *A quasi-partial b-metric on a non-empty set X is a function $qp_b : X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and all $\mu, \eta, \vartheta \in X$:*

(QPb$_1$) $qp_b(\mu, \mu) = qp_b(\mu, \eta) = qp_b(\eta, \eta)$ *implies* $\mu = \eta$,
(QPb$_2$) $qp_b(\mu, \mu) \leq qp_b(\mu, \eta)$,
(QPb$_3$) $qp_b(\mu, \mu) \leq qp_b(\eta, \mu)$,
(QPb$_4$) $qp_b(\mu, \eta) \leq s[qp_b(\mu, \vartheta) + qp_b(\vartheta, \eta)] - qp_b(\vartheta, \vartheta)$.

(X, qp_b) is called a quasi-partial b-metric space where X is a non-empty set and qp_b defines a quasi-partial b-metric on X. The number s is called the coefficient of (X, qp_b).

Let qp_b be a quasi-partial b-metric on the set X. Then

$$d_{qp_b}(\mu, \eta) = qp_b(\mu, \eta) + qp_b(\eta, \mu) - qp_b(\mu, \mu) - qp_b(\eta, \eta) \text{ is a b-metric on } X.$$

Let us see some new examples of quasi-partial b-metric space.

Example 1. *Let* $X = \left[0, \frac{\pi}{4k}\right]$. *Define the metric* $qp_b(\mu,\eta) = sink|\mu-\eta| + \mu$ *for any* $(\mu,\eta) \in X \times X$ *and* $k \geq 2$.

It can be shown here that (X, qp_b) *is a quasi-partial b-metric space. Actually, if* $qp_b(\mu,\mu) = qp_b(\mu,\eta) = qp_b(\eta,\eta)$, *that is,* $\mu = sink|\mu-\eta| + \mu = \eta$, *then it is obvious that* (QPb_1) *holds for any* $(\mu,\eta) \in X \times X$.

In addition, $sink|\mu-\eta| \geq 0$ *and* $sink|\mu-\eta| \geq |\mu-\eta|$ *when* $|\mu-\eta| \in \left[0, \frac{\pi}{4k}\right]$, *then* $qp_b(\mu,\mu) = \mu \leq sink|\mu-\eta| + \mu = qp_b(\mu,\eta)$

$$\begin{aligned} qp_b(\mu,\mu) &= \mu \\ &= |\mu-\eta+\eta| \\ &\leq |\mu-\eta| + |\eta| \\ &\leq sink|\eta-\mu| + \eta \\ &= qp_b(\eta,\mu) \text{ are true, hence } (QPb_2) \text{ and } (QPb_3) \text{ hold for any } (\mu,\eta) \in X \times X. \end{aligned}$$

Moreover, for any $\mu,\eta,\delta \in X$, $|\mu-\eta| \leq \frac{\pi}{4k} \leq \frac{\pi}{2k}$ *and* $[|\mu-\eta| + |\eta-\delta|] \leq \frac{\pi}{2k}$ *when* $k(|\mu-\delta| + |\delta-\mu|) \in \left[0, \frac{\pi}{2k}\right]$, *or* $k(|\mu-\delta| + |\delta-\mu|) \leq \frac{\pi}{2}$, *and since* $\sin\mu$ *is increasing on* $\left[0, \frac{\pi}{2}\right]$, *we get*

$$\begin{aligned} qp_b(\mu,\eta) + qp_b(\delta,\delta) &= sink|\mu-\eta| + \mu + \delta \\ &\leq sink(|\mu-\delta| + |\delta-\eta|) + \mu + \delta \\ &\leq k(|\mu-\delta| + |\delta-\eta|) + \mu + \delta \\ &\leq ksink|\mu-\delta| + ksink|\delta-\eta| + \mu + \delta \\ &= k(sink|\mu-\delta| + sink|\delta-\eta| + \mu + \delta) \\ &\leq s(qp_b(\mu,\delta) + qp_b(\delta,\eta)) \text{ for all } \mu,\eta,\delta \in X \end{aligned}$$

and $s \geq k$, (QPb_4) *holds, hence* (X, qp_b) *is a quasi-partial b-metric space with* $s \geq k$.

Lemma 1 ([29])**.** *Let* (X, qp_b) *be a quasi-partial b-metric space. Then the following hold:*

(a) *If* $qp_b(\mu,\eta) = 0$ *then* $\mu = \eta$.
(b) *If* $\mu \neq \eta$, *then* $qp_b(\mu,\eta) > 0$ *and* $qp_b(\eta,\mu) > 0$.

Definition 2 ([29])**.** *Let* (X, qp_b) *be a quasi-partial b-metric. Then*

(i) *A sequence* $\{\mu_n\} \subset X$ *converges to* $\mu \in X$ *if and only if*

$$qp_b(\mu,\mu) = \lim_{n\to\infty} qp_b(\mu,\mu_n) = \lim_{n\to\infty} qp_b(\mu_n,\mu).$$

(ii) *A sequence* $\{\mu_n\} \subset X$ *is called a Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} qp_b(\mu_n,\mu_m) \text{ and } \lim_{m,n\to\infty} qp_b(\mu_m,\mu_n) \text{ exist (and are finite).}$$

(iii) *The quasi partial b-metric space* (X, qp_b) *is said to be complete if every Cauchy sequence* $\{\mu_n\} \subset X$ *converges with respect to* τ_{qp_b} *to a point* $\mu \in X$ *such that*

$$qp_b(\mu,\mu) = \lim_{n,m\to\infty} qp_b(\mu_n,\mu_m) = \lim_{m,n\to\infty} qp_b(\mu_m,\mu_n).$$

(iv) *A mapping $f : X \to X$ is said to be continuous at $\mu_0 \in X$ if, for every $\varepsilon > 0$, there exist*

$$\delta > 0 \ \text{ such that } \ f(B(\mu_0, \delta)) \subset B(f(\mu_0), \varepsilon).$$

Lemma 2 ([29])**.** *Let (X, qp_b) be a quasi-partial b-metric space and (X, d_{qp_b}) be the corresponding b-metric space. Then (X, d_{qp_b}) is complete if (X, qp_b) is complete.*

Definition 3 ([30])**.** *Let (X, qp_b) be a quasi-partial b-metric space and $J : X \to X$ be a given mapping. Then J is said to be sequentially continuous at $z \in X$ if for each sequence $\{\mu_n\}$ in X converging to z, we have $J\mu_n \to Jz$, that is, $\lim_{n\to\infty} qp_b(J\mu_n, Jz) = qp_b(Jz, Jz)$. Similarly, let $S : X \to X$ be a given mapping. S is said to be sequentially continuous at $z \in X$ if for each sequence $\{\mu_n\}$ in X converging to z, we have $\lim_{n\to\infty} qp_b(Sz, S\mu_n) = qp_b(Sz, Sz)$. Then J is said to be sequentially continuous on X if J is sequentially continuous at each $z \in X$.*

3. Main Result

Let us discuss the main result.

Theorem 3. *Let (X, qp_b) be a complete quasi-partial b-metric space. Let $J, S : X \to X$ be self mappings. Assume that there are some $\rho \in [0, 1)$, $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$, and $s \geq 1$ such that the condition*

$$qp_b(J\mu, S\eta) \leq \rho[qp_b(\mu, \eta)]^\beta [qp_b(\mu, J\mu)]^\alpha \left[\frac{1}{s} qp_b(\eta, S\eta)\right]^{1-\alpha-\beta} \tag{1}$$

is satisfied for all $\mu, \eta \in X$ such that $J\mu \neq \mu$ whenever $S\eta \neq \eta$. Then S and J posses a common fixed point.

Proof. Let $\mu_0 \in X$. Define the sequence $\{\mu_n\}$ by $\mu_{2n+1} = J\mu_{2n}$, $\mu_{2n+2} = S\mu_{2n+1}$ for all $n = 0, 1, 2, \ldots$. If there exist $n \in 0, 1, 2, \ldots$ such that $\mu_n = \mu_{n+1} = \mu_{n+2}$ then μ_n is a common fixed point of S and J. Suppose that there are no three consecutive identical terms in the sequence μ_n and that $\mu_0 \neq \mu_1$. Now using (1), we deduce that

$$\begin{aligned}
qp_b(\mu_{2n+1}, \mu_{2n+2}) &= qp_b(J\mu_{2n}, S\mu_{2n+1}) \\
&\leq \rho[qp_b(\mu_{2n}, \mu_{2n+1})]^\beta \cdot [qp_b(\mu_{2n}, J\mu_{2n})]^\alpha \cdot \left[\frac{1}{s} qp_b(\mu_{2n+1}, S\mu_{2n+1})\right]^{1-\alpha-\beta} \\
&\leq \rho[qp_b(\mu_{2n}, \mu_{2n+1})]^\beta \cdot [qp_b(\mu_{2n}, \mu_{2n+1})]^\alpha \cdot \left[\frac{1}{s} qp_b(\mu_{2n+1}, \mu_{2n+2})\right]^{1-\alpha-\beta} \\
&\leq \rho[qp_b(\mu_{2n}, \mu_{2n+1})]^\beta \cdot [qp_b(\mu_{2n}, \mu_{2n+1})]^\alpha \cdot [qp_b(\mu_{2n+1}, \mu_{2n+2})]^{1-\alpha-\beta} \\
[qp_b(\mu_{2n+1}, \mu_{2n+2})]^{\alpha+\beta} &\leq \rho[qp_b(\mu_{2n}, \mu_{2n+1})]^{\alpha+\beta}
\end{aligned}$$

or

$$[qp_b(\mu_{2n+1}, \mu_{2n+2}) \leq \rho[qp_b(\mu_{2n}, \mu_{2n+1})]$$

Hence,

$$qp_b(\mu_{2n+1}, \mu_{2n+2}) \leq \rho qp_b(\mu_{2n}, \mu_{2n+1}) \leq \rho^2 qp_b(\mu_{2n-1}, \mu_{2n}).$$

Thus,

$$qp_b(\mu_{2n+1}, \mu_{2n+2}) \leq \rho^{2n+1} qp_b(\mu_0, \mu_1) \tag{2}$$

Similarly,

$$\begin{aligned}
qp_b(\mu_{2n+1},\mu_{2n}) &= qp_b(J\mu_{2n},S\mu_{2n-1}) \\
&\leq \rho[qp_b(\mu_{2n},\mu_{2n-1})]^{\beta}\cdot[qp_b(\mu_{2n},J\mu_{2n})]^{\alpha}\cdot\left[\frac{1}{s}qp_b(\mu_{2n-1},S\mu_{2n-1})\right]^{1-\alpha-\beta} \\
&\leq \rho[qp_b(\mu_{2n},\mu_{2n-1})]^{\beta}\cdot[qp_b(\mu_{2n},\mu_{2n+1})]^{\alpha}\cdot\left[\frac{1}{s}qp_b(\mu_{2n-1},\mu_{2n})\right]^{1-\alpha-\beta} \\
[qp_b(\mu_{2n+1},\mu_{2n})]^{1-\alpha} &\leq \rho[qp_b(\mu_{2n-1},\mu_{2n})]^{1-\alpha} \\
[qp_b(\mu_{2n+1},\mu_{2n})] &\leq \rho[qp_b(\mu_{2n-1},\mu_{2n})]
\end{aligned}$$

Hence,

$$qp_b(\mu_{2n+1},\mu_{2n}) \leq \rho qp_b(\mu_{2n-1},\mu_{2n}) \leq \rho^2 qp_b(\mu_{2n-1},\mu_{2n-2}).$$

Thus,

$$qp_b(\mu_{2n+1},\mu_{2n}) \leq \rho^{2n} qp_b(\mu_0,\mu_1). \tag{3}$$

From (2) and (3), we can deduce that

$$qp_b(\mu_n,\mu_{n+1}) \leq \rho^{n} qp_b(\mu_0,\mu_1). \tag{4}$$

To prove sequence $\{\mu_n\}$is Cauchy, let $n,k \in N$

$$\begin{aligned}
qp_b(\mu_n,\mu_{n+k}) &\leq sqp_b(\mu_n,\mu_{n+1}) + s^2 qp_b(\mu_{n+1},\mu_{n+2}) + \ldots + s^k qp_b(\mu_{n+k-1},\mu_{n+k}) \\
&\leq [s\rho^n + s^2\rho^{n+1} + \ldots + s^k\rho^{n+k-1}]qp_b(\mu_0,\mu_1) \\
&\leq s^k \sum_{i=n}^{n+k-1} \rho^i qp_b(\mu_0,\mu_1) \\
&\leq s^k \sum_{i=n}^{\infty} \rho^i qp_b(\mu_0,\mu_1) \qquad (5)
\end{aligned}$$

From (5),

$$qp_b(\mu_{n+m},\mu_{n+m+k}) \leq s^k \sum_{i=m}^{\infty} \rho^i qp_b(\mu_n,\mu_{n+1})$$

$$\begin{aligned}
\lim_{m\to\infty,n\to\infty} qp_b(\mu_{n+m},\mu_{n+m+k}) &\leq s^k \lim_{m\to\infty}\sum_{i=m}^{\infty}\lim_{n\to\infty}\rho^i qp_b(\mu_n,\mu_{n+1}) \\
&= 0
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} qp_b(\mu_n,\mu_{n+k}) = \lim_{m\to\infty,n\to\infty} qp_b(\mu_{n+m},\mu_{n+m+k}) = 0. \tag{6}$$

We conclude that $\{\mu_n\}$ is a Cauchy sequence. Since (X,qp_b) is complete, there exists $z \in X$ such that $\lim_{n\to\infty}\mu_n = z$. Next, we shall prove that z is a common fixed point of S and T.

$$\begin{aligned}
qp_b(Jz,\mu_{2n+2}) &= qp_b(Jz,S\mu_{2n+1}) \\
&\leq \rho[qp_b(z,\mu_{2n+1})]^{\beta}\cdot[qp_b(z,Jz)]^{\alpha}\cdot\left[\frac{1}{s}qp_b(\mu_{2n+1},S\mu_{2n+1})\right]^{1-\alpha-\beta} \\
&\leq \rho[qp_b(z,\mu_{2n+1})]^{\beta}\cdot[qp_b(z,Jz)]^{\alpha}\cdot[qp_b(\mu_{2n+1},S\mu_{2n+1})]^{1-\alpha-\beta}
\end{aligned}$$

Letting $n \to \infty$, we get, $Jz = z$.
Similarly,

$$\begin{aligned} qp_b(\mu_{2n+1}, Sz) &= qp_b(J\mu_{2n}, Sz) \\ &\leq \rho[qp_b(\mu_{2n}, Sz)]^\beta \cdot [qp_b(\mu_{2n}, J\mu_{2n})]^\alpha \cdot \left[\frac{1}{s} qp_b(z, Sz)\right]^{1-\alpha-\beta} \\ &\leq \rho[qp_b(\mu_{2n}, Sz)]^\beta \cdot [qp_b(\mu_{2n}, \mu_{2n+1})]^\alpha \cdot [qp_b(z, Sz)]^{1-\alpha-\beta} \end{aligned}$$

Letting $n \to \infty$, we get $Sz = z$. Hence S and J attain a common fixed point. □

The following fixed point result in the setting of complete quasi-partial b-metric space can be obtained from our main result Theorem 3.

Corollary 1. *Let (X, qp_b) be a complete quasi-partial b-metric space and $J, S : X \to X$ be self mappings such that $qp_b(J\mu, S\eta) \leq [qp_b(\mu, J\mu)]^\alpha [qp_b(\eta, S\eta)]^{1-\alpha}$ for all $\mu, \eta \in X$, $\rho \in [0, 1)$, $\alpha \in (0, 1)$, $J\mu \neq \mu$ whenever $S\eta \neq \eta$. Then S and J own a common fixed point.*

Proof. Taking $\beta = 0$ in Theorem 1. □

We now justify our result by illustrating it with an example below.

Example 2. *Let $X = \{1, 2, 3, 4\}$. Define complete quasi-partial b-metric as $qp_b(\mu, \eta) = \max\{\mu, \eta\} + |\mu - \eta|$ that is:*

$qp_b(\mu, \eta)$	1	2	3	4
1	1	3	5	7
2	3	2	4	6
3	5	4	3	5
4	7	6	5	4

We define self mappings J and S on X as $J : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}$, $S : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix}$ *as shown in Figure 1. Choose $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, and $\rho = \frac{7}{10}$.*

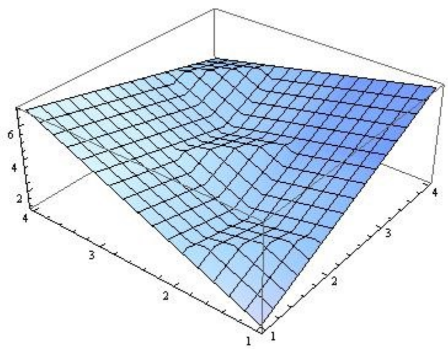

Figure 1. 1 is the common fixed point of S and J.

Case 1: *Let $(\mu, \eta) = (3, 4)$. Without loss of generality, we have*

$$qp_b(J\mu, S\eta) \leq \rho[qp_b(\mu, \eta)]^\beta [qp_b(\mu, J\mu)]^\alpha \left[\frac{1}{s} qp_b(\eta, S\eta)\right]^{1-\alpha-\beta}$$

$$qp_b(J3, S4) = 1 \leq \rho[qp_b(3, 4)]^{1/3} [qp_b(3, J3)]^{1/2} \left[\frac{1}{s} qp_b(4, S4)\right]^{1/6}$$

***Case 2**: Let* $(\mu, \eta) = (1, 4)$

$$qp_(J1, S4) = 1 \leq \rho[qp_b(1,4)]^{1/3}[qp_b(1, J1)]^{1/2}\left[\frac{1}{s}qp_b(4, S4)\right]^{1/6}$$

Thus, 1 is the common fixed point of S and J in the setting of interpolative Reich–Rus–Ćirić type contraction. Many more common fixed points can be obtained in similar manner. Hence, a fixed point exists but is not unique.

In our next theorem, we extend our obtained result for Hardy–Rogers contraction by using interpolative approach.

Theorem 4. *Let (X, qp_b) be a complete quasi-partial b-metric space. Let $J, S : X \to X$ be self mappings. Assume that there are some $\rho \in [0, 1)$, $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ and $s \geq 1$ such that the condition*

$$\begin{aligned} qp_b(J\mu, S\eta) \leq \rho[qp_b(\mu, \eta)]^{\beta}[qp_b(\mu, J\mu)]^{\alpha} \cdot [qp_b(\eta, S\eta)]^{\gamma} \\ \cdot \left[\frac{1}{2s}(qp_b(\mu, S\eta) + qp_b(\eta, J\mu))\right]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{7}$$

is satisfied for all $\mu, \eta \in X$ such that $J\mu \neq \mu$ whenever $S\eta \neq \eta$. Then S and J posses a common fixed point.

Proof. For any arbitrary initial point $\mu_0 \in (X, qp_b)$, we construct an iterative sequence $\{\mu_n\}_{n\geq 1}$ by $\mu_{2n+1} = J\mu_{2n}$, $\mu_{2n+2} = S\,\mu_{2n+1}$. If there exist $n_0 \in N$ such that $\mu_{n_0} = \mu_{n_0+1} = \mu_{n_0+2}$ then μ_{n_0} is a common fixed point of S and J. Suppose that there are no three consecutive identical terms in the sequence. Substituting μ by μ_{2n+1} and η by μ_{2n+2} in (7), we have

$$\begin{aligned} qp_b(\mu_{2n+1}, \mu_{2n+2}) &= qp_b(J\mu_{2n}, S\mu_{2n+1}) \\ &\leq \rho[qp_b(\mu_{2n}, \mu_{2n+1})]^{\beta}[qp_b(\mu_{2n}, J\mu_{2n})]^{\alpha} \cdot [qp_b(\mu_{2n+1}, S\mu_{2n+1})]^{\gamma} \\ &\quad \cdot \left[\frac{1}{2s}(qp_b(\mu_{2n}, S\mu_{2n+1}) + qp_b(\mu_{2n+1}, J\mu_{2n}))\right]^{1-\alpha-\beta-\gamma} \\ &\leq \rho[qp_b(\mu_{2n}, \mu_{2n+1})]^{\beta}[qp_b(\mu_{2n}, \mu_{2n+1})]^{\alpha} \cdot [qp_b(\mu_{2n+1}, \mu_{2n+2})]^{\gamma} \\ &\quad \cdot \left[\frac{1}{2s}(qp_b(\mu_{2n}, \mu_{2n+2}) + qp_b(\mu_{2n+1}, \mu_{2n+1}))\right]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{8}$$

By (QPb$_1$) and (8),

$$\begin{aligned} &\leq \rho[qp_b(\mu_{2n}, \mu_{2n+1})]^{\beta}[qp_b(\mu_{2n}, \mu_{2n+1})]^{\alpha} \cdot [qp_b(\mu_{2n+1}, \mu_{2n+2})]^{\gamma} \\ &\quad \cdot \left[\frac{1}{2s} \cdot s((qp_b(\mu_{2n+2}, \mu_{2n+1}) + qp_b(\mu_{2n+1}, \mu_{2n})))\right]^{1-\alpha-\beta-\gamma} \\ &\leq \rho[qp_b(\mu_{2n}, \mu_{2n+1})]^{\beta}[qp_b(\mu_{2n}, \mu_{2n+1})]^{\alpha} \cdot [qp_b(\mu_{2n+1}, \mu_{2n+2})]^{\gamma} \\ &\quad \cdot \left[\frac{1}{2}(qp_b(\mu_{2n+2}, \mu_{2n+1}) + qp_b(\mu_{2n+1}, \mu_{2n}))\right]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{9}$$

Suppose that

$$\begin{aligned} &qp_b(\mu_{2n}, \mu_{2n+1}) < qp_b(\mu_{2n+1}, \mu_{2n+2}) \\ &\frac{1}{2}(qp_b(\mu_{2n+2}, \mu_{2n+1}) + qp_b(\mu_{2n+1}, \mu_{2n})) \leq qp_b(\mu_{2n+1}, \mu_{2n+2}). \end{aligned}$$

By (8),

$$\begin{aligned} qp_b(\mu_{2n+1},\mu_{2n+2}) &\leq \rho[qp_b(\mu_{2n},\mu_{2n+1})]^{\beta}[qp_b(\mu_{2n},\mu_{2n+1})]^{\alpha}\cdot[qp_b(\mu_{2n+1},\mu_{2n+2})]^{\gamma} \\ &\quad\cdot[qp_b(x_{2n+1},x_{2n+2})]^{1-\alpha-\beta-\gamma} \\ [qp_b(\mu_{2n+1},\mu_{2n+2})]^{\alpha+\beta} &\leq \rho[qp_b(\mu_{2n},\mu_{2n+1})]^{\alpha+\beta} \end{aligned}$$

or

$$[qp_b(\mu_{2n+1},\mu_{2n+2})] \leq \rho[qp_b(\mu_{2n},\mu_{2n+1})]$$

Therefore, we obtain $[qp_b(\mu_{2n+1},\mu_{2n+2})] \leq [qp_b(\mu_{2n},\mu_{2n+1})]$, which is a contradiction. Thus, we have

$$\begin{aligned} &\frac{1}{2}(qp_b(\mu_{2n+2},\mu_{2n+1})+qp_b(\mu_{2n+1},\mu_{2n})) \leq qp_b(\mu_{2n+1},\mu_{2n}) \\ qp_b(\mu_{2n+1},\mu_{2n+2}) &\leq \rho[qp_b(\mu_{2n},\mu_{2n+1})]^{\beta}[qp_b(\mu_{2n},\mu_{2n+1})]^{\alpha}\cdot[qp_b(\mu_{2n+1},\mu_{2n+2})]^{\gamma} \\ &\quad\cdot[qp_b(\mu_{2n+1},\mu_{2n})]^{1-\alpha-\beta-\gamma} \\ [qp_b(\mu_{2n+1},\mu_{2n+2})]^{1-\gamma} &\leq \rho[qp_b(\mu_{2n},\mu_{2n+1})]^{1-\gamma} \end{aligned}$$

We deduce that

$$\begin{aligned} [qp_b(\mu_{2n+1},\mu_{2n+2}) &\leq \rho[qp_b(\mu_{2n},\mu_{2n+1})] \\ qp_b(\mu_{2n+1},\mu_{2n+2}) &\leq \rho qp_b(\mu_{2n+1},\mu_{2n+2}) \leq \rho^2[qp_b(\mu_{2n+1},\mu_{2n+2}) \\ &\leq \rho^3[qp_b(\mu_{2n+1},\mu_{2n+2})\rho^{2n+1}qp_b(\mu_0,\mu_1) \\ qp_b(\mu_{2n+1},\mu_{2n+2}) &\leq \rho^{2n+1}qp_b(\mu_0,\mu_1). \end{aligned} \tag{10}$$

Similarly

$$\begin{aligned} qp_b(\mu_{2n+1},\mu_{2n}) &= qp_b(J\mu_{2n},S\mu_{2n-1}) \\ &\leq \rho[qp_b(\mu_{2n},\mu_{2n-1})]^{\beta}[qp_b(\mu_{2n},Jx_{2n})]^{\alpha}\cdot[qp_b(\mu_{2n-1},S\mu_{2n-1})]^{\gamma} \\ &\quad\cdot\left[\frac{1}{2s}(qp_b(\mu_{2n},S\mu_{2n-1})+qp_b(\mu_{2n-1},J\mu_{2n}))\right]^{1-\alpha-\beta-\gamma} \\ &\leq \rho[qp_b(\mu_{2n},\mu_{2n-1})]^{\beta}[qp_b(\mu_{2n},\mu_{2n+1})]^{\alpha}\cdot[qp_b(\mu_{2n-1},\mu_{2n})]^{\gamma} \\ &\quad\cdot\left[\frac{1}{2s}(qp_b(\mu_{2n},\mu_{2n})+qp_b(\mu_{2n-1},\mu_{2n+1}))\right]^{1-\alpha-\beta-\gamma} \\ &\leq \rho[qp_b(\mu_{2n},\mu_{2n-1})]^{\beta}[qp_b(\mu_{2n},\mu_{2n+1})]^{\alpha}\cdot[qp_b(\mu_{2n-1},\mu_{2n})]^{\gamma} \\ &\quad\cdot\left[\frac{1}{2s}\cdot s((qp_b(\mu_{2n},\mu_{2n+1})+qp_b(\mu_{2n-1},\mu_{2n})))\right]^{1-\alpha-\beta-\gamma} \\ &\leq \rho[qp_b(\mu_{2n},\mu_{2n-1})]^{\beta}[qp_b(\mu_{2n},\mu_{2n+1})]^{\alpha}\cdot[qp_b(\mu_{2n-1},\mu_{2n})]^{\gamma} \\ &\quad\cdot\left[\frac{1}{2}(qp_b(\mu_{2n},\mu_{2n+1})+qp_b(\mu_{2n-1},\mu_{2n}))\right]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{11}$$

$$\begin{aligned} &\frac{1}{2}(qp_b(\mu_{2n+2},\mu_{2n+1})+qp_b(\mu_{2n+1},\mu_{2n})) \leq qp_b(\mu_{2n-1},\mu_{2n}) \\ qp_b(\mu_{2n+1},\mu_{2n}) &\leq \rho[qp_b(\mu_{2n},\mu_{2n-1})]^{\beta}[qp_b(\mu_{2n},\mu_{2n+1})]^{\alpha} \\ &\quad\cdot[qp_b(\mu_{2n-1},\mu_{2n})]^{\gamma}\cdot[qp_b(\mu_{2n-1},\mu_{2n})]^{1-\alpha-\beta-\gamma} \\ [qp_b(\mu_{2n+1},\mu_{2n})]^{1-\gamma} &\leq \rho[qp_b(\mu_{2n-1},\mu_{2n})]^{1-\gamma} \end{aligned}$$

We deduce that

$$
\begin{aligned}
&[qp_b(\mu_{2n+1},\mu_{2n}) \leq \rho[qp_b(\mu_{2n-1},\mu_{2n})] \\
&qp_b(\mu_{2n+1},\mu_{2n}) \leq \rho qp_b(\mu_{2n-1},\mu_{2n}) \leq \rho^2[qp_b(\mu_{2n-2},\mu_{2n-1}) \leq \cdots \leq \rho^{2n} qp_b(\mu_0,\mu_1) \\
&qp_b(\mu_{2n+1},\mu_{2n}) \leq \rho^{2n} qp_b(\mu_0,\mu_1)
\end{aligned} \tag{12}
$$

By (6), sequence $\{\mu_n\}$ is Cauchy. By completeness property, there exists $z \in X$ such that

$$\lim_{n\to\infty} \mu_n = z.$$

Again, we shall show that S and T attain z as a common fixed point of X.

$$
\begin{aligned}
qp_b(Jz,\mu_{2n+2}) &= qp_b(Jz, S\mu_{2n+1}) \\
&\leq \rho[qp_b(z,\mu_{2n+1})]^{\beta}[qp_b(z,Jz)]^{\alpha} \cdot [qp_b(\mu_{2n+1},S\mu_{2n+1})]^{\gamma} \\
&\quad \cdot \left[\frac{1}{2s}(qp_b(z,S\mu_{2n+1}) + qp_b(\mu_{2n+1},Jz))\right]^{1-\alpha-\beta-\gamma} \\
&\leq \rho[qp_b(z,\mu_{2n+1})]^{\beta}[qp_b(z,Tz)]^{\alpha} \cdot [qp_b(\mu_{2n+1},\mu_{2n+2})]^{\gamma} \\
&\quad \cdot \left[\frac{1}{2s}(qp_b(z,\mu_{2n+2}) + qp_b(\mu_{2n+1},Jz))\right]^{1-\alpha-\beta-\gamma}
\end{aligned}
$$

Letting $n \to \infty$, we get, $Jz = z$.
Similarly

$$
\begin{aligned}
qp_b(\mu_{2n+1},Sz) &= qp_b(J\mu_{2n},Sz) \\
&\leq \rho[qp_b(\mu_{2n},z)]^{\beta}[qp_b(x_{2n},J\mu_{2n})]^{\alpha} \cdot [qp_b(z,Sz)]^{\gamma} \\
&\quad \cdot \left[\frac{1}{2s}(qp_b(\mu_{2n},Sz) + qp_b(z,J\mu_{2n}))\right]^{1-\alpha-\beta-\gamma}
\end{aligned}
$$

Letting $n \to \infty$, we get $Sz = z$. Hence self mappings S and J posses common fixed points in complete metric space. □

The above result (Theorem 3) motivates us to generalize the interpolative Reich–Rus–Ćirić contraction for a family of maps. More precisely:

Problem 1. *Let (X, qp_b) be a complete quasi-partial b-metric space. Consider a family of self maps $S_n : X \to X$, $n \geq 1$, and $s \geq 1$ such that*

$$qp_b(S_i\mu, S_j\eta) \leq \rho_{i,j}\,[qp_b(\mu,\eta)]^{\beta_j} \cdot [qp_b(\mu,J\mu)]^{\alpha_i} \cdot \left[\frac{1}{s} qp_b(\eta,S_\eta)\right]^{1-\alpha_i-\beta_j}$$

What are the conditions on $\rho_{i,j}, \alpha_i, \beta_j$ for S_n to have a common fixed point?

4. Conclusions

The significant contribution of the paper proves the existence of common fixed points for interpolative Reich–Rus–Ćirić and Hardy–Rogers contraction mappings on quasi-partial b-metric space. Many real world problems and experimental signals lack a sensation of smoothness in their traces. Therefore, to model these signals, interpolants are required that are non-differentiable in a dense set of points in the domain. Fractal interpolation, which is based on the theory of iterated function system, is used to solve such problems. The uniqueness property of fixed points for these mappings and their application in the study of nonlinear integral equations will be an interesting concept for subsequent work.

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