Closed injective ideals of multilinear operators, related measures and interpolation

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Abstract

We introduce and discuss several ways of extending the inner measure arisen from the closed injective hull of an ideal of linear operators to the multilinear case. In particular, we consider new measures that allow to characterize the operators that belong to a closed injective ideal of multilinear operators as those having measure equal to zero. Some interpolation formulas for these measures, and consequently interpolation results involving ideals of multilinear operators, are established. Examples and applications related to summing multilinear operators are also shown.

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1 Introduction

A fruitful classical way of studying some properties of a linear operator is by considering functionals or measures (of the operator) related to operator ideals. An example of this is the inner measure $\beta_{\mathcal{I}}$ associated to an arbitrary ideal $\mathcal{I}$ of linear operators. We recall that for a continuous linear operator $T : E \to F$,

$\beta_{\mathcal{I}}(T) = \beta_{\mathcal{I}}(T : E \to F) := \inf \{\varepsilon > 0 :$ there are a Banach space Z and $R \in \mathcal{I}(E; Z)$ such that $\|Tx\|_F \leq \varepsilon \|x\|_E + \|Rx\|_Z$, for any $x \in E\}$. This measure was introduced by Tylli [34] in 1995 and determines the operators T that belong to the closed injective hull $\overline{\mathcal{I}}^{inj}$ (i.e. the smallest closed injective ideal containing $\mathcal{I}$) of $\mathcal{I}$ as those for which $\beta_{\mathcal{I}}(T) = 0$ (see [20, Theorem 20.7.3]). In particular when $\mathcal{I}$ is closed and injective, $T \in \mathcal{I}$ if and only if $\beta_{\mathcal{I}}(T) = 0$. Therefore, the inner measure provides a way of characterizing when a given operator belongs to the closed injective hull of a linear operator ideal, and so it allows to quantify (in some sense) how far is the operator from such a hull.

As far as we know, there is no such notion in the literature as the inner measure in the setting of multilinear operators. In the present paper we introduce some functions that extend the inner measure to the multilinear case. The paper is organized in five sections. After the preliminary Sections 1 and 2, we introduce in Section 3 the definitions and main properties of two measures, naturally given and associated to an ideal of multilinear operators, that generalize the aforementioned measure defined by Tylli. In addition, in this section, we establish results concerning the closed injective hull of certain classes of ideals of multilinear operators. Section 4 is devoted to establish interpolation formulas for the new measures and to obtain certain consequences of them. Finally, we show in Section 5 examples and applications related to summing multilinear operators, using the Jarchow-Matter interpolation procedure (see [21]). This point of view of Jarchow and Matter [21] has turned out to be very useful in the study of new (and other well-known) ideals of linear operators and different properties of Banach spaces (see for example [23] and references therein). Other interpolation ideas also used in the multilinear setting giving succesfull results can be seen, for instance, in [10], [14] and [29].

2 Preliminaries

Throughout the paper we consider real or complex Banach spaces without distinction. If $E_1, \dots, E_n$ and F are Banach spaces, then $\mathcal{L}(E_1, \dots, E_n; F)$ stands for the Banach space of all continuous n-linear operators $T : E_1 \times \cdots \times E_n \to F$ with the norm

$$\|T\| := \sup\{\|T(x_1, \dots, x_n)\|_F \, : \, x_1 \in B_{E_1}, \dots, x_n \in B_{E_n}\},$$

where B_{E_j} is the closed unit ball of E_j, $j = 1, \dots, n$. In particular, $\mathcal{L}(E; F)$ is the Banach space of all continuous linear operators from E into F.

Let $E_1 \otimes \cdots \otimes E_n$ denote the tensor product of $E_1, \dots, E_n$ and let π be the projective norm given by

$$\pi(\theta) := \inf \sum_{j=1}^{m} \|x_1^j\| \cdots \|x_n^j\|, \quad \theta \in E_1 \otimes \cdots \otimes E_n,$$

where the infimum is taken over all possible representations of θ of the form $\theta = \sum_{j=1}^{m} x_1^j \otimes \cdots \otimes x_n^j$, $x_i^j \in E_i \, (i = 1, \dots, n)$. The completed projective tensor product is denoted by $E_1\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$.

Given $T \in \mathcal{L}(E_1, \ldots, E_n; F)$, T_L stands for the linearization of T, that is, the unique continuous linear operator $T_L : E_1\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n \to F$ such that $T_L(x_1 \otimes \cdots \otimes x_n) = T(x_1, \ldots, x_n)$, for any $x_1 \in E_1, \ldots, x_n \in E_n$.

The notion of linear operator ideal (see [30]) extends to multilinear operators as follows.

Let $n \in N$ be fixed. An *ideal of n-linear operators*, or an *n-ideal*, is a class $\mathcal{M}_n$ of n-linear maps such that for all Banach spaces $E_1, \ldots, E_n$ and F, the components $\mathcal{M}_n(E_1, \ldots, E_n; F) := \mathcal{L}(E_1, \ldots, E_n; F) \cap \mathcal{M}_n$ satisfy

(i) $\mathcal{M}_n(E_1, \ldots, E_n; F)$ is a linear subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ that contains the n-linear maps of finite type.

(ii) If $R \in \mathcal{L}(F; H)$, $T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$ and $S_j \in \mathcal{L}(G_j; E_j)$, for $j = 1, \ldots, n$, then $R \circ T \circ (S_1, \ldots, S_n) \in \mathcal{M}_n(G_1, \ldots, G_n; H)$.

If for each $n \in N$, $\mathcal{M}_n$ is an ideal of n-linear operators, the class

$$\mathcal{M} := \bigcup_{n=1}^{\infty} \mathcal{M}_n$$

is called an *ideal of multilinear operators* or a *multi-ideal*.

The multi-ideal of all continuous multilinear operators is denoted by $\mathcal{L}$. Let us recall the construction of two examples of ideals of n-linear operators that can be found in [31] and are related to the classical notion of operator ideal [30]. To avoid confusions, we will use the letter $\mathcal{I}$ to denote an ideal of *linear* operators (instead of $\mathcal{M}_1$ or $\mathcal{I}_1$). Thus, a sequence as $\mathcal{I}_1, \ldots, \mathcal{I}_n$ means a sequence of n ideals of linear operators.

On the other hand, throughout the paper the symbol $\overset{[i]}{\cdots}$ means that the i-th term, or the i-th coordinate, does not appear.

Linearization ideal. Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be linear operator ideals. The ideal of n-linear operators $[\mathcal{I}_1, \ldots, \mathcal{I}_n]$ is defined as follows: Let $T \in \mathcal{L}(E_1, \ldots, E_n; F)$,

$$T \in [\mathcal{I}_1, \ldots, \mathcal{I}_n](E_1, \ldots, E_n; F) if and only if T_i \in \mathcal{I}_i(E_i; \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F)), i = 1, 2, \ldots, n,$$

where $T_i : E_i \to \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F)$ is defined as

$$T_i(x_i)(x_1, \overset{[i]}{\ldots}, x_n) := T(x_1, \ldots, x_n),\ x_1 \in E_1, \ldots, x_n \in E_n.$$

Factorization ideal. Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be linear operator ideals. The ideal of n-linear operators $\mathcal{L}(\mathcal{I}_1, \ldots, \mathcal{I}_n)$ is defined as follows: Let $T \in \mathcal{L}(E_1, \ldots, E_n; F)$,

$$T \in \mathcal{L}(\mathcal{I}_1, \ldots, \mathcal{I}_n)(E_1, \ldots, E_n; F) if and only if T factors as T = S \circ (R_1, \ldots, R_n),$$

for some $R_j \in \mathcal{I}_j(E_j; G_j)$ $(j = 1, 2, \ldots, n)$ and $S \in \mathcal{L}(G_1, \ldots, G_n; F)$.

Although both procedures (linearization and factorization) give, in general, different ideals of multilinear operators (see [10, p.741]), the inclusion

$\mathcal{L}(\mathcal{I}_1,\ldots,\mathcal{I}_n) \subset [\mathcal{I}_1,\ldots,\mathcal{I}_n]$ always holds. However, there are examples of ideals for which the inclusion becomes an equality. For instance, if $\mathcal{I}_i$ is the ideal of compact operators $\mathcal{K}$ for $i = 1, 2, \ldots, n$, or if $\mathcal{I}_i$ is the ideal of weakly compact operators $\mathcal{W}$ for $i = 1, \ldots, n$, then $\mathcal{L}(\mathcal{I}_1,\ldots,\mathcal{I}_n) = [\mathcal{I}_1,\ldots,\mathcal{I}_n]$. This is not a mère coincidence: the ideals of compact and weakly compact operators share the properties of being closed and injective. González and Gutiérrez proved that both procedures (linearization and factorization) coincide when they are applied to a single closed injective operator ideal $\mathcal{I}_1 = \cdots = \mathcal{I}_n = \mathcal{I}$ (see [16, Theorem 4] and [17]). More recently Braunss and Junek [10, Theorem 3.4] have shown that $\mathcal{L}(\mathcal{I}_1,\ldots,\mathcal{I}_n) = [\mathcal{I}_1,\ldots,\mathcal{I}_n]$ holds for different closed injective operator ideals $\mathcal{I}_1,\ldots,\mathcal{I}_n$. Under such a hypothesis the ideals of n-linear operators $\mathcal{L}(\mathcal{I}_1,\ldots,\mathcal{I}_n)$ and $[\mathcal{I}_1,\ldots,\mathcal{I}_n]$ turn out to be both closed and injective as well.

Let $\mathcal{M}_n$ be an ideal of n-linear operators. It will be denoted by $\overline{\mathcal{M}_n}$ the class of n-linear operators formed by components $\overline{\mathcal{M}_n}(E_1,\ldots,E_n;F)$ that are given by the closure of $\mathcal{M}_n(E_1,\ldots,E_n;F)$ in $\mathcal{L}(E_1,\ldots,E_n;F)$. $\mathcal{M}_n$ is said to be *closed* when $\overline{\mathcal{M}_n} = \mathcal{M}_n$.

The *injective hull* $\mathcal{M}_n^{inj}$ of $\mathcal{M}_n$ is defined as follows: $T \in \mathcal{L}(E_1,\ldots,E_n;F)$ belongs to $\mathcal{M}_n^{inj}(E_1,\ldots,E_n;F)$ if $J_F \circ T \in \mathcal{M}_n(E_1,\ldots,E_n;\ell_\infty(B_{F^*}))$, where $J_F : F \to \ell_\infty(B_{F^*})$ is the natural metric injection given by $J_F(y) = (\langle y, y^*\rangle)_{y^*\in B_{F^*}}$. $\mathcal{M}_n$ is called *injective* if $\mathcal{M}_n = \mathcal{M}_n^{inj}$, i.e. if for any Banach spaces $E_1,\ldots,E_n,F$ and each n-linear operator $T \in \mathcal{L}(E_1,\ldots,E_n;F)$, it holds that $T \in \mathcal{M}_n(E_1,\ldots,E_n;F)$ whenever $J_F \circ T \in \mathcal{M}_n(E_1,\ldots,E_n;\ell_\infty(B_{F^*}))$.

The closed injective hull $\overline{\mathcal{I}}^{inj}$ of an ideal $\mathcal{I}$ of linear operators can be characterized as follows (see [20, Theorem 20.7.3(i)] or [21, Section 1,(2)(a)]): Take $T \in \mathcal{L}(E;F)$, then $T \in \overline{\mathcal{I}}^{inj}(E;F)$ if and only if for each $\varepsilon > 0$ there are a Banach space Z and an operator $R \in \mathcal{I}(E;Z)$ such that $\|Tx\|_F \le \varepsilon\|x\|_E + \|Rx\|_Z$, for all $x \in E$.

Hence, as it was said in Introduction, the inner measure $\beta_{\mathcal{I}}$ of $T \in \mathcal{L}(E;F)$, given by $\beta_{\mathcal{I}}(T) = \beta_{\mathcal{I}}(T : E \to F) := \inf\{\varepsilon > 0 :$ there are a Banach space Z and $R \in \mathcal{I}(E;Z)$ such that $\|Tx\|_F \le \varepsilon\|x\|_E + \|Rx\|_Z$, for any $x \in E\}$, satisfies that $\beta_{\mathcal{I}}(T) = 0 \iff T \in \overline{\mathcal{I}}^{inj}(E;F)$. Therefore, $\beta_{\mathcal{I}}(T) = 0$ if and only if $T \in \mathcal{I}(E;F)$ when $\mathcal{I}$ is closed and injective.

3 Measures associated to ideals of multilinear operators

It is natural to investigate if it is possible to generalize the notion of inner measure to the setting of ideals of multilinear operators. We will deal with this issue in this section.

We start by stating two lemmas on the injective hull and the closed hull of an n-ideal $[\mathcal{I}_1,\ldots,\mathcal{I}_n]$ that are known results (see [8, p.309]). Because this type of n-ideal will play an important role in this paper, and for the sake of completeness, we include such lemmas and their proofs. First recall that a Banach space H is said to be *injective*, or it has the *metric extension property*, if for any Banach space G, any closed linear subspace E of G, and any $R \in \mathcal{L}(E;H)$, there exists an extension $S \in \mathcal{L}(G;H)$ of R with $\|S\| = \|R\|$. Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be linear operator ideals. Then,

(a) $[\mathcal{I}_1, \ldots, \mathcal{I}_n]^{inj} \subset [\mathcal{I}_1^{inj}, \ldots, \mathcal{I}_n^{inj}]$.

(b) $\overline{[\mathcal{I}_1, \ldots, \mathcal{I}_n]} \subset [\overline{\mathcal{I}_1}, \ldots, \overline{\mathcal{I}_n}]$.

(a) Take $T \in [\mathcal{I}_1, \cdots, \mathcal{I}_n]^{inj}(E_1, \ldots, E_n; F)$. For each $i = 1, \ldots, n$, consider the mapping

$$j_i : \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F) \to \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; \ell_\infty(B_{F^*}))$$

given by $j_i(A) := J_F \circ A$, $A \in \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F)$. Since J_F is an isometry, the map j_i is a metric injection. For any $x_i \in E_i$, we have $\mathrm{j}_i(T_i(x_i))(x_1, \overset{[i]}{\ldots}, x_n) = \big(J_F \circ (T_i(x_i))\big)(x_1, \overset{[i]}{\ldots}, x_n)$
$= J_F(T(x_1, \ldots, x_n)) = (J_F \circ T)_i(x_i)(x_1, \overset{[i]}{\ldots}, x_n)$. Using the metric extension property of $\ell_\infty(B_{\mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F)^*})$, there is a continuous linear mapping

$$\phi_i : \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; \ell_\infty(B_{F^*})) \to \ell_\infty(B_{\mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F)^*})$$

such that $\phi_i \circ j_i = J_{\mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F)}$. Then,

$$J_{\mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F)} \circ T_i = \phi_i \circ (J_F \circ T)_i \in \mathcal{I}_i\big(E_i; \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; \ell_\infty\big(B_{\mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F)^*}\big)\big).$$

Hence, $T_i \in \mathcal{I}_i^{inj}\big(E_i; \mathcal{L}(E_1, , \overset{[i]}{\ldots}, E_n; F)\big)$ for all $i = 1, \ldots, n$. Thus, $T \in [\mathcal{I}_1^{inj}, \cdots, \mathcal{I}_n^{inj}]$.

(b) Let $T \in \overline{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}(E_1, \ldots, E_n; F)$. Given $\varepsilon > 0$, we find n-linear operators $A, B \in \mathcal{L}(E_1, \ldots, E_n; F)$ such that $A \in [\mathcal{I}_1, \ldots, \mathcal{I}_n](E_1, \ldots, E_n; F)$, $\|B\| < \varepsilon$ and $T = A + B$. For each $i = 1, \ldots, n$, we have $T_i = A_i + B_i$, $A_i \in \mathcal{I}_i(E_i; \mathcal{L}(E_1 \overset{[i]}{\ldots}, E_n; F))$ and $\|B_i\| < \varepsilon$. Then, $T_i \in \overline{\mathcal{I}_i}(E_i; \mathcal{L}(E_1 \overset{[i]}{\ldots}, E_n; F))$. Hence, $T \in [\overline{\mathcal{I}_1}, \ldots, \overline{\mathcal{I}_n}]$.

As a direct consequence of Lemma 3 we derive the following result.

Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be linear operator ideals.

(a) If $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are injective, then $[\mathcal{I}_1, \ldots, \mathcal{I}_n]$ is injective too.

(b) If $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are closed, then $[\mathcal{I}_1, \ldots, \mathcal{I}_n]$ is closed too.

(c) If $\mathcal{I}_i = \overline{\mathcal{I}_i}^{inj}$, $i = 1, \ldots, n$, then $[\mathcal{I}_1, \ldots, \mathcal{I}_n] = \overline{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}^{inj}$.

(a) It follows from

$$[\mathcal{I}_1, \ldots, \mathcal{I}_n]^{inj} \subset [\mathcal{I}_1{}^{inj}, \ldots, \mathcal{I}_n{}^{inj}] = [\mathcal{I}_1, \ldots, \mathcal{I}_n].$$

(b) The next inclusion gives the result:

$$\overline{[\mathcal{I}_1, \ldots, \mathcal{I}_n]} \subset [\overline{\mathcal{I}_1}, \ldots, \overline{\mathcal{I}_n}] = [\mathcal{I}_1, \ldots, \mathcal{I}_n].$$

Now (c) is obvious.

Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be linear operator ideals. By [8, p.309] we trivially have

$$\overline{\mathcal{L}(\mathcal{I}_1, \ldots, \mathcal{I}_n)}^{inj} \subset \mathcal{L}(\overline{\mathcal{I}_1}^{inj}, \ldots, \overline{\mathcal{I}_n}^{inj}).$$

Let us show that for some particular operators the other inclusion also holds. To prove the next result, we need to extend a continuous multilinear operator $S : E_1 \times \cdots \times E_n \to F$ to some continuous multilinear operator $ext(S) : E_1^{**} \times \cdots \times E_n^{**} \to F^{**}$. Continuous bilinear operators $A : E_1 \times E_2 \to F$ were extended to continuous bilinear operator from $E_1^{**} \times E_2^{**}$ into F^{**} by Arens [1]. This extension is built by considering three times in a row the following transpose: $\begin{array}{llll} A^t: & F^* \times E_1 & \to & E_2^* \\ & (y^*, x_1) & & A^t(y^*, x_1)(x_2) = y^*(A(x_1, x_2)), \end{array}$ $x_1 \in E_1, x_2 \in E_2$ and $y^* \in F^*$. This procedure gives two, in general different, extensions: A^{ttt} and A^{TtttT}, where $B^T(x_1, x_2) = B(x_2, x_1)$ for any bilinear mapping B, and are known as Arens products. This procedure was generalized by Aron and Berner [2] to arbitrary multilinear mappings. Given a continuous multilinear operator $S : E_1 \times \cdots \times E_n \to F$ we will denote $AB(S) : E_1^{**} \times \cdots \times E_n^{**} \to F^{**}$ one of the Aron and Berner extensions of S.

Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be linear operator ideals and let $E_1, \ldots, E_n, F$ be Banach spaces. If $S \in \mathcal{L}(c_0, \ldots, c_0; F)$ and $R_i \in \overline{\mathcal{I}_i}^{inj}(E_i; c_0)$ for each $i = 1, \ldots, n$, then $S \circ (R_1, \ldots, R_n) \in \overline{\mathcal{L}(\mathcal{I}_1, \ldots, \mathcal{I}_n)}^{inj}$.

We can trivially assume $S \neq 0$. Fix $\varepsilon > 0$ and $i \in \{1, \ldots, n\}$. Since $R_i \in \overline{\mathcal{I}_i}^{inj}(E_i; c_0)$, there exist continuous linear operators $A_i, B_i \in \mathcal{L}(E_i; c_0)$ such that

$$A_i \in \mathcal{I}_i^{inj}(E_i; c_0), \ \|B_i\| < \frac{\varepsilon^{1/n}}{\|S\|^{1/n}} \ \text{ and } R_i = A_i + B_i.$$

Let $\{e_j : \ j \in N\}$ be the usual canonical basis in ℓ_1. Having in mind that $(c_0^*)^* = (\ell_1)^* = \ell_\infty$, define the map

$$P : \ell_\infty(B_{\ell_1}) \to \ell_\infty, \quad P(\eta) = (\eta_{e_j})_{j=1}^\infty,$$

for any $\eta := (\eta_{y^*})_{y^* \in B_{\ell_1}} \in \ell_\infty(B_{\ell_1})$. Clearly the map P is well-defined, linear and continuous, with $\|P\| \leq 1$. Moreover, $P \circ J_{c_0} = I_{c_0}$, where $I_{c_0} : c_0 \to \ell_\infty$ is the canonical injection.

Take any of the Aron and Berner extensions of $S : c_0 \times \cdots \times c_0 \to F$, denoted by $AB(S) : \ell_\infty \times \cdots \times \ell_\infty \to F^{**}$. Consider the canonical isometric inclusions $I_F : F \to F^{**}$ and $K_F : F^{**} \to \ell_\infty(B_{F^*})$, given by $y^{**} \in F^{**} \to (\langle y^*, y^{**} \rangle)_{y^* \in B_{F^*}}$. These mappings are related via the equality $J_F = K_F \circ I_F$. Since $J_{c_0} \circ A_i \in \mathcal{I}_i(E_i; \ell_\infty(B_{\ell_1}))$ for an arbitrary $i \in \{1, \ldots, n\}$, it follows that the map

$$T_0 := K_F \circ AB(S) \circ (P \circ J_{c_0} \circ A_1, \ldots, P \circ J_{c_0} \circ A_n)$$

belongs to $\mathcal{L}(\mathcal{I}_1, \ldots, \mathcal{I}_n)(E_1, \ldots, E_n; \ell_\infty(B_{F^*}))$. Besides,

$$T_0 = K_F \circ AB(S) \circ (I_{c_0} \circ A_1, \ldots, I_{c_0} \circ A_n) = J_F \circ S \circ (A_1, \ldots, A_n).$$

Then, $S\circ(A_1,\dots,A_n)\in\mathcal{L}(\mathcal{I}_1,\dots,\mathcal{I}_n)^{inj}(E_1,\dots,E_n;F)$. Since $\|S\circ(R_1,\dots,R_n)-S\circ(A_1,\dots,A_n)\|\le\|S\|\|R_1-A_1\|\cdots\|R_n-A_n\|$ $=\|S\|\|B_1\|\cdots\|B_n\|\le\|S\|\frac{\varepsilon}{\|S\|}=\varepsilon$, we conclude that $S\circ(R_1,\dots,R_n)\in\overline{\mathcal{L}(\mathcal{I}_1,\dots,\mathcal{I}_n)}^{inj}(E_1,\dots,E_n;F)$.

Let us denote by $\mathcal{L}_{c_0}(\mathcal{I}_1,\dots,\mathcal{I}_n)$ those elements in $\mathcal{L}(\mathcal{I}_1,\dots,\mathcal{I}_n)$ that factor through $c_0\times\cdots\times c_0$, i.e. $T\in\mathcal{L}_{c_0}(\mathcal{I}_1,\dots,\mathcal{I}_n)(E_1,\dots,E_n;F)$ if $T=S\circ(R_1,\dots,R_n)$ for some $S\in\mathcal{L}(c_0,\dots,c_0;F)$ and some $R_i\in\mathcal{I}_i(E_i;c_0)$, $i=1,\dots,n$. Then, Theorem 3 can be rephrased as follows:

$$\mathcal{L}_{c_0}(\overline{\mathcal{I}_1}^{inj},\dots,\overline{\mathcal{I}_n}^{inj})\subset\overline{\mathcal{L}(\mathcal{I}_1,\dots,\mathcal{I}_n)}^{inj}.$$

Next let us extend the inner measure introduced by Tylli [34] to the setting of multilinear operators. Let $\mathcal{I}_1,\dots,\mathcal{I}_n$ be linear operator ideals. For $T\in\mathcal{L}(E_1,\dots,E_n;F)$, $\beta_{[\mathcal{I}_1,\dots,\mathcal{I}_n]}(T)=\beta_{[\mathcal{I}_1,\dots,\mathcal{I}_n]}(T:E_1\times\cdots\times E_n\to F):=$
$\inf\Big\{\varepsilon>0: there are Banach spaces \mathrm{Z}_i$ and $\mathrm{R}_i\in\mathcal{I}_i(E_i;Z_i) so that if x_1\in E_1,\dots x_n\in E_n,$
$\|T(x_1,\dots,x_n)\|_F\le\varepsilon\|x_1\|\cdots\|x_n\|+\min_{i\in\{1,\dots,n\}}\left\{\|R_i(x_i)\|\|x_1\|\overset{[i]}{\cdots}\|x_n\|\right\}\Big\}$.

The following result generalizes the well-known characterization of a closed injective linear operator ideal (that can be found in [20, Theorem 20.7.3(i)]). Let $\mathcal{I}_1,\dots,\mathcal{I}_n$ be linear operator ideals and let $T\in\mathcal{L}(E_1,\dots,E_n;F)$. The following statements are equivalent.

(a) $T\in[\overline{\mathcal{I}_1}^{inj},\dots,\overline{\mathcal{I}_n}^{inj}](E_1,\dots,E_n;F)$.

(b) For every $\varepsilon>0$ there are Banach spaces Z_i and operators $R_i\in\mathcal{I}_i(E_i;Z_i)$, $i=1,\dots,n$, such that for all $x_1\in E_1,\dots x_n\in E_n$

$$\|T(x_1,\dots,x_n)\|_F\le\varepsilon\|x_1\|\cdots\|x_n\|+\min_{i\in\{1,\dots,n\}}\left\{\|R_i(x_i)\|\|x_1\|\overset{[i]}{\cdots}\|x_n\|\right\}.$$

(c) $\beta_{[\mathcal{I}_1,\dots,\mathcal{I}_n]}(T)=0$.

If (a) holds, for every $\varepsilon>0$ there exists a Banach space Z_1 and an operator $R_1\in\mathcal{I}_1(E_1;Z_1)$ such that for any $x_1\in E_1,\dots,x_n\in E_n$

$$\|T(x_1,\dots,x_n)\|_F=\|T_1(x_1)(x_2,\dots,x_n)\|_F\le\varepsilon\|x_1\|\cdots\|x_n\|+\|R_1(x_1)\|\|x_2\|\cdots\|x_n\|.$$

Since this also holds for every $i=2,\dots,n$, we get

$$\|T(x_1,\dots,x_n)\|_F\le\varepsilon\|x_1\|\cdots\|x_n\|+\min_{i\in\{1,\dots,n\}}\left\{\|R_i(x_i)\|\|x_1\|\overset{[i]}{\cdots}\|x_n\|\right\},$$

for each $x_1\in E_1,\cdots,x_n\in E_n$, that is, we obtain (b).

Now assume (b) and let us prove (a). Fix $\varepsilon>0$, then for $x_1\in E_1,\dots,x_n\in E_n$, we have $\|T(x_1,\dots,x_n)\|_F\le\varepsilon\|x_1\|\cdots\|x_n\|+\|R_1(x_1)\|\|x_2\|\cdots\|x_n\|$ $=\big(\varepsilon\|x_1\|+\|R_1(x_1)\|\big)\|x_2\|\cdots\|x_n\|$. Thus

$$\|T_1(x_1)\|_{\mathcal{L}(E_2,\dots,E_n;F)}=\sup_{x_2\in B_{E_2},\dots,x_n\in B_{E_n}}\big\|T_1(x_1)(x_2,\dots,x_n)\big\|_F\le\varepsilon\|x_1\|+\|R_1(x_1)\|.$$

Hence, $T_1 \in \overline{\mathcal{I}_1}^{inj}(E_1; \mathcal{L}(E_2, \ldots, E_n; F))$. Reasoning similarly for $i = 2, \ldots, n$, we can conclude that $T_i \in \overline{\mathcal{I}_i}^{inj}(E_i; \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F))$ for all $i = 1, \ldots, n$. Then, it holds that $T \in [\overline{\mathcal{I}_1}^{inj}, \ldots, \overline{\mathcal{I}_n}^{inj}](E_1, \ldots, E_n; F)$, so we get (a).

It is obvious that (b) $\Longleftrightarrow$ (c).

Observe that the proof of Theorem 3 also allows to ensure that for any $T \in \mathcal{L}(E_1, \ldots, E_n; F)$,

$$\beta_{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}(T) = \max\{\beta_{\mathcal{I}_i}(T_i) : i = 1, \ldots n\}.$$

We have established in Theorem 3 that $\beta_{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}$ just characterizes when an n-linear operator belongs to the ideal $[\overline{\mathcal{I}_1}^{inj}, \ldots, \overline{\mathcal{I}_n}^{inj}]$. However, note that the inclusion $\overline{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}^{inj} \subset [\overline{\mathcal{I}_1}^{inj}, \ldots, \overline{\mathcal{I}_n}^{inj}]$ always holds, but it is not an equality in general. In fact, for $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{A}$, the ideal of (linear) approximable operators, it holds that $\overline{[\mathcal{A}, \mathcal{A}]}^{inj} \neq [\overline{\mathcal{A}}^{inj}, \overline{\mathcal{A}}^{inj}]$. It is well-known that $\overline{\mathcal{A}}^{inj} = \mathcal{K}$ (see [30, Proposition 4.2.5, Remarks 4.6.13 and 4.7.13]). By [8, Example 3.4] (see also [3, proof of Theorem 4.5]), there exists a Banach space E without the approximation property and an operator $u \in \mathcal{L}(E; E^*)$ that is compact, symmetric and non-approximable. Let A be the bilinear form on $E \times E$ considered in [8, Example 3.4], defined by $A(x, y) = u(x)(y)$. It is immediate that $A \in [\overline{\mathcal{A}}^{inj}, \overline{\mathcal{A}}^{inj}]$ since $A_1 = A_2 = u \in \mathcal{K}(E; E^*)$. Nevertheless, if $A \in \overline{[\mathcal{A}, \mathcal{A}]}^{inj}(E, E; F)$, taking into account that F is an injective space and $\mathcal{A}$ is closed, it would follow (see [8, Corollary 2.6]) that

$$A \in \overline{[\mathcal{A}, \mathcal{A}]}^{inj}(E, E; F) = \overline{[\mathcal{A}, \mathcal{A}]}(E, E; F) = [\mathcal{A}, \mathcal{A}](E, E; F),$$

but this is a contradiction because $A_1 = A_2 = u \notin \mathcal{A}(E; E^*)$.

In order to establish the next results, we recall that if $\mathcal{I}$ is a *Banach linear operator ideal* (see definition for example in [20, 19.3] or [15, Chapter I, Section 9]), then the closed injective hull of $\mathcal{I}$ can be characterized as follows (see [20, Theorem 20.7.3(ii)] or [21, Section 1,(3)(a)]): An operator $T \in \mathcal{L}(E; F)$ belongs to $\overline{\mathcal{I}}^{inj}(E; F)$ if and only if there are a function $N : R^+ \to R^+$, a Banach space G and an operator $S \in \mathcal{I}(E; G)$ such that

$$\|T(x)\|_F \leq N(\varepsilon)\|S(x)\|_G + \varepsilon\|x\|_E, \; forevery$$

$\varepsilon > 0$ and each $x \in E$. (1)

Let us see how the inequality (1) results in a general factorization theorem. We will proceed with the multilinear case directly.

Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be Banach linear operator ideals, let $E_1, \ldots, E_n$ be Banach spaces and let $T \in \mathcal{L}(E_1, \ldots, E_n; F)$. Then, $T \in [\overline{\mathcal{I}_1}^{inj}, \ldots, \overline{\mathcal{I}_n}^{inj}](E_1, \ldots, E_n; F)$ if and only if for each $i = 1, \ldots, n$, there exist a function $N_i : R^+ \to R^+$ and a linear operator $S_i \in \mathcal{I}_i(E_i; G_i)$ such that

$$\|T(x_1,\ldots,x_n)\| \le \big(N_1(\varepsilon_1)\|S_1(x_1)\| + \varepsilon_1\|x_1\|\big)\cdots\big(N_n(\varepsilon_n)\|S_n(x_n)\| + \varepsilon_n\|x_n\|\big), \tag{2}$$

for all $\varepsilon_1 > 0, \ldots, \varepsilon_n > 0$ and $x_1 \in E_1, \ldots, x_n \in E_n$.

Assume first that $T \in [\overline{\mathcal{I}_1}^{inj}, \ldots, \overline{\mathcal{I}_n}^{inj}](E_1, \ldots, E_n; F)$. Then, there are Banach spaces $H_1, \ldots, H_n$, linear operators $R_i \in \overline{\mathcal{I}_i}^{inj}(E_i; H_i)$ and a continuous n-linear operator $S \in \mathcal{L}(H_1, \ldots, H_n; F)$ such that $T = S \circ (R_1, \ldots, R_n)$. By (1), there are functions $N_i : R^+ \to R^+$ and a linear operator $S_i \in \mathcal{I}_i(E_i, G_i)$ such that for any $\varepsilon_i > 0$ and $x_i \in E_i$ $(i = 1, \ldots, n)$

$$\|R_i(x_i)\| \le N_i(\varepsilon_i')\|S_i(x_i)\| + \varepsilon_i'\|x_i\|,$$

where $\varepsilon_1' = \frac{\varepsilon_1}{\|S\|}$ and $\varepsilon_i' := \varepsilon_i$ if $i = 2, \ldots, n$. Hence, $\|T(x_1, \ldots, x_n)\|_F \le \|S\|\|R_1(x_1)\| \cdots \|R_n(x_n)\|$
$\le \|S\|\big(N_1(\varepsilon_1')\|S_1(x_1)\| + \varepsilon_1'\|x_1\|\big) \cdots \big(N_n(\varepsilon_n)\|S_n(x_n)\| + \varepsilon_n\|x_n\|\big)$.
The functions $\|S\|N_1(\varepsilon_1/\|S\|), N_2(\varepsilon_2), \ldots, N_n(\varepsilon_n)$ and $S_1, \ldots, S_n$ are what we were looking for.

Let us proceed with the converse. Suppose that for every $i = 1, \ldots, n$ there exist a function $N_i : R^+ \to R^+$ and an operator $S_i \in \mathcal{I}_i(E_i; G_i)$ such that $\|T(x_1, \ldots, x_n)\| \le \big(N_1(\varepsilon_1)\|S_1(x_1)\| + \varepsilon_1\|x_1\|\big) \cdots \big(N_n(\varepsilon_n)\|S_n(x_n)\| + \varepsilon_n\|x_n\|\big)$, for all $x_1 \in E_1, \ldots, x_n \in E_n$ and all $\varepsilon_1 > 0, \ldots, \varepsilon_n > 0$. We will prove that $T_i \in \overline{\mathcal{I}_i}^{inj}(E_i; \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F))$, for $i = 1, \ldots, n$. It is enough to see it for a fixed i, since the argument is the same for the rest. Take for instance $i = 1$. Let $\varepsilon > 0$ and $x_1 \in E_1$. Given arbitrary $x_2 \in B_{E_2}, \ldots, x_n \in B_{E_n}$, choosing in (2) any $\varepsilon_j > 0\,(j = 2, \ldots, n)$ and $\varepsilon_1 = \varepsilon/K$, with $K = (N_2(\varepsilon_2)\|S_2\| + \varepsilon_2) \cdots (N_n(\varepsilon_n)\|S_n\| + \varepsilon_n)$, it follows that $\|T_1(x_1)(x_2, \ldots, x_n)\| = \|T(x_1, \ldots, x_n)\|$
$\le \big(N_1(\varepsilon_1)\|S_1(x_1)\| + \varepsilon_1\|x_1\|\big) \cdots \big(N_n(\varepsilon_n)\|S_n(x_n)\| + \varepsilon_n\|x_n\|\big)$
$\le K\big(N_1(\varepsilon_1)\|S_1(x_1)\| + \varepsilon_1\|x_1\|\big) = KN_1(\varepsilon_1)\|S_1(x_1)\| + \varepsilon\|x_1\|$. Hence, denoting $\widehat{N}_1(\varepsilon) = KN_1(\varepsilon/K)$, we conclude that

$$\|T_1(x_1)\|_{\mathcal{L}(E_2,\ldots,E_n;F)} \le \widehat{N}_1(\varepsilon)\|S_1(x_1)\| + \varepsilon\|x_1\|, \ forevery$$

$\varepsilon > 0$ and each $x_1 \in E_1$.

$$Thus$$

$\mathrm{T}_1 \in \overline{\mathcal{I}_1}^{inj}(E_1; \mathcal{L}(E_2, \ldots, E_n; F))$, and the proof is complete.

If T satisfies the domination inequality (2) of Theorem 3, then $T \in \mathcal{L}(\overline{\mathcal{I}_1}^{inj}, \ldots, \overline{\mathcal{I}_n}^{inj})$. We will just give a sketch of this. For each $i = 1, \ldots, n$, consider the positively homogeneous function

$$\Phi_i(x) := \inf_{\varepsilon > 0}\big\{N_i(\varepsilon)\|S_i(x)\| + \varepsilon\|x\|\big\}, \quad x \in E_i,$$

and its convexification

$$\|x_i\|_{N_i,S_i} := \inf\Big\{\sum_{j=1}^m \Phi_i(x_{ij}) : \sum_{j=1}^m x_{ij} = x_i\Big\}, \quad x_i \in E_i.$$

Let E_{N_i,S_i} be the Banach space defined as the completion of the quotient space formed by the equivalence classes $x \equiv y \leftrightarrow \|x-y\|_{N_i,S_i} = 0$. Note that the quotient map $j_i : E_i \to E_{N_i,S_i}$ is continuous and so

$$\|x_i\|_{N_i,S_i} \leq K_i \Phi_i(x_i) \leq K_i N_i\Big(\frac{\varepsilon_i}{K_i}\Big)\|S_i(x_i)\| + \varepsilon_i\|x_i\|,$$

for some constant $K_i > 0$ and all $\varepsilon_i > 0$ and all $x_i \in E_i$. Using (1), $j_i \in \overline{\mathcal{I}_i}^{inj}(E_i; E_{N_i,S_i})$, $i = 1,\ldots,n$. Moreover, T admits the following factorization through the product of the Banach spaces E_{N_i,S_i}

$$\begin{tikzcd}
E_1 \times \cdots \times E_n \arrow[rr, "T"] \arrow[dr, dotted, "{(j_1,\ldots,j_n)}"'] & & F \\
& E_{N_1,S_1} \times \cdots \times E_{N_n,S_n} \arrow[ur, dotted, "S"'] &
\end{tikzcd}$$

where S is a continuous multilinear map.

Our aim now is to introduce a multilinear measure that characterizes the operators that belong to $\overline{\mathcal{M}_n}^{inj}$, for a given n-ideal $\mathcal{M}_n$. It turns out that this measure will coincide with the inner measure of the linearization of the multilinear mapping for certain class of multi-ideals. The following result is a preliminary step before our objective.

Let $\mathcal{M}_n$ be an ideal of n-linear operators and $T \in \mathcal{L}(E_1,\ldots,E_n;F)$. The following assertions are equivalent.

(a) $T \in \mathcal{M}_n^{inj}(E_1,\ldots,E_n;F)$.

(b) There are a Banach space G and an operator $R \in \mathcal{M}_n(E_1,\ldots,E_n;G)$ such that
$$\Big\|\sum_{j=1}^m T(x_1^j,\ldots,x_n^j)\Big\| \leq \Big\|\sum_{j=1}^m R(x_1^j,\ldots,x_n^j)\Big\|,$$
for all $m \in N$ and all $x_1^j \in E_1,\ldots,x_n^j \in E_n, j = 1,\ldots,m$.

(a)$\Longrightarrow$(b) It is enough to take $R := J_F \circ T$.

(b)$\Longrightarrow$(a) Consider the normed space G_0 defined as the linear span of $R(E_1 \times \cdots \times E_n) \subseteq G$, i.e. the normed space of all vectors of the form $\sum_{j=1}^m R(x_1^j,\ldots,x_n^j) \in G$, with $m \in N$ and $x_1^j \in E_1,\ldots,x_n^j \in E_n, j = 1,\ldots,m$. Now observe that the linear operator $S_0 : G_0 \to F$ given by

$$S_0\Big(\sum_{j=1}^m R(x_1^j,\ldots,x_n^j)\Big) := \sum_{j=1}^m T(x_1^j,\ldots,x_n^j),$$

is well-defined. This can be checked using the assumption in (b) and taking into account that, for each j, it holds that $\mathrm{T}(\mathrm{x}_1^j,\ldots,x_n^j) - T(y_1^j,\ldots,y_n^j) = T(x_1^j - y_1^j, x_2^j,\ldots,x_n^j) + T(y_1^j, x_2^j - y_2^j, x_3^j,\ldots,x_n^j) +$
$+T(y_1^j, y_2^j, x_3^j - y_3^j, x_4^j,\ldots,x_n^j) + \cdots + T(y_1^j, y_2^j,\ldots,y_{n-1}^j, x_n^j - y_n^j)$. We can extend S_0 to the completion $\overline{G}_0$ of G_0: write S for this extension. Since $\ell^\infty(B_{F^*})$ has

the metric extension property, we obtain a new operator $S^{ext} : G \to \ell^\infty(B_{F^*})$ which satisfies that $J_F \circ T = S^{ext} \circ R$. Due to $R \in \mathcal{M}_n(E_1, \ldots, E_n; G)$, we get that $J_F \circ T \in \mathcal{M}_n(E_1, \ldots, E_n; \ell^\infty(B_{F^*}))$.

In 2010 Botelho, Galindo and Pellegrini [8, Theorem 2.4] proved that, given $T \in \mathcal{L}(E_1, \ldots, E_n; F)$, $T \in \overline{\mathcal{M}_n}^{inj}(E_1, \ldots, E_n; F)$ if, and only if, for each $\varepsilon > 0$ there is a Banach space Z and $R \in \mathcal{M}_n(E_1, \ldots, E_n; Z)$ such that

$$\Big\| \sum_{j=1}^m T(x_1^j, \ldots, x_n^j) \Big\|_F \leq \varepsilon \sum_{j=1}^m \|x_1^j\|_{E_1} \cdots \|x_n^j\|_{E_n} + \Big\| \sum_{j=1}^m R(x_1^j, \ldots, x_n^j) \Big\|_Z, \quad (3)$$

for any $m \in N$ and $x_1^j \in E_1, \ldots, x_n^j \in E_n, j = 1, \ldots, m$.

Inspired by (3), we next introduce a measure $\widetilde{\beta}_{\mathcal{M}_n}$ which satisfies that $\widetilde{\beta}_{\mathcal{M}_n}(T) = 0$ if and only if $T \in \overline{\mathcal{M}_n}^{inj}$. Thus, in particular, $\widetilde{\beta}_{\mathcal{M}_n}(T) = 0$ if and only if $T \in \mathcal{M}_n$, whenever $\mathcal{M}_n$ is a closed injective ideal of n-linear operators.

Let $\mathcal{M}_n$ be an ideal of n-linear operators. For $T \in \mathcal{L}(E_1, \ldots, E_n; F)$, $\widetilde{\beta}_{\mathcal{M}_n}(T) = \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) :=$
$\inf \Big\{ \varepsilon > 0 \, : \, thereisaBanachspace \mathrm{Z} and R \in \mathcal{M}_n(E_1, \ldots, E_n; Z) \; such that$
$\Big\| \sum_{j=1}^m T(x_1^j, \ldots, x_n^j) \Big\|_F \leq \varepsilon \sum_{j=1}^m \|x_1^j\| \cdots \|x_n^j\| + \Big\| \sum_{j=1}^m R(x_1^j, \ldots, x_n^j) \Big\|,$
for all $m \in N$ and all $x_1^j \in E_1, \ldots, x_n^j \in E_n, j = 1, \ldots, m \Big\}$.

It is very easy to check that if $\mathcal{M}_n = [\mathcal{I}_1, \ldots, \mathcal{I}_n]$, where $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are linear operator ideals, it holds that

$$\beta_{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}(T) \leq \widetilde{\beta}_{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}(T), \; forevery T \in \mathcal{L}(E_1, \ldots, E_n; F).$$

Nevertheless, $\beta_{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}$ and $\widetilde{\beta}_{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}$ do not coincide in general. In fact, if $\varepsilon > \widetilde{\beta}_{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}(T)$, there are a Banach space Z and an operator $R \in [\mathcal{I}_1, \ldots, \mathcal{I}_n](E_1, \ldots, E_n; Z)$ such that, for all $x_1 \in E_1, \ldots, x_n \in E_n$, $\|T(x_1, \ldots, x_n)\|_F \leq \varepsilon \|x_1\| \cdots \|x_n\| + \|R(x_1, \ldots, x_n)\|$
$\leq \varepsilon \|x_1\| \cdots \|x_n\| + \|R_i(x_i)\| \|x_1\| \overset{[i]}{\cdots} \|x_n\|$ whatever $i = 1, \ldots, n$. Therefore,

$$\|T(x_1, \ldots, x_n)\|_F \leq \varepsilon \|x_1\| \cdots \|x_n\| + \min_{i \in \{1, \ldots, n\}} \big\{ \|R_i(x_i)\| \|x_1\| \overset{[i]}{\cdots} \|x_n\| \big\}.$$

This implies $\beta_{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}(T) \leq \widetilde{\beta}_{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}(T)$. However, choosing $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{A}$, the ideal of approximable operators, and taking A the bilinear form considered in Remark 3, we know that $\beta_{[\mathcal{A}, \mathcal{A}]}(A) = 0$ because $A \in [\overline{\mathcal{A}}^{inj}, \overline{\mathcal{A}}^{inj}](E, E; F)$, but $\widetilde{\beta}_{[\mathcal{A}, \mathcal{A}]}(A) > 0$ since $A \notin \overline{[\mathcal{A}, \mathcal{A}]}^{inj}(E, E; F)$.

We note that both measures we have introduced so far, $\beta_{[\mathcal{I}_1, \ldots, \mathcal{I}_n]}$ and $\widetilde{\beta}_{\mathcal{M}_n}$, coincide with the measure $\beta_{\mathcal{I}}$ when $n = 1$ and $\mathcal{I}$ is any ideal of linear operators. For instance, if $\varepsilon > \beta_{\mathcal{I}}(T)$ there exist a Banach space Z and an operator $R \in \mathcal{I}(E; Z)$ such that

$$\|T(x)\|_F \leq \varepsilon \|x\|_E + \|R(x)\|_Z, \; forall x \in E.$$

Thus, for any $m \in N$ and $x^j \in E, j = 1, \dots m$,

$$\Big\| \sum_{j=1}^m T(x^j) \Big\|_F = \Big\| T\Big(\sum_{j=1}^m x^j \Big) \Big\|_F \leq \varepsilon \sum_{j=1}^m \|x^j\|_E + \Big\| \sum_{j=1}^m R(x^j) \Big\|_Z,$$

and so $\varepsilon > \widetilde{\beta}_{\mathcal{I}}(T)$. The other inequality $\beta_{\mathcal{I}}(T) \leq \widetilde{\beta}_{\mathcal{I}}(T)$ is also trivial.

Any $R \in \mathcal{L}(\mathcal{I}_1, \dots, \mathcal{I}_n)(E_1, \dots, E_n; F)$ factors as $R = B \circ (S_1, \dots, S_n)$, for some $S_i \in \mathcal{I}_i(E_i; G_i)$, some $B \in \mathcal{L}(G_1, \dots, G_n; F)$ and some Banach space G_i, $i = 1, \dots, n$. Then

$$\|R(x_1, \dots, x_n)\| \leq \|B\| \|S_1(x_1)\| \cdots \|S_n(x_n)\|.$$

Hence, it is easy to conclude that $\beta_{[\mathcal{I}_1, \dots, \mathcal{I}_n]} \leq \widetilde{\beta}_{\mathcal{L}(\mathcal{I}_1, \dots, \mathcal{I}_n)}$.

Let $\mathcal{M}_n$ be an ideal of n-linear operators. For $T \in \mathcal{L}(E_1, \dots, E_n; F)$, $\widetilde{\beta}_{\mathcal{M}_n}(T) = \inf \Big\{ \varepsilon > 0 : there are a Banach space Z and R \in \mathcal{M}_n(E_1, \dots, E_n; Z) so that$ $\Big\| \sum_{j=1}^m T(x_1^j, \dots, x_n^j) \Big\|_F \leq \varepsilon \cdot \pi \Big(\sum_{j=1}^m x_1^j \otimes \cdots \otimes x_n^j \Big) + \Big\| \sum_{j=1}^m R(x_1^j, \dots, x_n^j) \Big\|,$ for any $m \in N$ and all $x_1^j \in E_1, \dots, x_n^j \in E_n, j = 1, \dots, m \Big\}$.

Clearly, $\widetilde{\beta}_{\mathcal{M}_n}$ is less than or equal to the above infimum. To show the converse inequality take $T \in \mathcal{L}(E_1, \dots, E_n; F)$. Let $\varepsilon > \widetilde{\beta}_{\mathcal{M}_n}(T)$ and let $x_1^j \in E_1, \dots, x_n^j \in E_n, j = 1, \dots, m$. Consider the tensor $\theta = \sum_{j=1}^m x_1^j \otimes \cdots \otimes x_n^j$ and take $\delta > 0$. We can find a representation of $\theta = \sum_{j=1}^l y_1^j \otimes \cdots \otimes y_n^j$ such that

$$\sum_{j=1}^l \|y_1^j\| \cdots \|y_n^j\| \leq \pi(\theta) + \delta.$$

Then, for some Banach space Z and some operator $R \in \mathcal{M}_n(E_1, \dots, E_n; Z)$, we have (adding zeros if necessary) $\Big\| \sum_{j=1}^m T(x_1^j, \dots, x_n^j) \Big\| = \Big\| T_L \Big(\sum_{j=1}^m x_1^j \otimes \cdots \otimes x_n^j \Big) \Big\| = \Big\| T_L \Big(\sum_{j=1}^l y_1^j \otimes \cdots \otimes y_n^j \Big) \Big\|$
$= \Big\| \sum_{j=1}^l T(y_1^j, \dots, y_n^j) \Big\| \leq \varepsilon \sum_{j=1}^l \|y_1^j\| \cdots \|y_n^j\| + \Big\| \sum_{j=1}^l R(y_1^j, \dots, y_n^j) \Big\|$
$\leq \varepsilon(\pi(\theta) + \delta) + \Big\| \sum_{j=1}^m R(x_1^j, \dots, x_n^j) \Big\|$. As δ is arbitrary, the conclusion follows.

As mentioned before, in [8, Theorem 2.4] it is proved that given $T \in \mathcal{L}(E_1, \dots, E_n; F)$, then $\widetilde{\beta}_{\mathcal{M}_n}(T) = 0$ if, and only if, $T \in \overline{\mathcal{M}_n}^{inj}(E_1, \dots, E_n; F)$. Proposition 3 seems to indicate that, given a multi-ideal $\mathcal{M}$, $\widetilde{\beta}_{\mathcal{M}_n}(T)$ is close to $\beta_{\mathcal{M}_1}(T_L)$, and one could even think that both values coincide. This will be the case if $\mathcal{M}$ is closed under linearization, i.e. if $T_L \in \mathcal{M}_1$ whenever $T \in \mathcal{M}$. However, it cannot be ensured in general that $T_L \in \mathcal{M}_1$ when $T \in \mathcal{M}$. Indeed, Botelho [7] gave an example of a p-dominated n-homogeneous polynomial P_0 which is not weakly compact. Therefore, its linearization L_0 is not weakly compact. Given $T \in \mathcal{L}(^nE; F)$ we denote $\widehat{T}$ the polynomial $\widehat{T}(x) = T(x, \dots, x)$. Consider the ideal $\mathcal{M}_0$ of all continuous n-linear operators T such that $\widehat{T}$ is

p-dominated. Then, the unique symmetric n-linear operator T_0 associated to P_0 satisfies that $T_0 \in \mathcal{M}_0$, but its linearization L_0 is not absolutely p-summing, that is, $L_0 \notin (\mathcal{M}_0)_1$.

Let us show a class of multi-ideals $\mathcal{M}$ for which $T_L \in \mathcal{M}_1$ if and only if $T \in \mathcal{M}$. Let $\mathcal{I}$ be a linear operator ideal. The *composition* multi-ideal $\mathcal{I} \circ \mathcal{L}$ is formed by all compositions of continuous multilinear mappings with elements of $\mathcal{I}$; that is, an n-linear operator T belongs to the component $\mathcal{I} \circ \mathcal{L}(E_1, \dots, E_n; F)$ if there are a Banach space G, an n-linear operator $S \in \mathcal{L}(E_1, \dots, E_n; G)$ and a linear operator $R \in \mathcal{I}(G; F)$ such that $T = R \circ S$. By [9, Proposition 3.2] an n-linear operator $T \in \mathcal{L}(E_1, \dots, E_n; F)$ belongs to $\mathcal{I} \circ \mathcal{L}$ if and only if its linearization T_L belongs to $\mathcal{I}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; F)$. Therefore,

$$\widetilde{\beta}_{\mathcal{I} \circ \mathcal{L}}(T) = \beta_{\mathcal{I}}(T_L), for any T \in \mathcal{L}(E_1, \dots, E_n; F).$$

Examples of this kind of composition multi-ideals are the compact multilinear operators, the weakly compact multilinear operators (both as consequences of the work of Pełczyński [27]), that are composition of continuous multilinear operators with compact operators and weakly compact operators respectively, and the factorable strongly p-summing multilinear operators introduced in [28], that can be seen as composition of continuous multilinear operators with absolutely p-summing linear operators (see also [32]). Other examples can be found in [9].

Let $\mathcal{I}$ be a linear operator ideal and let $T \in \mathcal{L}(E_1, \dots, E_n; F)$. The following statements are equivalent:

(a) $T \in \overline{\mathcal{I} \circ \mathcal{L}}^{inj}(E_1, \dots, E_n; F)$.

(b) $T_L \in \overline{\mathcal{I}}^{inj}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; F)$.

(c) $\widetilde{\beta}_{\mathcal{I} \circ \mathcal{L}}(T) = 0$.

(d) $\beta_{\mathcal{I}}(T_L) = 0$.

Let $\mathcal{I}$ be a linear operator ideal. Then, $\overline{\mathcal{I} \circ \mathcal{L}}^{inj} = \overline{\mathcal{I}}^{inj} \circ \mathcal{L}$.

This last Corollary has been already stated in [8, Proposition 4.6].

4 Interpolation properties of the measures

Before of establishing the results of this section, we recall some basic definitions about interpolation theory.

It is said to be that $\bar{A} = (A_0, A_1)$ is a *Banach couple* if A_0 and A_1 are Banach spaces which are continuously embedded in some Hausdorff topological vector space. The spaces $\Sigma(\bar{A}) := A_0 + A_1$ and $\Delta(\bar{A}) := A_0 \cap A_1$ become Banach spaces when endowed with the norms $K(1, \cdot)$ and $J(1, \cdot)$, respectively, where the K and J functionals are defined, for $t > 0$, by

$$K(t, a) = K(t, a; \bar{A}) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \ : \ a = a_0 + a_1, \ a_i \in A_i\}, \ a \in \Sigma(\bar{A}).$$

$$J(t,a) = J(t,a;\bar{A}) := \max\{\|a\|_{A_0},\, t\|a\|_{A_1}\},\, a \in \Delta(\bar{A}).$$

A Banach space A is called an *intermediate space* with respect to $\bar{A} = (A_0, A_1)$ if $\Delta(\bar{A}) \hookrightarrow A \hookrightarrow \Sigma(\bar{A})$, where "$\hookrightarrow$" means continuous inclusion. An intermediate space A with respect to $\bar{A} = (A_0, A_1)$ is said to be *of class* $C_J(\theta;\bar{A})$, where $0 < \theta < 1$, if there exists a constant $C > 0$ such that for all $t > 0$ and $a \in A_0 \cap A_1$,

$$\|a\|_A \leq Ct^{-\theta} J(t,a). \tag{4}$$

The real interpolation space $(A_0, A_1)_{\theta,q}$ and the complex interpolation space $(A_0, A_1)_{[\theta]}$ are important examples of spaces of class $C_J(\theta;\bar{A})$. We refer to the books [5] and [33] for wide information about fundamentals of interpolation theory.

Next we investigate the behavior under interpolation of the measures $\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}$ and $\widetilde{\beta}_{\mathcal{M}_n}$, introduced in Section 3. Our techniques are inspired by ideas used in [11, Theorem 3.3] and [12, Theorem 3.1] (see also [13]) for the linear case.

Let $\mathcal{M}_n$ be any ideal of n-linear operators, let $E_1, \ldots, E_n$ be Banach spaces and let $\bar{F} = (F_0, F_1)$ be a Banach couple. Assume that F is of class $\mathcal{C}_J(\theta, \bar{F})$ with constant C. For $T \in \mathcal{L}(E_1, \ldots, E_n; \Delta(\bar{F}))$, $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) \leq C\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F_0)^{1-\theta}\, \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F_1)^{\theta}$.

Let $\varepsilon_k > \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F_k)$, $k = 0, 1$. Then, for certain Banach space Z_k and n-linear operator $R_k \in \mathcal{M}_n(E_1, \ldots, E_n; Z_k)$, it holds that

$$\Big\|\sum_{j=1}^{m} T(x_1^j, \ldots, x_n^j)\Big\|_{F_k} \leq \varepsilon_k \sum_{j=1}^{m} \|x_1^j\| \ldots \|x_n^j\| + \Big\|\sum_{j=1}^{m} R_k(x_1^j, \ldots, x_n^j)\Big\|_{Z_k} \quad (k = 0, 1),$$

for all $m \in N$ and $x_1^j \in E_1, \ldots, x_n^j \in E_n$, $j = 1, \ldots, m$.
Let $t > 0$. We denote $Z := (Z_0 \oplus Z_1)_1$ and $\varepsilon := \min\{\varepsilon_0, \varepsilon_1\}$. Moreover, we define the n-linear operator $R \in \mathcal{L}(E_1, \ldots, E_n; Z)$ by

$$R(x_1, \ldots, x_n) := \frac{Ct^{-\theta}\max\{\varepsilon_0, t\varepsilon_1\}}{\varepsilon}(R_0(x_1, \ldots, x_n), R_1(x_1, \ldots, x_n)).$$

Since $R = \frac{Ct^{-\theta}\max\{\varepsilon_0, t\varepsilon_1\}}{\varepsilon}(\psi_0 \circ R_0 + \psi_1 \circ R_1)$, where $\psi_k : Z_k \to Z$ is the natural inclusion $(k = 0, 1)$, then $R \in \mathcal{M}_n(E_1, \ldots, E_n; Z)$.
For any $m \in N$ and $x_1^j \in E_1, \ldots, x_n^j \in E_n$, $j = 1, \ldots, m$, it follows from (4) that
$\Big\|\sum_{j=1}^m T(x_1^j, \cdots, x_n^j)\Big\|_F \leq Ct^{-\theta} J\Big(t, \sum_{j=1}^m T(x_1^j, \ldots, x_n^j)\Big)$
$\leq Ct^{-\theta}\max\Big\{t^k\Big(\varepsilon_k \sum_{j=1}^m \|x_1^j\| \cdots \|x_n^j\| + \Big\|\sum_{j=1}^m R_k(x_1^j, \ldots, x_n^j)\Big\|_{Z_k}\Big) : k = 0, 1\Big\}$
$\leq Ct^{-\theta}\max\Big\{t^k\varepsilon_k\Big(\sum_{j=1}^m \|x_1^j\| \cdots \|x_n^j\| + \frac{1}{\varepsilon}\Big\|\sum_{j=1}^m R_k(x_1^j, \ldots, x_n^j)\Big\|_{Z_k}\Big) : k = 0, 1\Big\}$
$\leq Ct^{-\theta}\max\{\varepsilon_0, t\varepsilon_1\}\max\Big\{\sum_{j=1}^m \|x_1^j\| \cdots \|x_n^j\| + \frac{1}{\varepsilon}\Big\|\sum_{j=1}^m R_k(x_1^j, \ldots, x_n^j)\Big\|_{Z_k} : k = 0, 1\Big\}$

$\leq Ct^{-\theta}\max\{\varepsilon_0, t\varepsilon_1\}\Big[\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\|+$

$$+\tfrac{1}{\varepsilon}\Big(\Big\|\sum_{j=1}^m R_0(x_1^j,\ldots,x_n^j)\Big\|_{Z_0}+\Big\|\sum_{j=1}^m R_1(x_1^j,\ldots,x_n^j)\Big\|_{Z_1}\Big)\Big]$$

$= Ct^{-\theta}\max\{\varepsilon_0, t\,\varepsilon_1\}\Big[\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\|+\frac{1}{\varepsilon}\frac{\varepsilon}{Ct^{-\theta}\max\{\varepsilon_0,t\varepsilon_1\}}\Big\|\sum_{j=1}^m R(x_1^j,\ldots,x_n^j)\Big\|_{Z}\Big].$

Therefore, for any $t>0$, it holds that

$$\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F)\leq Ct^{-\theta}\max\Big\{t^k\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F_k):k=0,1\Big\}. \tag{5}$$

We consider three possibilities:

i) If $\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F_0)=0$, then

$$\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F)\leq Ct^{1-\theta}\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F_1)$$

for each $t>0$. Hence, $\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F)=0$.

ii) If $\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F_1)=0$, then

$$\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F)\leq Ct^{-\theta}\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F_0),$$

for every $t>0$, and so $\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F)=0$.

iii) Assume that $\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F_k)>0$ for $k=0,1$. Then, for the particular choice $t:=\dfrac{\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F_0)}{\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F_1)}>0$ in (5), it turns out that $\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F)\leq$ $C\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F_0)^{1-\theta}\,\widetilde{\beta}_{\mathcal{M}_n}(T:E_1\times\cdots\times E_n\to F_1)^{\theta}$.

Let $\mathcal{M}_n$ be a closed injective ideal of n-linear operators. Assume that $\bar{F}=(F_0,F_1)$ is a Banach couple and F is of class $\mathcal{C}_J(\theta,\bar{F})$. For $T\in\mathcal{L}(E_1,\ldots,E_n;\Delta(\bar{F}))$, it follows that $T\in\mathcal{M}_n(E_1,\ldots,E_n;F)$ whenever $T\in\mathcal{M}_n(E_1,\ldots,E_n;F_0)$ or $T\in\mathcal{M}_n(E_1,\ldots,E_n;F_1)$.

Theorem 4 recovers [11, Theorem 3.3] in the particular case $n=1$ and Corollary 4 can be read as a version of [19, Proposition 1.6] in the multilinear case.

We remark that even for $n=1$ a similar result to Theorem 4 does not hold in general if $T\in\mathcal{L}(\Sigma(\bar{E});F)$, where $\bar{E}=(E_0,E_1)$ is a Banach couple and F is a Banach space. To see it, we first recall that if (Ω,Σ) is a measurable space and μ is a σ-finite measure on (Ω,Σ), then it holds with equivalence of norms that (see [5, Theorem 5.3.1])

$$(L_\infty,L_1)_{\theta,q}=L_{p,q},\ for 0<\theta=1/p<1,\ 1\leq q\leq\infty. \tag{6}$$

As usual, the Lorentz space for the particular case $\Omega=[0,1]$ or $\Omega=[0,\infty)$, with the usual Lebesgue measure, will be denoted by $L_{p,q}[0,1]$ or $L_{p,q}[0,\infty)$, respectively.

Take $\mathcal{I}=\mathcal{S}$, the ideal of *strictly singular operators*, which is a closed injective operator ideal (see [22]). Let $\bar{E}=(L_\infty[0,1],L_1[0,1])$, let $F=L_1[0,1]$ and let T be the identity operator. Then $T:L_\infty[0,1]\to L_1[0,1]$ belongs to $\mathcal{I}$ (see

[18]). However, by (6), if $0 < \theta = 1/p < 1$, then $L_p[0,1] = (L_\infty[0,1], L_1[0,1])_{\theta,p}$, but as it was pointed out by Beucher [6, Counterexample 2.4] the operator $T : L_p[0,1] \to L_1[0,1]$ does not belong to the ideal $\mathcal{I}$ since, according to Khintchine's inequality, the span of the Rademacher functions in $L_p[0,1]$ and $L_1[0,1]$ is isomorphic to ℓ_2. Thus, the restriction of the identity operator T to this subspace of $L_p[0,1]$ is an isomorphism into $L_1[0,1]$.

The following result provides an estimate in terms of the measures of the extreme restrictions $T : E_1 \times \cdots \times E_n \to \Delta(\bar{F})$ and $T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F})$. It extends [13, Theorem 3.3]. Note that $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F})) \leq \|T\|_{E_1,\ldots,E_n,\bar{F}} := \max\{\|T\|_{\mathcal{L}(E_1,\ldots,E_n;F_k)} : k = 0,1\}$, and that our proof does not involve duality arguments.

Let $\mathcal{M}_n$ be any ideal of n-linear operators, let $E_1, \ldots, E_n$ be Banach spaces and let $\bar{F} = (F_0, F_1)$ be a Banach couple. Assume that F is of class $\mathcal{C}_J(\theta, \bar{F})$ with constant C. For $T \in \mathcal{L}(E_1, \ldots, E_n; \Delta(\bar{F}))$, $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) \leq 4C\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))^{1-\Theta}\, \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))^{\Theta}$, where $\Theta = \min\{\theta, 1-\theta\}$.

Let $\eta > 0$. We take any $t \geq 1$ such that

$$t^{-\theta} \leq \eta \quad \textit{and} \quad t^{\theta-1} \leq \eta. \tag{7}$$

Let $\sigma > \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))$. Then, it is possible to find a Banach space H and an n-linear operator $R \in \mathcal{M}_n(E_1, \ldots, E_n; H)$ such that

$$\Big\|\sum_{j=1}^m T(x_1^j, \ldots, x_n^j)\Big\|_{\Sigma(\bar{F})} \leq \sigma \sum_{j=1}^m \|x_1^j\| \cdots \|x_n^j\| + \Big\|\sum_{j=1}^m R(x_1^j, \ldots, x_n^j)\Big\|_H, \tag{8}$$

for any $m \in N$ and $x_1^j \in E_1, \ldots, x_n^j \in E_n, j = 1, \ldots, m$.
On the other hand, if $\delta > \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))$ then, for certain Banach space G and n-linear operator $S \in \mathcal{M}_n(E_1, \ldots, E_n; G)$, it holds that

$$\Big\|\sum_{j=1}^m T(x_1^j, \ldots, x_n^j)\Big\|_{\Delta(\bar{F})} \leq \delta \sum_{j=1}^m \|x_1^j\| \cdots \|x_n^j\| + \Big\|\sum_{j=1}^m S(x_1^j, \ldots, x_n^j)\Big\|_G, \tag{9}$$

for every $m \in N$ and $x_1^j \in E_1, \ldots, x_n^j \in E_n, j = 1, \ldots, m$.
Let $\varepsilon > 0$. We define $V := (H \oplus G)_1$ and $P \in \mathcal{L}(E_1, \ldots, E_n; V)$ given by

$$P(x_1, \ldots, x_n) := (2+\varepsilon)t\,(R(x_1, \ldots, x_n), S(x_1, \ldots, x_n)).$$

Due to $P = (2+\varepsilon)t(\psi_0 \circ R + \psi_1 \circ S)$, where $\psi_0 : H \to V$ and $\psi_1 : G \to V$ are the natural inclusions, it follows that $P \in \mathcal{M}_n(E_1, \ldots, E_n; V)$.
For any $m \in N$ and $x_1^j \in E_1, \ldots, x_n^j \in E_n, j = 1, \ldots, m$, there exists a decomposition of $\sum_{j=1}^m T(x_1^j, \ldots, x_n^j)$ as $\sum_{j=1}^m T(x_1^j, \ldots, x_n^j) = y_0 + y_1$, with $y_k \in F_k$ and

$$\|y_k\|_{F_k} \leq \|y_0\|_{F_0} + \|y_1\|_{F_1} \leq (1+\varepsilon)\Big\|\sum_{j=1}^m T(x_1^j, \ldots, x_n^j)\Big\|_{\Sigma(\bar{F})}, \quad k = 0, 1. \tag{10}$$

It follows from (10) and (8) that

$$\|y_k\|_{F_k} \le (1+\varepsilon)\sigma \sum_{j=1}^{m} \|x_1^j\| \cdots \|x_n^j\| + (1+\varepsilon)\Big\| \sum_{j=1}^{m} R(x_1^j,\dots,x_n^j)\Big\|_H, \quad k=0,1. \tag{11}$$

Since also $y_k \in \Delta(\bar F)$, $k=0,1$, using (10) we obtain that $\|y_k\|_{F_{1-k}} = \Big\|\sum_{j=1}^m T(x_1^j,\dots,x_n^j) - y_{1-k}\Big\|_{F_{1-k}} \le \Big\|\sum_{j=1}^m T(x_1^j,\dots,x_n^j)\Big\|_{F_{1-k}} + \|y_{1-k}\|_{F_{1-k}}$
$\le \Big\|\sum_{j=1}^m T(x_1^j,\dots,x_n^j)\Big\|_{\Delta(\bar F)} + (1+\varepsilon)\Big\|\sum_{j=1}^m T(x_1^j,\dots,x_n^j)\Big\|_{\Sigma(\bar F)}$
$\le (2+\varepsilon)\Big\|\sum_{j=1}^m T(x_1^j,\dots,x_n^j)\Big\|_{\Delta(\bar F)}$, for $k=0,1$. By (9),

$$\|y_k\|_{F_{1-k}} \le (2+\varepsilon)\delta \sum_{j=1}^{m} \|x_1^j\| \cdots \|x_n^j\| + (2+\varepsilon)\Big\| \sum_{j=1}^{m} S(x_1^j,\dots,x_n^j)\Big\|_G, \quad k=0,1. \tag{12}$$

Taking into account (4), (7), (11), (12) and the fact that $t \ge 1$, it holds that $\Big\|\sum_{j=1}^m T(x_1^j,\dots,x_n^j)\Big\|_F \le \|y_0\|_F + \|y_1\|_F \le Ct^\theta J(t^{-1},y_0) + Ct^{-\theta}J(t,y_1)$
$\le C\eta t \max\big\{\|y_0\|_{F_0}, t^{-1}\|y_0\|_{F_1}\big\} + C\eta\max\big\{\|y_1\|_{F_0}, t\|y_1\|_{F_1}\big\}$
$\le C\eta t\max\Big\{(1+\varepsilon)\sigma\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\| + (1+\varepsilon)\Big\|\sum_{j=1}^m R(x_1^j,\dots,x_n^j)\Big\|_H,$
$t^{-1}\Big[(2+\varepsilon)\delta\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\| + (2+\varepsilon)\Big\|\sum_{j=1}^m S(x_1^j,\dots,x_n^j)\Big\|_G\Big]\Big\}$
$+ C\eta\max\Big\{(2+\varepsilon)\delta\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\| + (2+\varepsilon)\Big\|\sum_{j=1}^m S(x_1^j,\dots,x_n^j)\Big\|_G,$
$t\Big[(1+\varepsilon)\sigma\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\| + (1+\varepsilon)\Big\|\sum_{j=1}^m R(x_1^j,\dots,x_n^j)\Big\|_H\Big]\Big\}$
$= 2C\eta\max\Big\{(1+\varepsilon)\sigma t\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\| + (1+\varepsilon)t\Big\|\sum_{j=1}^m R(x_1^j,\dots,x_n^j)\Big\|_H,$
$(2+\varepsilon)\delta\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\| + (2+\varepsilon)\Big\|\sum_{j=1}^m S(x_1^j,\dots,x_n^j)\Big\|_G\Big\}$
$\le 2C\eta\max\big\{(1+\varepsilon)\sigma t, (2+\varepsilon)\delta\big\}\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\| +$
$+ (1+\varepsilon)t\Big\|\sum_{j=1}^m R(x_1^j,\dots,x_n^j)\Big\|_H + (2+\varepsilon)\Big\|\sum_{j=1}^m S(x_1^j,\dots,x_n^j)\Big\|_G$
$\le 2C\eta\max\big\{(1+\varepsilon)\sigma t, (2+\varepsilon)\delta\big\}\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\| +$
$+ (2+\varepsilon)t\Big(\Big\|\sum_{j=1}^m R(x_1^j,\dots,x_n^j)\Big\|_H + \Big\|\sum_{j=1}^m S(x_1^j,\dots,x_n^j)\Big\|_G\Big)$
$= 2C\eta\max\big\{(1+\varepsilon)\sigma t, (2+\varepsilon)\delta\big\}\sum_{j=1}^m \|x_1^j\|\cdots\|x_n^j\| + \Big\|\sum_{j=1}^m P(x_1^j,\dots,x_n^j)\Big\|_V.$
Hence,

$$\widetilde{\beta}_{\mathcal{M}_n}(T : E_1\times\cdots\times E_n \to F) \le 2C\eta\max\{(1+\varepsilon)\sigma t, (2+\varepsilon)\delta\}.$$

Therefore

$$\widetilde{\beta}_{\mathcal{M}_n}(T : E_1\times\cdots\times E_n \to F) \le 2C\eta\cdot \quad \cdot\max\{\widetilde{\beta}_{\mathcal{M}_n}(T : E_1\times\cdots\times E_n \to \Sigma(\bar F))\, t, 2\widetilde{\beta}_{\mathcal{M}_n}(T : E_1\times\cdots\times E_n \to \Delta(\bar F))\} \tag{13}$$

We consider the following two cases:

i) If $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F})) = 0$, then $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) = 0$ as well, since η is arbitrary.
ii) Assume that $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F})) > 0$. Note that $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F})) > 0$ too, because $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F})) \leq \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))$. Take

$$\eta := \max\left\{ \left(\frac{\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))}{\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))} \right)^{\theta}, \left(\frac{\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))}{\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))} \right)^{1-\theta} \right\}.$$

The real number

$$t := \frac{\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))}{\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))} \geq 1$$

satisfies (7). If we denote $\Theta := \min\{\theta, 1-\theta\}$ and substitute these concrete choices of η and t in (13), we obtain that $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) \leq 4C\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F})) \cdot$
$\cdot \max\left\{ \left(\frac{\widetilde{\beta}_{\mathcal{M}_n}(T:E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))}{\widetilde{\beta}_{\mathcal{M}_n}(T:E_1 \times \cdots \times E_n \to \Delta(\bar{F}))} \right)^{\theta}, \left(\frac{\widetilde{\beta}_{\mathcal{M}_n}(T:E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))}{\widetilde{\beta}_{\mathcal{M}_n}(T:E_1 \times \cdots \times E_n \to \Delta(\bar{F}))} \right)^{1-\theta} \right\}$
$= 4C\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F})) \left(\frac{\widetilde{\beta}_{\mathcal{M}_n}(T:E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))}{\widetilde{\beta}_{\mathcal{M}_n}(T:E_1 \times \cdots \times E_n \to \Delta(\bar{F}))} \right)^{\Theta}$
$= 4C\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))^{1-\Theta} \, \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))^{\Theta}$.

Let $\mathcal{M}_n$ be a closed injective ideal of n-linear operators. Assume that $\bar{F} = (F_0, F_1)$ is a Banach couple and F is of class $\mathcal{C}_J(\theta, \bar{F})$. For $T \in \mathcal{L}(E_1, \ldots, E_n; \Delta(\bar{F}))$, it follows that $T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$ if and only if $T \in \mathcal{M}_n(E_1, \ldots, E_n; \Sigma(\bar{F}))$.

We can use Corollaries 4 and 4 to establish results on the interpolation of certain classes of multilinear operators. Namely when $\mathcal{M}_n = [\overline{\mathcal{I}_1}^{inj}, \ldots, \overline{\mathcal{I}_n}^{inj}]$, where $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are ideals of linear operators, it holds that $\mathcal{M}_n$ is a closed injective ideal of n-linear operators and so Corollaries 4 and 4 can be applied to $\mathcal{M}_n$. In particular, let us consider $\mathcal{I}$ to be any of the following ideals: (the closed injective ideal of) compact operators, weakly compact operators, strictly singular operators, Rosenthal operators, Banach-Saks operators, or decomposing operators (also called Asplund operators). Then, if $\mathcal{M}_n = [\mathcal{I}, \ldots, \mathcal{I}]$ we obtain an extension to the multilinear case of some interpolation results for these ideals of linear operators established in the literature (see for example [5, Theorem 3.8.1(ii)], [19, Proposition 1.6], [6, Proposition 2.1] and [25, Proposition 5]).

On the other hand, the previous interpolation formulas can be applied to provide, for instance, upper estimates for the measure $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_{p,q}[0,\infty))$ for any Lorentz space $L_{p,q}[0,\infty)$. Thus, because of (6), the following logarithmically convex inequalities hold (for adequate $C > 0$ in each case): $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_{p,q}[0,\infty)) \leq$
$C\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_{\infty}[0,\infty))^{1-\frac{1}{p}} \, \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_1[0,\infty))^{\frac{1}{p}}$
and $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_{p,q}[0,\infty)) \leq$

$C\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_1[0,\infty) \cap L_\infty[0,\infty))^{\max\left\{\frac{1}{p}, 1-\frac{1}{p}\right\}}$ ·
$\cdot\, \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_1[0,\infty) + L_\infty[0,\infty))^{\min\left\{\frac{1}{p}, 1-\frac{1}{p}\right\}}$.

Analogously, taking into account that if $F_0 = L_1[0,\infty) \cap L_\infty[0,\infty)$ and $F_1 = L_1[0,\infty) + L_\infty[0,\infty)$, then

$$(F_0, F_1)_{[\theta]} = \{ L_p[0,\infty) \cap L_{p'}[0,\infty), \;\; 1p = 1-\theta, \; 0 < \theta \leq 1/2, L_p[0,\infty) + L_{p'}[0,\infty), \;\; 1p = 1-\theta, \; 1/2 \leq \theta < 1,$$

where $1/p + 1/p' = 1$ (see [24]), we obtain for $1 < p \leq 2$ (and some $C > 0$) that
$\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_p[0,\infty) \cap L_{p'}[0,\infty)) \leq$
$C\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_1[0,\infty) \cap L_\infty[0,\infty))^{\frac{1}{p}}$ ·
$\cdot\, \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_1[0,\infty) + L_\infty[0,\infty))^{1-\frac{1}{p}}$, and for $2 \leq p < \infty$ (and some $C > 0$) that $\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_p[0,\infty) + L_{p'}[0,\infty)) \leq$
$C\widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_1[0,\infty) \cap L_\infty[0,\infty))^{\frac{1}{p}}$ ·
$\cdot\, \widetilde{\beta}_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to L_1[0,\infty) + L_\infty[0,\infty))^{1-\frac{1}{p}}$.

The following result can be proved in a similar way to Theorems 4 and 4. We include the proof for the sake of completeness.

Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be linear operator ideals, let $E_1, \ldots, E_n$ be Banach spaces and let $\bar{F} = (F_0, F_1)$ be a Banach couple. Assume that F is of class $\mathcal{C}_J(\theta, \bar{F})$ with constant C. For any $T \in \mathcal{L}(E_1, \ldots, E_n; \Delta(\bar{F}))$,

(a) $\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to F) \leq$
$C\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to F_0)^{1-\theta} \beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to F_1)^{\theta}$.

(b) $\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to F) \leq$
$4C\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))^{\Theta} \beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))^{1-\Theta}$,
where $\Theta = \min\{\theta, 1-\theta\}$.

We start by proving (a). Let $\varepsilon_k > \beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to F_k)$, $k = 0, 1$. We have that for Banach spaces Z_i^k and operators $R_i^k \in \mathcal{I}_i(E_i; Z_i^k)$ $(i = 1, \ldots, n)$, it holds that for all $x_1 \in E_1, \ldots, x_n \in E_n$

$$\|T(x_1, \cdots, x_n)\|_{F_k} \leq \varepsilon_k \|x_1\| \cdots \|x_n\| + \min_{i \in \{1,\ldots,n\}} \left\{ \|R_i^k(x_i)\|_{Z_i^k} \|x_1\| \overset{[i]}{\cdots} \|x_n\| \right\}, \; k = 0, 1.$$

We write Z_i for $(Z_i^0 \oplus Z_i^1)_1$ and $\varepsilon := \min\{\varepsilon_0, \varepsilon_1\}$. Take $t > 0$ and consider the operator given by $R_i x := \frac{Ct^{-\theta} \max\{\varepsilon_0, t\varepsilon_1\}}{\varepsilon}(R_i^0 x, R_i^1 x)$, $x \in E_i$ $(i = 1, \ldots, n)$. The operators R_i^0 and R_i^1 belong to $\mathcal{I}_i$, and so R_i also belongs to $\mathcal{I}_i$ $(i = 1, \ldots, n)$.
By (4) we have, for any $x_1 \in E_1, \ldots, x_n \in E_n$, $\|T(x_1, \ldots, x_n)\|_F \leq Ct^{-\theta} J(t, T(x_1, \ldots, x_n))$
$\leq Ct^{-\theta} \max\Big\{ t^k \Big(\varepsilon_k \|x_1\| \cdots \|x_n\| + \min_{i \in \{1,\ldots,n\}} \left\{ \|R_i^k(x_i)\|_{Z_i^k} \|x_1\| \overset{[i]}{\cdots} \|x_n\| \right\} \Big)$:
$k = 0, 1 \Big\}$
$\leq Ct^{-\theta} \max\Big\{ t^k \varepsilon_k \Big(\|x_1\| \cdots \|x_n\| + \frac{1}{\varepsilon} \min_{i \in \{1,\ldots,n\}} \left\{ \|R_i^k(x_i)\|_{Z_i^k} \|x_1\| \overset{[i]}{\cdots} \|x_n\| \right\} \Big)$:
$k = 0, 1 \Big\}$

$\leq Ct^{-\theta}\max\{\varepsilon_0, t\varepsilon_1\}\cdot$
$\cdot \max\Big\{\|x_1\|\cdots\|x_n\| + \frac{1}{\varepsilon}\min_{i\in\{1,\dots,n\}}\big\{\|R_i^k(x_i)\|_{Z_i^k}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\} \;:\; k=0,1\Big\}.$
Thus, $\|T(x_1,\dots,x_n)\|_F \leq Ct^{-\theta}\max\{\varepsilon_0, t\varepsilon_1\}\cdot\Big[\|x_1\|\cdots\|x_n\|+$
$+\frac{1}{\varepsilon}\min_{i\in\{1,\dots,n\}}\Big\{\big(\|R_i^0x_i\|_{Z_i^0}+\|R_i^1x_i\|_{Z_i^1}\big)\|x_1\| \overset{[i]}{\cdots} \|x_n\|\Big\}\Big]$
$= Ct^{-\theta}\max\{\varepsilon_0, t\varepsilon_1\}\cdot\Big[\|x_1\|\cdots\|x_n\|+$
$+\frac{1}{\varepsilon}\frac{\varepsilon}{Ct^{-\theta}\max\{\varepsilon_0,t\varepsilon_1\}}\min_{i\in\{1,\dots,n\}}\big\{\|R_i(x_i)\|_{Z_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}\Big]$
$= Ct^{-\theta}\max\{\varepsilon_0, t\,\varepsilon_1\}\|x_1\|\cdots\|x_n\|+\min_{i\in\{1,\dots,n\}}\big\{\|R_i(x_i)\|_{Z_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}$. So we obtain

$$\beta_{[\mathcal{I}_1,\dots,\mathcal{I}_n]}(T: E_1\times\cdots\times E_n\to F) \leq Ct^{-\theta}\max\Big\{t^k\beta_{[\mathcal{I}_1,\cdots,\mathcal{I}_n]}(T: E_1\times\cdots\times E_n\to F_k) : k=0,1\Big\},$$

for any $t>0$. Finally using an analogous reasoning to that used in the last part of the proof of Theorem 4, the estimate given in (a) is proved.

Now we establish (b). Fix $\eta>0$ and let $t\geq 1$ be such that

$$t^{-\theta}\leq\eta \quad and \quad t^{\theta-1}\leq\eta. \tag{14}$$

Consider $\sigma>\beta_{[\mathcal{I}_1,\dots,\mathcal{I}_n]}(T: E_1\times\cdots\times E_n\to\Sigma(\bar F))$. We can find Banach spaces H_i and operators $R_i\in\mathcal{I}_i(E_i;H_i)$ $(i=1,\dots,n)$ so that, for all $x_1\in E_1,\dots,x_n\in E_n$,

$$\|T(x_1,\dots,x_n)\|_{\Sigma(\bar F)} \leq \sigma\|x_1\|\cdots\|x_n\| + \min_{i\in\{1,\dots,n\}}\big\{\|R_i(x_i)\|_{H_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}. \tag{15}$$

Moreover, if $\delta>\beta_{[\mathcal{I}_1,\dots,\mathcal{I}_n]}(T: E_1\times\cdots\times E_n\to\Delta(\bar F))$ then there are Banach spaces G_i and operators $S_i\in\mathcal{I}_i(E_i;G_i)$ $(i=1,\dots,n)$ for which we have, for $x_1\in E_1,\dots,x_n\in E_n$, that

$$\|T(x_1,\dots,x_n)\|_{\Delta(\bar F)} \leq \delta\|x_1\|\cdots\|x_n\| + \min_{i\in\{1,\dots,n\}}\big\{\|S_i(x_i)\|_{G_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}. \tag{16}$$

Now for $\varepsilon>0$ and $x_1\in E_1,\dots,x_n\in E_n$, $T(x_1,\dots,x_n)$ can be written as $T(x_1,\dots,x_n)=y_0+y_1$, $y_k\in F_k$, and

$$\|y_k\|_{F_k}\leq\|y_0\|_{F_0}+\|y_1\|_{F_1}\leq(1+\varepsilon)\|T(x_1,\dots,x_n)\|_{\Sigma(\bar F)}, \quad k=0,1, \tag{17}$$

what implies, by (17) and (15), that

$$\|y_k\|_{F_k}\leq(1+\varepsilon)\sigma\|x_1\|\cdots\|x_n\|+(1+\varepsilon)\min_{i\in\{1,\dots,n\}}\big\{\|R_i(x_i)\|\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}, k=0,1. \tag{18}$$

Taking into account that also $y_k\in\Delta(\bar F)$, and using (17), we get that

$$\|y_k\|_{F_{1-k}}\leq\|T(x_1,\dots,x_n)\|_{F_{1-k}}+\|y_{1-k}\|_{F_{1-k}}\leq(2+\varepsilon)\|T(x_1,\dots,x_n)\|_{\Delta(\bar F)},\ k=0,1.$$

By (16) we have finally, for $k = 0, 1$,

$$\|y_k\|_{F_{1-k}} \le (2+\varepsilon)\delta\|x_1\|_{E_1} \cdots \|x_n\|_{E_n} + (2+\varepsilon) \min_{i\in\{1,\ldots,n\}} \big\{\|S_i(x_i)\|\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}. \tag{19}$$

Therefore, for any $x_1 \in E_1, \ldots, x_n \in E_n$, we obtain from (4), (14), (18), (19) and the fact that $t \ge 1$ $\|T(x_1, \ldots, x_n)\|_F \le \|y_0\|_F + \|y_1\|_F \le Ct^{\theta} J(t^{-1}, y_0) + Ct^{-\theta} J(t, y_1)$
$\le C\eta t \max\big\{\|y_0\|_{F_0}, t^{-1}\|y_0\|_{F_1}\big\} + C\eta \max\big\{\|y_1\|_{F_0}, t\|y_1\|_{F_1}\big\}$
$\le C\eta t \max\Big\{(1+\varepsilon)\sigma\|x_1\| \cdots \|x_n\| + (1+\varepsilon)\min_{i\in\{1,\ldots,n\}} \big\{\|R_i(x_i)\|_{H_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\},$
$t^{-1}\big[(2+\varepsilon)\delta\|x_1\| \cdots \|x_n\| + (2+\varepsilon)\min_{i\in\{1,\ldots,n\}} \big\{\|S_i(x_i)\|_{G_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}\big]\Big\}$
$+C\eta \max\Big\{(2+\varepsilon)\delta\|x_1\| \cdots \|x_n\| + (2+\varepsilon)\min_{i\in\{1,\ldots,n\}} \big\{\|S_i(x_i)\|_{G_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\},$
$t\big[(1+\varepsilon)\sigma\|x_1\| \cdots \|x_n\| + (1+\varepsilon)\min_{i\in\{1,\ldots,n\}} \big\{\|R_i(x_i)\|_{H_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}\big]\Big\}$
$= 2C\eta \max\Big\{(1+\varepsilon)\sigma t\|x_1\| \cdots \|x_n\| + (1+\varepsilon)t\min_{i\in\{1,\ldots,n\}} \big\{\|R_i(x_i)\|_{H_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\},$
$(2+\varepsilon)\delta\|x_1\| \cdots \|x_n\| + (2+\varepsilon)\min_{i\in\{1,\ldots,n\}} \big\{\|S_i(x_i)\|_{G_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}\Big\}$
$\le 2C\eta \max\big\{(1+\varepsilon)\sigma t, (2+\varepsilon)\delta\big\}\|x_1\| \cdots \|x_n\| +$
$+ (2+\varepsilon)t\min_{i\in\{1,\ldots,n\}} \big\{\big(\|R_i(x_i)\|_{H_i} + \|S_i(x_i)\|_{G_i}\big)\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}$
$= 2C\eta \max\big\{(1+\varepsilon)\sigma t, (2+\varepsilon)\delta\big\}\|x_1\| \cdots \|x_n\| + \min_{i\in\{1,\ldots,n\}} \big\{\|U_i(x_i)\|_{V_i}\|x_1\| \overset{[i]}{\cdots} \|x_n\|\big\}$, where $V_i := (H_i \oplus G_i)_1$ and $U_i x := (2+\varepsilon)t(R_i x, S_i x)$ for all $x \in E_i\,(i = 1, \ldots, n)$. Since $R_i \in \mathcal{I}_i(E_i; H_i)$ and $S_i \in \mathcal{I}_i(E_i; G_i)$, it follows that $U_i \in \mathcal{I}_i(E_i; V_i)$ for every $i = 1, \ldots, n$, and then

$$\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to F) \le 2C\eta \max\{(1+\varepsilon)\sigma t, (2+\varepsilon)\delta\}.$$

Whence $\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to F) \le 2C\eta \cdot$
$\cdot \max\{\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))t, 2\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))\}$. Now similar arguments to those used in the final part of the proof of Theorem 4 allow to establish the validity of (b).

Using Theorem 4, analogous estimates to those obtained just before that theorem for $\widetilde{\beta}_{\mathcal{M}_n}$ also hold for the measure $\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}$.

5 Some examples and applications related to summing operators

It is well-known that the notion of summing operator can be generalized in different ways to the multilinear setting, and each of them has shown to be useful depending on the particular application. We will center our attention in generalizations that allow to get factorization theorems for the corresponding multilinear map. These are mainly variants of dominated multilinear operators and factorable summing multilinear operators.

5.1 Examples of multilinear operators belonging to the closed injective hull of summing multilinear operators

Recall that $T \in \mathcal{L}(E; F)$ is *absolutely summing* if there exists $K > 0$ such that

$$\sum_{k=1}^{n} \|T(x_k)\| \leq K \sup_{x^* \in B_{E^*}} \Big(\sum_{k=1}^{n} |\langle x_k, x^* \rangle| \Big)$$

for every finitely many $x_1, \ldots, x_n \in E$. The set of all absolutely summing linear operators, which is denoted by Π_1, is an injective Banach linear operator ideal (see for instance [20, Theorem 19.5.3] or [15, Chapter I, Section 11]). We will use the following useful characterization [20, Corollary 20.7.5] (see also [30, Theorem 17.3.2]):

The following assertions are equivalent for any $T \in \mathcal{L}(E; F)$.

(a) $T \in \overline{\Pi_1}^{inj}(E; F)$.

(b) There is a function $N : R^+ \to R^+$ and a regular Borel probability measure η on B_{E^*} such that

$$\|T(x)\| \leq N(\varepsilon) \int_{B_{E^*}} |\langle x, x^* \rangle| \, d\eta(x^*) + \varepsilon \|x\|, \quad \text{for each } \varepsilon > 0 \text{ and } x \in E. \tag{20}$$

The characterization given by Lemma 5.1 allows to define a measure associated to the ideal of absolutely summing operators as follows: given $T \in \mathcal{L}(E; F)$, consider the function

$$\beta_{\mathcal{N}}(T) := \inf \Big\{ \varepsilon > 0 : \|T(x)\| \leq N(\varepsilon) \int_{B_{E^*}} |\langle x, x^* \rangle| d\eta(x^*) + \varepsilon \|x\| \, , \, x \in E, \; \textit{for a given function}$$
$$N : R^+ \to R^+ \textit{ and a given regular Borel probability measure } \eta \textit{ both depending only on } T \Big\}.$$

Consequently, T belongs to $\overline{\Pi}_1^{inj}$ if and only $\beta_{\mathcal{N}}(T) = 0$.

Jarchow and Matter [21] considered that concrete choices of the function N in Lemma 5.1(b) provide better descriptions of classes of operators that are included in the closed injective hull of the ideal of summing operators. For any $0 < \sigma < 1$ and $K > 0$ let $r = \sigma/(1 - \sigma)$ and define $N_1(\varepsilon) = \frac{K}{\varepsilon^r}$. It is not hard to check that (20) is equivalent to (see [21, p.47] and also [26, p.195]) $\|T(x)\| \leq K \|x\|^{\sigma} \Big(\int_{B_{E^*}} |\langle x, x^* \rangle| d\eta(x^*) \Big)^{1-\sigma}$, $forall x \in E$. This last inequality defines, for $0 \leq \sigma < 1$, the class of $(1, \sigma)$-absolutely continuous operators. Therefore, the class of $(1, \sigma)$-absolutely continuous operators, that trivially contains the class Π_1 of all absolutely summing operators, is actually contained in its closed injective hull $\overline{\Pi_1}^{inj}$. A similar treatment can be done for arbitrary $1 \le p < \infty$. The class $\Pi_{(p,\sigma)}$ is formed by all (p,σ)-absolutely continuous operators, that is, all $T \in \mathcal{L}(E;F)$ for which there exist a constant $K > 0$ and a regular Borel probability measure η such that

$$\|T(x)\| \le K\|x\|^{\sigma}\Big(\int_{B_{E^*}} |\langle x, x^*\rangle|^p d\eta(x^*)\Big)^{(1-\sigma)/p},\ foranyx \in E.$$

In this general case, Lemma 5.1 reads as follows (see [20, Corollary 20.7.5]): T belongs to $\overline{\Pi_p}^{inj}(E;F)$ if and only if there is a function $N : R^+ \to R^+$ and a regular Borel probability measure η on B_{E^*} such that

$$\|T(x)\| \le N(\varepsilon)\Big(\int_{B_{E^*}} |\langle x, x^*\rangle|^p\, d\eta(x^*)\Big)^{1/p} + \varepsilon\|x\|,\ foreach$$

$\varepsilon > 0$ and $x \in E$.

Replacing

N(ε) with a suitable $N_1(\varepsilon)$ and doing similar calculations as for $p = 1$, we get that $\Pi_{(p,\sigma)} \subset \overline{\Pi_p}^{inj}$.

Now we use this information in the multilinear case. Take $1 \le p \le p_1, \ldots, p_n < \infty$ such that $1/p = \sum_{i=1}^n 1/p_i$ and $0 \le \sigma < 1$. According [14, Theorem 3.3], an n-linear operator $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is $(p; p_1, \ldots, p_n; \sigma)$-*absolutely continuous* (in symbols $T \in \mathcal{L}^{\sigma}_{as,(p;p_1,\cdots,p_n)}$) if there are regular Borel probability measures $\mu_1, \ldots, \mu_n$ on $B_{E_1^*}, \ldots, B_{E_n^*}$, respectively, and a constant $K > 0$, in such a way that for every $x_1 \in E_1, \ldots, x_n \in E_n$,

$$\|T(x_1, \ldots, x_n)\| \le K \prod_{i=1}^{n} \|x_i\|^{\sigma}\Big(\int_{B_{E_i^*}} \big|\langle x_i, x_i^*\rangle\big|^{p_i}\, d\mu_i(x_i^*)\Big)^{(1-\sigma)/p_i}.$$

The infimum of all $K > 0$ is the norm $\|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_n)}}$. The linear case gives directly the inclusion:

$$\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_n)} \subset [\overline{\Pi}^{inj}_{p_1}, \ldots, \overline{\Pi}^{inj}_{p_n}].$$

Therefore, for every $T \in \mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_n)}$, it holds that

$$\beta_{[\Pi_{p_1},\ldots,\Pi_{p_n}]}(T) = \widetilde{\beta}_{[\overline{\Pi}^{inj}_{p_1},\ldots,\overline{\Pi}^{inj}_{p_n}]}(T) = 0.$$

Just as some examples, let us apply these ideas to get new classes of operators contained in $\overline{\Pi_1}^{inj}$.

(a) We start by considering a function N_0 such that $N_0 < N_1$. Then, when replacing N with N_0 in Lemma 5.1(b), we obtain a new class of operators Π_{N_0} that is contained in $\Pi_{(1,\sigma)}$ for all $0 < \sigma < 1$. This is the case if we take, for instance, $N_0(\varepsilon) = K\log(\frac{1}{\varepsilon})$, as we next show. Fix $x \in E$. The function $\phi_0(\varepsilon) = N_0(\varepsilon)\int_{B_{E^*}} |\langle x, x^*\rangle|\, d\eta(x^*) + \varepsilon\|x\|$ has a minimum at

$$\varepsilon_x := \frac{K\int_{B_{E^*}} |\langle x, x^*\rangle| d\eta(x^*)}{\|x\|}$$

and

$$\phi_0(\varepsilon_x) = K\Big(\log \frac{\|x\|}{K\int_{B_{E^*}} |\langle x, x^*\rangle| d\eta(x^*)} + 1\Big)\int_{B_{E^*}} |\langle x, x^*\rangle| d\eta(x^*).$$

Note that the above holds for arbitrary $x \in E$ with $\|T(x)\| \neq 0$. Therefore, the new class Π_{N_0} is defined by all $T \in \mathcal{L}(E; F)$ for which there is a probability measure η such that

$$\|T(x)\| \le K\Big(\log \frac{\|x\|}{K\int_{B_{E^*}} |\langle x, x^*\rangle| d\eta(x^*)} + 1\Big)\int_{B_{E^*}} |\langle x, x^*\rangle| d\eta(x^*)$$

for all $x \in E$ with $\|T(x)\| \neq 0$. Note that this domination shows in particular that $\Pi_1 \subset \Pi_{N_0}$.

(b) If we consider a function N_2 with $N_1 < N_2$, then we get a new class of linear operators that contains $\Pi_{(1,\sigma)}$ for all $0 < \sigma < 1$ but it is still contained in $\overline{\Pi_1}^{inj}$. For instance, take $N_2(\varepsilon) = Ke^{\frac{1}{\varepsilon}}$. Fix $x \in E$ with $T(x) \neq 0$. In this case, it is not possible to give an explicit formula for the point ε_x where the function $\phi_2(\varepsilon) = N_2(\varepsilon)\int_{B_{E^*}} |\langle x, x^*\rangle|\, d\eta(x^*) + \varepsilon\|x\|$ attains its minimum, as this point is given by the solution of the equation

$$Ke^{1/\varepsilon_x}\int_{B_{E^*}} |\langle x, x^*\rangle| d\eta(x^*) = \varepsilon_x^2\|x\|.$$

The next example refers to the multilinear case.

(a) Using arguments as in the second part of the proof of Theorem 3 and taking into account Lemma 5.1, we obtain the following: Let $E_1, \ldots, E_n$ be Banach spaces and let $T \in \mathcal{L}(E_1, \ldots, E_n; F)$. If there are functions $N_i : R^+ \to R^+$ and regular Borel probability measures η_i on $B_{E_i^*}$ $(i = 1, \ldots, n)$ such that $\|T(x_1, \ldots, x_n)\| \le \Big(N_1(\varepsilon_1)\int_{B_{E_1^*}} |\langle x_1, x_1^*\rangle| d\eta_1(x_1^*) + \varepsilon_1\|x_1\|\Big)\cdots$ $\cdots\Big(N_n(\varepsilon_n)\int_{B_{E_n^*}} |\langle x_n, x_n^*\rangle| d\eta_n(x_n^*) + \varepsilon_n\|x_n\|\Big)$, for all $\varepsilon_1 > 0, \ldots, \varepsilon_n > 0$ and $x_1 \in E_1, \ldots, x_n \in E_n$, then $T \in [\overline{\Pi_1}^{inj}, \ldots, \overline{\Pi_1}^{inj}](E_1, \ldots, E_n; F)$.

(b) When we now take, for instance, the functions N_i as the function N_0 considered in Example 5.1(a) we deduce in particular: Let $E_1, \ldots, E_n$ be Banach spaces and let $T \in \mathcal{L}(E_1, \ldots, E_n; F)$. If there are constants $K_1 > 0, \ldots, K_n > 0$ and regular Borel probability measures $\eta_1, \ldots, \eta_n$ on $B_{E_1^*}, \ldots, B_{E_n^*}$, respectively, such that $\|T(x_1, \ldots, x_n)\| \le K_1\Big(\log \frac{\|x_1\|_{E_1}}{K_1\int_{B_{E_1^*}} |\langle x_1, x_1^*\rangle| d\eta_1(x_1^*)} + 1\Big)\int_{B_{E_1^*}} |\langle x_1, x_1^*\rangle| d\eta_1(x_1^*)\cdots$ $\cdots K_n\Big(\log \frac{\|x_n\|_{E_n}}{K_n\int_{B_{E_n^*}} |\langle x_n, x_n^*\rangle| d\eta_n(x_n^*)} + 1\Big)\int_{B_{E_n^*}} |\langle x_n, x_n^*\rangle| d\eta_n(x_n^*)$, for all $x_1 \in E_1, \ldots, x_n \in E_n$, then $T \in [\overline{\Pi_1}^{inj}, \ldots, \overline{\Pi_1}^{inj}](E_1, \ldots, E_n; F)$.

5.2 Interpolation and closed injective hull of summing multilinear operators

The application of the corresponding interpolation formulas obtained in Section 4 allows to relate some classes of multilinear operators considered in Section 5. Let us finish the paper by showing a concrete example of this, concerning interpolation and multilinear operators belonging to $\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_n)}$ with values in Lorentz spaces. We will use (6). Direct consequences of Theorem 4 and Theorem 3 are the following results.

Let $\mathcal{I}_1,\ldots,\mathcal{I}_n$ be linear operator ideals and let $E_1,\ldots,E_n$ be Banach spaces. Suppose that $T\in\mathcal{L}(E_1,\ldots,E_n;L_\infty\cap L_1)$. For $0<\theta=1/p<1$, $1\le q\le\infty$, there is $C>0$ such that $\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T:E_1\times\cdots\times E_n\to L_{p,q})\le C\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T:E_1\times\cdots\times E_n\to L_\infty)^{1-\theta}\beta_{[\mathcal{I}_1,\ldots,\mathcal{I}_n]}(T:E_1\times\cdots\times E_n\to L_1)^{\theta}$.

Let $1/p=1/p_1+\cdots+1/p_n$, $1<p,p_i<\infty$, $\theta=1/p$ and $1\le q\le\infty$. Let $T\in\mathcal{L}(E_1,\ldots,E_n;L_\infty\cap L_1)$, with $T\in\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_n)}(E_1,\ldots,E_n;L_\infty)$ or $T\in\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_n)}(E_1,\ldots,E_n;L_1)$. Then $T\in[\overline{\Pi}^{inj}_{p_1},\ldots,\overline{\Pi}^{inj}_{p_n}](E_1,\ldots,E_n;L_{p,q})$. If for example $T\in\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_n)}(E_1,\ldots,E_n;L_1)$, then $T\in[\overline{\Pi}^{inj}_{p_1},\ldots,\overline{\Pi}^{inj}_{p_n}](E_1,\ldots,E_n;L_1)$. By Theorem 3,

$$\beta_{[\Pi_{p_1},\ldots,\Pi_{p_n}]}(T:E_1\times\cdots\times E_n\to L_1)=0.$$

It follows from Corollary 5.2 that

$$\beta_{[\Pi_{p_1},\ldots,\Pi_{p_n}]}(T:E_1\times\cdots\times E_n\to L_{p,q})=0,$$

and so, again by Theorem 3, we get the result.

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