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On fixed point index theory for the sum of operators and applications to a class of ODEs and PDEs

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Abstract

The aim of this work is two fold: first we extend some results concerning the computation of the fixed point index for the sum of an expansive mapping and a k-set contraction obtained in [3, 6], to the case of the sum $T+F$, where T is a mapping such that $(I-T)$ is Lipschitz invertible and F is a k-set contraction. Secondly, as illustration of some our theoretical results, we study the existence of non-negative solutions for two classes of differential equations, covering a class of first-order ordinary differential equations (ODEs for short) posed on the non-negative half-line as well as a class of partial differential equations (PDEs for short).

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1. Preliminaries

Many problems in science lead to nonlinear equations $Tx + Fx = x$ posed in some closed convex subset of a Banach space. In particular, ordinary, fractional, partial differential equations and integral equations can be formulated

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like these abstract equations. It is the reason for which it becomes desirable to develop fixed point theorems for such equations. When T is compact and F is a contraction there are many classical tools to deal with such problems (see [2], [5], [9], [11] and references therein). The main aim of this paper is to give some recent results for existence of fixed points for some operators that are of the form $T+F$, where T is an expansive operator and F is a k-set contraction. The positivity of solutions of nonlinear equations, especially ordinary, partial differential equations, and integral equations is a very important issue in applications, where a positive solution may represent a density, a temperature, a velocity, etc.

In this paper we extend some results concerning the computation of the fixed point index for the sum of an expansive mapping and a k-set contraction, obtained in [1, 3, 4, 8, 6, 7], to the case when T is a mapping such that $(I-T)$ is Lipschitz invertible and F is a k-set contraction. We illustrate some of our theoretical results. More precisely, we study the existence of non-negative solutions for the following IVP

$$x' = f(t,x), \quad t > 0,$$

$$x(0) = x_0,$$

where $x_0 \in \mathbb{R}$ is a given constant, $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying a general polynomial growth condition.

Moreover, we consider an application for an IVP subject to Burgers-Fisher equation:

$$u_t - u_{xx} + \alpha(t) u u_x = \beta(t) u(1-u), \quad t > 0, \quad x \geq 0,$$

$$u(0,x) = u_0(x), \quad x \geq 0,$$

where $u_0 \in \mathcal{C}^2([0,\infty))$ and $\alpha, \beta \in \mathcal{C}([0,\infty))$ with $\alpha < 0$, $\beta \geq 0$ on $[0,\infty)$.

The paper is organized as follows. In the next section, we give some auxiliary results. In sections 3 and 4, we will present our contribution in fixed point index theory for the sum of two operators of the form $T+F$, where T is a mapping such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$ and F is a k-set contraction when $0 \leq k < \gamma^{-1}$. We will consider separately two cases: firstly the computation of fixed point index on cones is treated in Section 3. Then in Section 4, we will discuss the computation of fixed point index on translates of cones. Applications are given in sections 4 and 5.

2. Auxiliary results

Let X be a linear normed space and I be the identity map of X. The following Lemmas give sufficient conditions for $I-T$ to be Lipschitz invertible.

Lemma 2.1 ([12, Lemma 2.1])**.** *Let $(X, \|.\|)$ be a normed linear space, $D \subset X$. If a mapping $T : D \to X$ is expansive with a constant $h > 1$, then the mapping*

$I-T: D \to (I-T)(D)$ is invertible and

$$\|(I-T)^{-1}x-(I-T)^{-1}y\| \leq (h-1)^{-1}\|x-y\| \text{ for all } x,y \in (I-T)(D).$$

Lemma 2.2 ([13, Lemma 2.3]). *Let $(E, \|.\|)$ be a Banach space and $T: E \to E$ be Lipschitzian map with constant $\beta > 0$. Assume that for each $z \in E$, the map $T_z : E \to E$ defined by $T_z x = Tx + z$ satisfies that T_z^p is expansive and onto for some $p \in \mathbb{N}$. Then $(I-T)$ maps E onto E, the inverse of $I-T: E \to E$ exists, and*

$$\|(I-T)^{-1}x-(I-T)^{-1}y\| \leq \gamma_p\|x-y\| \text{ for all } x,y \in E,$$

where

$$\gamma_p = \frac{\beta^p - 1}{(\beta-1)(lip(T^p)-1)}.$$

Lemma 2.3 ([13, Lemma 2.5]). *Let $(X, \|.\|)$ be a linear normed space, $M \subset X$. Assume that the mapping $T: M \to X$ is contractive with a constant $k < 1$, then the inverse of $I-T: M \to (I-T)(M)$ exist, and*

$$\|(I-T)^{-1}x-(I-T)^{-1}y\| \leq (1-k)^{-1}\|x-y\| \quad \text{for all } x,y \in (I-T)(M).$$

Lemma 2.4 ([13, Lemma 2.6]). *Let $(E, \|.\|)$ be a Banach space and $T: E \to E$ be Lipschitzian map with constant $\beta \geq 0$. Assume that for each $z \in E$, the map $T_z : E \to E$ defined by $T_z x = Tx + z$ satisfies that T_z^p is contractive for some $p \in \mathbb{N}$. Then $(I-T)$ maps E onto E, the inverse of $I-T: E \to E$ exists, and*

$$\|(I-T)^{-1}x-(I-T)^{-1}y\| \leq \rho_p\|x-y\| \text{ for all } x,y \in E,$$

where

$$\rho_p = \begin{cases} \frac{p}{1-Lip(T^p)}, & \text{if } \beta = 1; \\ \frac{1}{1-\beta}, & \text{if } \beta < 1; \\ \frac{\beta^p-1}{(\beta-1)(1-Lip(T^p))}, & \text{if } \beta > 1. \end{cases}$$

3. Fixed point index on cones

In all what follows, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$, Ω is a subset of $\mathcal{P}$, and U is a bounded open subset of $\mathcal{P}$. For $r > 0$ define the conical shell

$$\mathcal{P}_r = \mathcal{P} \bigcap \{x \in E : \|x\| < r\}.$$

Assume that $T : \Omega \to E$ is a mapping such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$ and $F : \overline{U} \to E$ is a k-set contraction.

Suppose that

$$0 \leq k < \gamma^{-1}, \tag{3.1}$$

$$F(\overline{U}) \subset (I-T)(\Omega), \tag{3.2}$$

and

$$x \neq Tx + Fx, \text{ for all } x \in \partial U \bigcap \Omega. \tag{3.3}$$

Then $x \neq (I-T)^{-1}Fx$, for all $x \in \partial U$ and the mapping $(I-T)^{-1}F : \overline{U} \to \mathcal{P}$ is a strict γk-set contraction. Indeed, $(I-T)^{-1}F$ is continuous and bounded; and for any bounded set B in U, we have

$$\alpha(((I-T)^{-1}F)(B)) \leq \gamma\, \alpha(F(B)) \leq \gamma k \alpha(B).$$

The fixed point index $i\,((I-T)^{-1}F, U, \mathcal{P})$ is so well defined. Thus we put

$$i_*\,(T+F, U \bigcap \Omega, \mathcal{P}) = i\,((I-T)^{-1}F, U, \mathcal{P}). \tag{3.4}$$

Proposition 3.1. *Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $tF(\overline{U}) \subset (I-T)(\Omega)$ for all $t \in [0,1]$. If $(I-T)^{-1}0 \in U$, and*

$$x - Tx \neq \lambda Fx \ \text{ for all } x \in \partial U \bigcap \Omega \ \text{ and } 0 \leq \lambda \leq 1, \tag{3.5}$$

then the fixed point index $i_*\,(T+F, U \bigcap \Omega, \mathcal{P}) = 1$.

Proof. Consider the homotopic deformation $H : [0,1] \times \overline{U} \to \mathcal{P}$ defined by

$$H(t,x) = (I-T)^{-1}tFx.$$

The operator H is continuous and uniformly continuous in t for each x. Moreover, $H(t,.)$ is a strict k-set contraction for each t and the mapping $H(t,.)$ has no fixed point on ∂U. Otherwise, there would exist some $x_0 \in \partial U \bigcap \Omega$ and $t_0 \in [0,1]$ such that

$$x_0 - Tx_0 = t_0 F x_0,$$

which contradicts our assumption.
From the invariance under homotopy and the normalization property of the index fixed point, we deduce that

$$i_*\,((I-T)^{-1}F, U, \mathcal{P}) = i_*\,((I-T)^{-1}0, U, \mathcal{P}) = 1.$$

Consequently, from (3.4), we deduce that

$$i_*\,(T+F, U \bigcap \Omega, \mathcal{P}) = 1,$$

which completes the proof. □

As a consequence of Proposition 3.1 , we have the two following results.

Corollary 3.2. *Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $tF(\overline{U}) \subset (I-T)(\Omega)$ for all $t \in [0,1]$. If $(I-T)^{-1}0 \in U$, and*

$$\|Fx\| \leq \|x - Tx\| \ \text{ and } Tx + Fx \neq x \ \text{ for all } x \in \partial U \bigcap \Omega,$$

then the fixed point index $i_*\,(T+F, U \bigcap \Omega, \mathcal{P}) = 1$.

Proof. It is sufficient to prove that Assumption (3.5) is satisfied. □

Corollary 3.3. *Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \le k < \gamma^{-1}$, and $tF(\overline{U}) \subset (I-T)(\Omega)$ for all $t \in [0,1]$. If $(I-T)^{-1}0 \in U$,*

$$Fx \in \mathcal{P} \;\; \textit{for all} \;\; x \in \partial U \bigcap \Omega,$$

and

$$Fx \not\geq x - Tx \quad \textit{for all} \quad x \in \partial U \bigcap \Omega,$$

then the fixed point index $i_*\left(T+F, \bigcap \Omega, \mathcal{P}\right) = 1$.

Proof. It is easy to see that Assumption (3.5) is satisfied. □

Proposition 3.4. *Let U be a bounded open subset of $\mathcal{P}$ with $0 \in U$. Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \le k < \gamma^{-1}$, and $F(\overline{U}) \subset (I-T)(\Omega)$. If*

$$Fx \neq (I-T)(\lambda x) \;\; \textit{for all } x \in \partial U,\ \lambda \ge 1 \;\; \textit{and} \;\; \lambda x \in \Omega,$$

then the fixed point index $i_*\left(T+F, U \bigcap \Omega, \mathcal{P}\right) = 1$.

Proof. The mapping $(I-T)^{-1}F : \overline{U} \to \mathcal{P}$ is a strict γk-set contraction and it is readily seen that the following condition of Leray-Schauder type is satisfied

$$(I-T)^{-1}Fx \neq \lambda x, \;\; \text{for all } x \in \partial U \;\; \text{and } \lambda \ge 1.$$

In fact, if there exist $x_0 \in \partial U$ and $\lambda_0 \ge 1$ such that $(I-T)^{-1}Fx_0 = \lambda_0 x_0$. Then $Fx_0 = (I-T)(\lambda_0 x_0)$, which contradicts our assumption. The claim then follows from (3.4) and [8, Theorem 1.3.7]. □

Proposition 3.5. *Let U be a bounded open subset of $\mathcal{P}$ with $0 \in U \bigcap \Omega$. Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \le k < \gamma^{-1}$, and $F(\overline{U}) \subset (I-T)(\Omega)$. If*

$$\gamma \|Fx + T0\| \le \|x\| \;\; \textit{and} \;\; Tx + Fx \neq x \;\; \textit{for all } x \in \partial U \bigcap \Omega, \tag{3.6}$$

then the fixed point index $i_*\left(T+F, U \bigcap \Omega, \mathcal{P}\right) = 1$.

Proof. The mapping $(I-T)^{-1}F : \overline{U} \to \mathcal{P}$ is a strict γk-set contraction.
$(I-T)$ being Lipschitz invertible with constant $\gamma > 0$, for each $x \in \overline{U}$

$$\begin{array}{rcl} \|(I-T)^{-1}Fx\| & = & \|(I-T)^{-1}Fx - (I-T)^{-1}(I-T)0\| \\ & \le & \gamma \|Fx + T0\|. \end{array} \tag{3.7}$$

Therefor, from (3.7) and Assumption (3.6), we conclude that for all $x \in \partial U$,

$$\|(I-T)^{-1}Fx\| \le \gamma \|Fx + T0\| \le \|x\|.$$

Our claim then follows from (3.4) and [8, Theorem 1.3.7]. □

The following result is as straightforward consequence of Proposition [8, Corollary 1.3.1].

Proposition 3.6. *Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{U}) \subset (I - T)(\Omega)$. If further*

$$(I - T)^{-1} F(\overline{U}) \subset U,$$

then the fixed point index $i_* (T + F, U \bigcap \Omega, \mathcal{P}) = 1$.

As a particular case, we obtain

Corollary 3.7. *Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{\mathcal{P}_r} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{\mathcal{P}_r}) \subset (I - T)(\Omega)$. If $0 \in \Omega$ and*

$$\gamma \|Fx + T0\| < r, \quad \textit{for all } x \in \overline{\mathcal{P}_r}, \tag{3.8}$$

then the fixed point index $i_* (T + F, \mathcal{P}_r \bigcap \Omega, \mathcal{P}) = 1$.

Proof. From (3.7) and Assumption (3.8), for any $x \in \overline{\mathcal{P}_r}$, we conclude that

$$\|(I - T)^{-1} F x\| \leq \gamma \|Fx + T0\| < r,$$

which implies that $(I - T)^{-1} F(\overline{\mathcal{P}_r}) \subset \mathcal{P}_r$. □

Taking $r > \frac{\gamma}{1-\gamma} \|T0\|$, we get

Corollary 3.8. *Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I - T)$ is Lipschitz invertible with constant $0 < \gamma < 1$, $F : \overline{\mathcal{P}_r} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{\mathcal{P}_r}) \subset (I - T)(\Omega)$. If $0 \in \Omega$ and*

$$\|Fx\| \leq \|x\|, \quad \textit{for all } x \in \overline{\mathcal{P}_r}, \tag{3.9}$$

then $T + F$ has at least one fixed point in $\mathcal{P}_r \bigcap \Omega$.

Proposition 3.9. *Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{U}) \subset (I - T)(\Omega)$. If there exists $u_0 \in \mathcal{P}^*$ such that*

$$Fx \neq (I - T)(x - \lambda u_0), \quad \textit{for all } \lambda \geq 0 \textit{ and } x \in \partial U \bigcap (\Omega + \lambda u_0), \tag{3.10}$$

then the fixed point index $i_* (T + F, U \bigcap \Omega, \mathcal{P}) = 0$.

Proof. The mapping $(I - T)^{-1} F : \overline{U} \to \mathcal{P}$ is a strict γk-set contraction and for some $u_0 \in \mathcal{P}^*$ this operator satisfies

$$x - (I - T)^{-1} F x \neq \lambda u_0, \ \forall\, x \in \partial U, \ \forall\, \lambda \geq 0.$$

By (3.4) and [8, Theorem 1.3.8], we deduce that

$$i_* (T + F, U \bigcap \Omega, \mathcal{P}) = i\left((I - T)^{-1} F, U, \mathcal{P}\right) = 0.$$

□

Proposition 3.10. *Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{U}) \subset (I - T)(\Omega)$. Suppose further that there exists $u_0 \in \mathcal{P}^*$ such that $T(x - \lambda u_0) \in \mathcal{P}$, for all $\lambda \geq 0$ and $x \in \partial U \bigcap (\Omega + \lambda u_0)$, and one of the following conditions holds:*
(a) $Fx \nleq x, \ \forall x \in \partial U$.
(b) $Fx \in \mathcal{P}$, $\|Fx\| > N\|x\|$, $\forall x \in \partial U$, and the cone $\mathcal{P}$ is normal with constant N.
Then the fixed point index $i_* (T + F, U \bigcap \Omega, \mathcal{P}) = 0$.

Proof. We show that conditions (a) or (b) imply that

$$Fx \neq (I - T)(x - \lambda u_0), \text{ for all } \lambda \geq 0 \text{ and } x \in \partial U \bigcap (\Omega + \lambda u_0).$$

On the contrary, assume the existence of $\lambda_0 \geq 0$ and $x_0 \in \partial U \bigcap (\Omega + \lambda_0 u_0)$ such that

$$Fx_0 = (I - T)(x_0 - \lambda_0 u_0).$$

Then $x_0 - Fx_0 = T(x_0 - \lambda_0 u_0) + \lambda_0 u_0 \in \mathcal{P}$. If condition (a) holds, then a contradiction is achieved. Otherwise, we deduce that

$$Fx_0 \leq x_0.$$

Since $\mathcal{P}$ is normal, we deduce that

$$\|Fx_0\| \leq N\|x_0\|,$$

contradicting condition (b) and ending the proof of our Proposition. □

4. Fixed point index on translates of cones

In this section, let E be a Banach space, $\mathcal{P}$ ($\mathcal{P} \neq \{0\}$) be a cone in it. Given $\theta \in E$, we consider the translate of $\mathcal{P}$, namely

$$\mathcal{K} = \mathcal{P} + \theta = \{x + \theta, \ \ x \in \mathcal{P}\}.$$

Then $\mathcal{K}$ is a closed convex of E, so it is a retract of E.

Let Ω be any subset of $\mathcal{K}$ and U be a bounded open of $\mathcal{K}$ such that $U \bigcap \Omega \neq \varnothing$. We denote by $\overline{U}$ and ∂U the closure and the boundary of U relative to $\mathcal{K}$.

The fixed point index $i_* (T + F, U \bigcap \Omega, \mathcal{K})$ defined by

$$i_* (T + F, U \bigcap \Omega, \mathcal{K}) = i\,((I - T)^{-1} F, U, \mathcal{K}). \tag{4.1}$$

is well defined whenever $T : \Omega \to E$ is a mapping such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$ and $F : \overline{U} \to E$ is a k-set contraction, $0 \leq k < \gamma^{-1}$ and $F(\overline{U}) \subset (I - T)(\Omega)$.

Proposition 4.1. *Let U be a bounded open subset of $\mathcal{K}$ with $\theta \in U$. Assume that the mapping $T : \Omega \subset \mathcal{K} \to E$ be such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{U}) \subset (I - T)(\Omega)$. If*

$$Fx \neq (I - T)(\lambda x + (1 - \lambda)\theta) \text{ for all } x \in \partial U,\ \lambda \geq 1 \text{ and } \lambda x + (1 - \lambda)\theta \in \Omega, \tag{4.2}$$

then the fixed point index $i_* (T + F, U \bigcap \Omega, \mathcal{P}) = 1$.

Proof. Define the homotopic deformation $H : [0,1] \times \overline{U} \to \mathcal{K}$ by

$$H(t,x) = t(I-T)^{-1}Fx + (1-t)\theta.$$

Then, the operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t,.)$ is a strict γk-set contraction for each t. Moreover, $H(t,.)$ has no fixed point on ∂U. Otherwise, there would exist some $x_0 \in \partial U$ and $t_0 \in [0,1]$ such that $\frac{1}{t_0}x_0 + (1-\frac{1}{t_0})\theta \in \Omega$ for $t_0 \neq 0$, and

$$t_0(I-T)^{-1}Fx_0 + (1-t_0)\theta = x_0.$$

We may distinguish between two cases:
(i) If $t_0 = 0$, then $x_0 = \theta$, which is a contradiction.
(ii) If $t_0 \in (0,1]$, then $Fx_0 = (I-T)(\frac{1}{t_0}x_0 + (1-\frac{1}{t_0})\theta)$, which contradicts our assumption.
The properties of invariance by homotopy and normalization of the fixed point index guarantee that

$$i\left((I-T)^{-1}F, U, \mathcal{K}\right) = i\left(\theta, U, \mathcal{K}\right).$$

Consequently, by (4.1), we deduce that $i_*\left(T+F, U\bigcap\Omega, \mathcal{K}\right) = 1$. □

Proposition 4.2. *Let U be a bounded open subset of $\mathcal{K}$ with $\theta \in U$. Assume that the mapping $T : \Omega \subset \mathcal{K} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{U}) \subset (I-T)(\Omega)$. If*

$$\|Fx - T\theta - \theta\| \leq \|x-\theta\| \text{ and } Tx + Fx \neq x, \text{ for all } x \in \partial U\bigcap\Omega, \tag{4.3}$$

then the fixed point index $i_*\left(T+F, U\bigcap\Omega, \mathcal{P}\right) = 1$.

Proof. The mapping $(I-T)^{-1}F : \overline{U} \to \mathcal{P}$ is a strict γk-set contraction.
Since $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, for each $x \in \overline{U}$

$$\begin{aligned} \|(I-T)^{-1}Fx - \theta\| &= \|(I-T)^{-1}Fx - (I-T)^{-1}(I-T)\theta\| \\ &\leq \gamma\|Fx + T\theta - \theta\|. \end{aligned} \tag{4.4}$$

Therefor, from (4.4) and Assumption (4.3), we conclude that for all $x \in \partial U$,

$$\|(I-T)^{-1}Fx - \theta\| \leq \gamma\|Fx + T\theta - \theta\| \leq \|x-\theta\|,$$

which implies the condition (4.5) in Proposition 4.1. This completes the proof. □

Remark 4.3. Propositions 4.1,4.2 can be proven directly by appealing to [4, proposition 2.2], and [4, Corollary 2.2], respectively.

Proposition 4.4. *Let U be a bounded open subset of $\mathcal{K}$. Assume that the mapping $T : \Omega \subset \mathcal{K} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $\left(tF(\overline{U}) + (1-t)\theta\right) \subset (I-T)(\Omega)$ for all $t \in [0,1]$. If $(I-T)^{-1}\theta \in U$, and*

$$x - Tx \neq \lambda Fx + (1-\lambda)\theta \text{ for all } x \in \partial U\bigcap\Omega \text{ and } 0 \leq \lambda \leq 1, \tag{4.5}$$

then the fixed point index $i_*\left(T+F, U\bigcap\Omega, \mathcal{K}\right) = 1$.

Proof. Define the homotopic deformation $H : [0,1] \times \overline{U} \to E$ by

$$H(t,x) = tFx + (1-t)\theta.$$

Then, the operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t,.)$ is a k-set contraction for each t. Moreover, $T + H(t,.)$ has no fixed point on $\partial U \bigcap \Omega$. Otherwise, there would exist some $x_0 \in \partial U \bigcap \Omega$ and $t_0 \in [0,1]$ such that

$$Tx_0 + t_0 Fx_0 + (1-t_0)\theta = x_0,$$

then $x_0 - Tx_0 = t_0 Fx_0 + (1-t_0)\theta$, leading to a contradiction with the hypothesis. By (4.1), property (c) in [3, Theorem 2.3] and the normalization property of the fixed point index, we conclude that

$$\begin{array}{rcl} i_*\left(T+F, U\bigcap\Omega, \mathcal{K}\right) & = & i_*\left(T+\theta, \mathcal{K}_r\bigcap\Omega, \mathcal{K}\right) \\ & = & \left((I-T)^{-1}\theta, U\bigcap\Omega, \mathcal{K}\right) = 1. \end{array}$$

$\square$

Corollary 4.5. *Let U be a bounded open subset of $\mathcal{K}$. Assume that the mapping $T : \Omega \subset \mathcal{K} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $\big(tF(\overline{U}) + (1-t)\theta\big) \subset (I-T)(\Omega)$ for all $t \in [0,1]$. If $(I-T)^{-1}\theta \in U$,*

$$Fx \in \mathcal{K} \ \textit{for all} \ x \in \Omega \bigcap \partial U,$$

and

$$Fx \ngeq x - Tx \quad \textit{for all} \quad x \in \partial U \bigcap \Omega, \tag{4.6}$$

then the fixed point index $i_*\left(T+F, U\bigcap\Omega, \mathcal{K}\right) = 1$.

Proof. It is easy to see that Assumption (4.5) is satisfied. Otherwise, there exist some $x_0 \in \partial U \bigcap \Omega$ and $0 \leq \lambda_0 \leq 1$ such that $x_0 - Tx_0 = \lambda_0 Fx_0 + (1-\lambda_0)\theta$. Then

$$Fx_0 - x_0 + Tx_0 = (1-\lambda_0)(Fx_0 - \theta) \in \mathcal{P},$$

which leads us to a contradiction with (4.6). $\square$

Proposition 4.6. *Let U be a bounded open subset of $\mathcal{K}$. Assume that the mapping $T : \Omega \subset \mathcal{K} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{U}) \subset (I-T)(\Omega)$. If there exists $u_0 \in \mathcal{P}^*$ such that*

$$Fx \neq (I-T)(x - \lambda u_0), \ \textit{ for all } \lambda \geq 0 \textit{ and } x \in \partial U \bigcap (\Omega + \lambda u_0), \tag{4.7}$$

then the fixed point index $i_*\left(T+F, U\bigcap\Omega, \mathcal{K}\right) = 0$.

Proof. The mapping $(I-T)^{-1}F : \overline{U} \to \mathcal{K}$ is a strict γk-set contraction. Suppose that $i_*\left(T+F, U\bigcap\Omega, \mathcal{K}\right) \neq 0$. Then,

$$i\left((I-T)^{-1}F, U, \mathcal{P}\right) \neq 0.$$

For each $r > 0$, define the homotopy:

$$H(t,x) = (I-T)^{-1}Fx + tru_0, \ \text{ for } x \in \overline{U} \text{ and } \ t \in [0,1].$$

The operator H is continuous and uniformly continuous in t for each x. Moreover, $H(t,.)$ is a strict k-set contraction for each t and

$$H([0,1]\times\overline{U}) = (I-T)^{-1}F(U) + tru_0 \subset \mathcal{K}.$$

We check that $H(t,x)\neq x$, for all $(t,x)\in[0,1]\times\partial U$. If $H(t_0,x_0)=x_0$ for some $(t_0,x_0)\in[0,1]\times\partial U$, then

$$x_0 - t_0 r u_0 = (I-T)^{-1}Fx_0,$$

and so $x_0 - t_0 r u_0 \in \Omega$. Hence

$$(I-T)(x_0 - t_0 r u_0) = Fx_0,$$

for $x_0\in\partial U\bigcap(\Omega + t_0 r u_0)$, contradicting Assumption (4.7).
By homotopy invariance property of the fixed point index, we deduce that

$$i\left((I-T)^{-1}F + ru_0, U\bigcap\Omega, \mathcal{P}\right) = i\left((I-T)^{-1}F, U, \mathcal{P}\right)\neq 0.$$

Thus the existence property of the fixed point index, for each $r>0$, there exists $x_r\in U$ such that

$$x_r - (I-T)^{-1}Fx_r = ru_0. \tag{4.8}$$

Letting $r\to+\infty$ in (4.8), the left-hand side of (4.8) is bounded while the right-hand side is not, which is a contradiction. Therefore

$$i_*\left(T+F, U\bigcap\Omega, \mathcal{P}\right) = 0,$$

which completes the proof. □

5. Applications to ODE

In this section we investigate the IVP

$$\begin{aligned} x' &= f(t,x), \quad t>0,\\ x(0) &= x_0, \end{aligned} \tag{5.1}$$

where $x_0\in\mathbb{R}$ is a given constant, $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a given function. Let $l\in\mathbb{N}$ and x_0, s, r, A_j, $j\in\{0,1,\ldots,l\}$, are positive constants such that

(H1):

$$x_0 + \sum_{j=0}^{l}\left(\frac{r}{2}\right)^j A_j < \frac{r}{2},$$

(H2): $f\in\mathcal{C}([0,\infty)\times\mathbb{R})$ and

$$0\le f(y,x)\le\sum_{j=0}^{l}a_j(y)|x|^j, \quad y\in[0,\infty), \quad x\in\mathbb{R},$$

where $a_j\in\mathcal{C}([0,\infty))$, $a_j\ge 0$ on $[0,\infty)$ and

$$\int_0^\infty a_j(y)dy\le A_j, \quad j\in\{0,1,\ldots,l\}.$$

Theorem 5.1. *Assume that* $(H1)$-$(H2)$ *hold. Then the IVP* (5.1) *has a solution* $x \in \mathcal{C}^1([0,\infty))$ *such that* $0 \le x(t) < \frac{r}{2}$, $t \in [0,\infty)$.

Proof. **Case 1.:** Let $t \in [0,1]$. Consider the IVP

$$(5.2)\qquad \begin{aligned} x' &= f(t,x), \quad t \in (0,1], \\ x(0) &= x_0. \end{aligned}$$

Take $\epsilon > 0$ arbitrarily. Let $E_1 = \mathcal{C}([0,1])$ be endowed with the maximum norm and

$$\begin{aligned} \mathcal{P}_1 &= \{x \in E_1 : x(t) \ge 0, \quad t \in [0,1]\}, \\ \Omega_1 &= \mathcal{P}_{1r} = \{x \in \mathcal{P}_1 : \|x\| < r\}, \\ U_1 &= \mathcal{P}_{1\frac{r}{2}} = \left\{x \in \mathcal{P}_1 : \|x\| < \frac{r}{2}\right\}. \end{aligned}$$

For $x \in E_1$, define the operators

$$\begin{aligned} T_1 x(t) &= (1+\epsilon)x(t), \\ F_1 x(t) &= -\epsilon\left(x_0 + \int_0^t f(y,x(y))dy\right), \quad t \in [0,1]. \end{aligned}$$

Note that for any fixed point $x \in E_1$ of the operator $T_1 + F_1$ we have that $x \in \mathcal{C}^1([0,1])$ and it is a solution of the IVP (5.2).

(1) For $x, y \in E_1$, we have

$$\|(I-T_1)^{-1}x - (I-T_1)^{-1}y\| = \frac{1}{\epsilon}\|x-y\|,$$

i.e., $(I-T_1) : E_1 \to E_1$ is Lipschitz invertible with constant $\frac{1}{\epsilon}$.

(2) For $x \in \overline{U}_1$ and $t \in [0,1]$, we have

$$\begin{aligned} |F_1 x(t)| &= \epsilon\left(x_0 + \int_0^t f(y,x(y))dy\right) \\ &\le \epsilon\left(x_0 + \int_0^t \sum_{j=0}^{l} a_j(y)(x(y))^j dy\right) \\ &\le \epsilon\left(x_0 + \sum_{j=0}^{l}\left(\frac{r}{2}\right)^j \int_0^t a_j(y)dy\right) \\ &\le \epsilon\left(x_0 + \sum_{j=0}^{l}\left(\frac{r}{2}\right)^j A_j\right) \end{aligned}$$

and

$$
\begin{aligned}
|(F_1x)'(t)| &= \epsilon f(t,x(t)) \\
&\leq \epsilon \sum_{j=0}^{l} a_j(y)(x(y))^j \\
&\leq \epsilon \sum_{j=0}^{l} \left(\frac{r}{2}\right)^j a_j(y) \\
&\leq \epsilon \sum_{j=0}^{l} \left(\frac{r}{2}\right)^j B_j
\end{aligned}
$$

Thus,

$$
\|F_1x\| \leq \epsilon \left(x_0 + \sum_{j=0}^{l} \left(\frac{r}{2}\right)^j A_j \right)
$$

and

$$
\|(F_1x)'\| \leq \epsilon \sum_{j=0}^{l} \left(\frac{r}{2}\right)^j B_j.
$$

Hence, using the Arzela-Ascoli theorem, we conclude that $F_1 : \overline{U}_1 \to E$ is a completely continuous mapping.
Therefore $F_1 : \overline{U}_1 \to E$ is a 0-set contraction.

(3) Let $\lambda \in [0,1]$ and $x \in \overline{U}_1$ be arbitrarily chosen. Then

$$
z(t) = \lambda \left(x_0 + \int_0^t f(y,x(y))ds \right) \in E_1
$$

and

$$
\begin{aligned}
z(t) &\leq \lambda \left(x_0 + \int_0^\infty f(y,x(y))dy \right) \\
&\leq \lambda \left(x_0 + \sum_{j=0}^{l} \int_0^\infty a_j(y)(x(y))^j dy \right) \\
&\leq \lambda \left(x_0 + \sum_{j=0}^{l} \left(\frac{r}{2}\right)^j A^j \right) \\
&< \lambda \frac{r}{2} \\
&\leq \frac{r}{2}, \quad t \in [0,1],
\end{aligned}
$$

i.e., $z \in \Omega_1$. Next,

$$\begin{aligned}\lambda F_1 x(t) &= -\lambda\epsilon\left(x_0 + \int_0^t f(y, x(y))dy\right)\\ &= -\epsilon z(t)\\ &= (I - T_1)z(t), \quad t \in [0,1].\end{aligned}$$

Thus, $\lambda F_1(\overline{U_1}) \subset (I - T_1)(\Omega_1)$.

(4) Note that

$$(I - T_1)^{-1}0 = 0 \in U_1.$$

(5) Assume that there are $x \in \partial U_1 \bigcap \Omega_1$ and $\lambda \in [0,1]$ such that

$$x - T_1 x = \lambda F_1 x.$$

If $\lambda = 0$, then

$$0 = x - T_1 x = -\epsilon x \quad \text{on} \quad [0,1],$$

whereupon $x(t) = 0$, $t \in [0,1]$. This is a contradiction because $x \in \partial U_1$. Therefore $\lambda \in (0,1]$. Let $t_1 \in [0,1]$ be such that $x(t_1) = \frac{r}{2}$. Then

$$\begin{aligned}(I - T_1)x(t_1) &= -\epsilon x(t_1)\\ &= -\epsilon \frac{r}{2}\\ &= -\lambda\epsilon\left(x_0 + \int_0^{t_1} f(y, x(y))dy\right),\end{aligned}$$

whereupon

$$\begin{aligned}
\frac{r}{2} &= \lambda\left(x_0+\int_0^{t_1} f(y,x(y))dy\right)\\
&\le \lambda\left(x_0+\int_0^{\infty} f(y,x(y))dy\right)\\
&\le \lambda\left(x_0+\sum_{j=0}^{l}\int_0^{\infty} a_j(y)(x(y))^j dy\right)\\
&\le \lambda\left(x_0+\sum_{j=0}^{l} A_j\left(\frac{r}{2}\right)^j\right)\\
&< \lambda\frac{r}{2}\\
&\le \frac{r}{2},
\end{aligned}$$

i.e., $\frac{r}{2}<\frac{r}{2}$, which is a contradiction.

By 1, 2, 3, 4, 5 and Proposition 3.1, it follows that the operator T_1+F_1 has a fixed point in U_1. Denote it by x_1. We have

$$0\le x_1(t)<\frac{r}{2},\quad t\in[0,1],$$

and $x_1\in\mathcal{C}^1([0,1])$ is a solution of the IVP (5.2).

Case 2.: Let $t\in[1,2]$. Consider the IVP

$$\begin{aligned}
x' &= f(t,x),\quad t\in(1,2],\\
x(1) &= x_1(1).
\end{aligned}\tag{5.3}$$

Take $\epsilon>0$ arbitrarily. Let $E_2=\mathcal{C}([1,2])$ be endowed with the maximum norm and

$$\begin{aligned}
\mathcal{P}_2 &= \{x\in E_2: x(t)\ge 0,\quad t\in[1,2]\},\\
\Omega_2 &= \mathcal{P}_{2r}=\{x\in\mathcal{P}_2:\|x\|<r\},\\
U_2 &= \mathcal{P}_{2\frac{r}{2}}=\left\{x\in\mathcal{P}_2:\|x\|<\frac{r}{2}\right\}.
\end{aligned}$$

For $x\in E_2$ define the operators

$$\begin{aligned}
T_2x(t) &= (1+\epsilon)x(t),\\
F_2x(t) &= -\epsilon\left(x_1(1)+\int_1^t f(s,x(s))ds\right),\quad t\in[1,2].
\end{aligned}$$

Note that for $x \in U_2$, we have

$$\begin{array}{rcl} x_1(1) + \displaystyle\int_1^t f(s, x(s))ds & = & x_0 + \displaystyle\int_0^t f(y, x(y))dy \\ & \leq & x_0 + \displaystyle\int_0^\infty f(y, x(y))dy \\ & \leq & x_0 + \displaystyle\sum_{j=0}^{l} a_j(y)(x(y))^j dy \\ & \leq & x_0 + \displaystyle\sum_{j=0}^{l} A_j r^j \\ & < & \dfrac{r}{2}, \quad t \in [1, 2]. \end{array}$$

As in Case 1 we prove that the operator $T_2 + F_2$ has a fixed point $x_2 \in U_2$. We have that

$$0 \leq x_2(t) < \frac{r}{2}, \quad t \in [1, 2], \quad x_2 \in \mathcal{C}^1([1, 2]).$$

Note that

$$\begin{array}{rcl} x_1(1) & = & x_2(1), \\ x_1'(1) & = & f(1, x_1(1)) \\ & = & f(1, x_2(1)) \\ & = & x_2'(1). \end{array}$$

Thus,

$$x(t) = \begin{cases} x_1(t) & t \in [0, 1] \\ x_2(t) & t \in [1, 2] \end{cases}$$

is a solution to the IVP

$$x' = f(t, x), \quad t \in (0, 2],$$

$$x(0) = x_0.$$

Case 3.: Consider the IVP

$$x' = f(t, x), \quad t \in (2, 3],$$

$$x(2) = x_2(2).$$

And so on, the function

$$x(t)=\begin{cases} x_1(t) & t\in[0,1]\\ x_2(t) & t\in[1,2]\\ x_3(t) & t\in[2,3]\\ x_4(t) & t\in[3,4]\\ \dots \end{cases}$$

is a solution to the IVP (5.1). This completes the proof. □

6. Applications to PDE

In this section we consider the IVP for Burgers-Fisher equation

$$u_t-u_{xx}+\alpha(t)uu_x=\beta(t)u(1-u),\quad t>0,\quad x\geq 0, \tag{6.1}$$

$$u(0,x)=u_0(x),\quad x\geq 0, \tag{6.2}$$

where

(A1): $u_0\in\mathcal{C}^2([0,\infty))$, $r_1\geq u_0\geq\frac{r_1}{2}$ on $[0,\infty)$, where $r_1\in\left(0,\frac{1}{2}\right)$ is a given constant,

(A2): $\alpha,\beta\in\mathcal{C}([0,\infty))$, $\alpha<0$, $\beta\geq 0$ on $[0,\infty)$, $A\in(0,1)$ is a constant and g is a positive continuous function on $[0,\infty)\times[0,\infty)$ such that

$$1-(1+2r_1)A>0,\quad \left(4+\frac{3}{2}r_1\right)A<\frac{1}{2},$$

and

$$120\left(1+t+t^2+t^3+t^4\right)\left(1+x+x^2+x^3+x^4+x^5+x^6\right)$$

$$\times\int_0^t\int_0^x g(t_1,x_1)\left(1+\int_0^{t_1}(\beta(t_2)-\alpha(t_2))dt_2\right)dx_1dt_1\leq A,$$

$t\geq 0$, $x\geq 0$.

Let $E=\mathcal{C}^1([0,\infty),\mathcal{C}^2([0,\infty)))$ be endowed with the norm

$$\begin{aligned}\|u\| \;=\; \Bigg\{ & \sup_{(t,x)\in[0,\infty)\times[0,\infty)}|u(t,x)|,\quad \sup_{(t,x)\in[0,\infty)\times[0,\infty)}\left|\frac{\partial}{\partial t}u(t,x)\right|,\\ & \sup_{(t,x)\in[0,\infty)\times[0,\infty)}\left|\frac{\partial}{\partial x}u(t,x)\right|,\quad \sup_{(t,x)\in[0,\infty)\times[0,\infty)}\left|\frac{\partial^2}{\partial x^2}u(t,x)\right|\Bigg\},\end{aligned}$$

provided it exists.

Lemma 6.1. *Suppose* $(A1)$ *and* $(A2)$*. If a function* $u \in E$ *is a solution of the integral equation*

$$
\begin{aligned}
0 \quad = \quad & \int_0^t \int_0^x (t-t_1)^4 (x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \int_0^{x_1} (x_1-x_2)\beta(t_2) \\
& \times u(t_2,x_2)(1-u(t_2,x_2)) dx_2 dt_2 dx_1 dt_1 \\
& -\frac{1}{2} \int_0^t \int_0^x (t-t_1)^4 (x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \int_0^{x_1} \alpha(t_2)(u(t_2,x_2))^2 \\
& \times dx_2 dt_2 dx_1 dt_1 \\
& + \int_0^t \int_0^x (t-t_1)^4 (x-x_1)^4 g(t_1,x_1) \int_0^{t_1} u(t_2,x_1) dt_2 dx_1 dt_1 \\
& + \int_0^t \int_0^x (t-t_1)^4 (x-x_1)^4 g(t_1,x_1) \int_0^{x_1} (x_1-x_2) \\
& \times (u_0(x_2) - u(t_1,x_2)) dx_2 dx_1 dt_1,
\end{aligned}
$$

$(t,x) \in [0,\infty) \times [0,\infty)$*, then it is a solution to the IVP* (6.1)-(6.2)*.*

Proof. We differentiate the considered integral equation five times in t and five times in x and using that $g > 0$ on $[0,\infty) \times [0,\infty)$, we get

$$
\begin{aligned}
0 \quad = \quad & g(t,x) \int_0^t \int_0^x \int_0^{x_1} \beta(t_1) u(t_1,x_2)(1-u(t_1,x_2)) dx_2 dx_1 dt_1 \\
& -\frac{1}{2} g(t,x) \int_0^t \int_0^x \alpha(t_1)(u(t_1,x_1))^2 dx_1 dt_1 \\
& + g(t,x) \int_0^t u(t_1,x) dt_1 \\
& + g(t,x) \int_0^x \int_0^{x_1} (u_0(x_2) - u(t_1,x_2)) dx_2 dx_1, \quad (t,x) \in [0,\infty) \times [0,\infty),
\end{aligned}
$$

whereupon

$$\begin{aligned} 0 &= \int_0^t \int_0^x \int_0^{x_1} \beta(t_1)u(t_1,x_2)(1-u(t_1,x_2))dx_2dx_1dt_1 \\ &\quad -\frac{1}{2}\int_0^t \int_0^x \alpha(t_1)(u(t_1,x_1))^2dx_1dt_1 \\ &\quad +\int_0^t u(t_1,x)dt_1 \\ &\quad +\int_0^x \int_0^{x_1}(u_0(x_2)-u(t_1,x_2))dx_2dx_1, \quad (t,x)\in[0,\infty)\times[0,\infty). \end{aligned}$$

The last equation we differentiate twice in x and we get

$$\begin{aligned} 0 &= \textstyle\int_0^t \beta(t_1)u(t_1,x)(1-u(t_1,x))dt_1 \\ &\quad -\textstyle\int_0^t \alpha(t_1)u(t_1,x)u_x(t_1,x)dt_1+\int_0^t u_{xx}(t_1,x)dt_1 \\ &\quad +u_0(x)-u(t,x), \quad (t,x)\in[0,\infty)\times[0,\infty), \end{aligned} \tag{6.3}$$

which we differentiate in t and we obtain

$$\begin{aligned} 0 &= \beta(t)u(t,x)(1-u(t,x))-\alpha(t)u(t,x)u_x(t,x) \\ &\quad +u_{xx}(t,x)-u_t(t,x), \quad (t,x)\in[0,\infty)\times[0,\infty), \end{aligned}$$

i.e., u satisfies (6.1). Now we put $t=0$ in (6.3) and we get

$$u(0,x)=u_0(x), \quad x\in[0,\infty).$$

This completes the proof. □

For $u \in E$, define the operators

$$\begin{aligned}
F_1 u(t,x) &= \int_0^t \int_0^x (t-t_1)^4 (x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \int_0^{x_1} (x_1-x_2)\beta(t_2) \\
&\quad \times (u(t_2,x_2))^2 dx_2 dt_2 dx_1 dt_1 \\
&\quad + \int_0^t \int_0^x (t-t_1)^4 (x-x_1)^4 g(t_1,x_1) \int_0^{x_1} (x_1-x_2) \\
&\quad \times u(t_1,x_2) dx_2 dx_1 dt_1, \\
F_2 u(t,x) &= \int_0^t \int_0^x (t-t_1)^4 (x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \int_0^{x_1} (x_1-x_2)\beta(t_2) \\
&\quad \times u(t_2,x_2) dx_2 dt_2 dx_1 dt_1 \\
&\quad - \frac{1}{2} \int_0^t \int_0^x (t-t_1)^4 (x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \int_0^{x_1} \alpha(t_2)(u(t_2,x_2))^2 \\
&\quad \times dx_2 dt_2 dx_1 dt_1 \\
&\quad + \int_0^t \int_0^x (t-t_1)^4 (x-x_1)^4 g(t_1,x_1) \int_0^{t_1} u(t_2,x_1) dt_2 dx_1 dt_1 \\
&\quad + \int_0^t \int_0^x (t-t_1)^4 (x-x_1)^4 g(t_1,x_1) \int_0^{x_1} (x_1-x_2) \\
&\quad \times u_0(x_2) dx_2 dx_1 dt_1,
\end{aligned}$$

$(t,x) \in [0,\infty) \times [0,\infty)$. Note that if $u \in E$ is a fixed point of the operator $F_2 - F_1$, then it is a solution of the IVP (6.1)-(6.2).

Lemma 6.2. *Suppose* (A1), (A2) *and* $r > 0$. *If* $u \in E$ *and* $\|u\| \le r$, *then*

$$\|F_1 u\| \le (1+r)A\|u\|, \quad \|F_2 u\| \le \left(3 + \frac{r}{2}\right) rA$$

and $F_2 : \{u \in E : \|u\| \le r\} \to E$ *is a completely continuous operator. Moreover,*

$$\|F_1 u_1 - F_1 u_2\| \le (2r+1)A\|u_1 - u_2\|$$

for any $u_1, u_2 \in \{u \in E : \|u\| \le r\}$.

Proof. Take $u \in \{E : \|u\| \le r\}$ arbitrarily. Then

$$\begin{aligned}
|F_1u(t,x)| &\le \int_0^t \int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \int_0^{x_1} (x_1-x_2)\beta(t_2) \\
&\quad \times (u(t_2,x_2))^2 dx_2 dt_2 dx_1 dt_1 \\
&\quad + \int_0^t \int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1) \int_0^{x_1} (x_1-x_2) \\
&\quad \times |u(t_1,x_2)| dx_2 dx_1 dt_1 \\
&\le r\|u\| \int_0^t \int_0^x x_1^2 (t-t_1)^4(x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \beta(t_2) dt_2 dx_1 dt_1 \\
&\quad + \|u\| \int_0^t \int_0^x x_1^2 (t-t_1)^4(x-x_1)^4 g(t_1,x_1) dx_1 dt_1 \\
&\le r\|u\| t^4 x^6 \int_0^t \int_0^x g(t_1,x_1) \int_0^{t_1} \beta(t_2) dt_2 dx_1 dt_1 \\
&\quad + \|u\| t^4 x^6 \int_0^t \int_0^x g(t_1,x_1) dx_1 dt_1 \\
&\le (1+r)A\|u\|, \quad t \ge 0, \quad x \ge 0,
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial}{\partial t} F_1u(t,x)\right| &\le 4\int_0^t \int_0^x (t-t_1)^3(x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \int_0^{x_1} (x_1-x_2)\beta(t_2) \\
&\quad \times (u(t_2,x_2))^2 dx_2 dt_2 dx_1 dt_1 \\
&\quad + 4\int_0^t \int_0^x (t-t_1)^3(x-x_1)^4 g(t_1,x_1) \int_0^{x_1} (x_1-x_2) \\
&\quad \times |u(t_1,x_2)| dx_2 dx_1 dt_1 \\
&\le 4r\|u\| \int_0^t \int_0^x x_1^2 (t-t_1)^3(x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \beta(t_2) dt_2 dx_1 dt_1 \\
&\quad + 4\|u\| \int_0^t \int_0^x x_1^2 (t-t_1)^3(x-x_1)^4 g(t_1,x_1) dx_1 dt_1
\end{aligned}$$

$$
\begin{aligned}
&\leq 4r\|u\|t^3x^6\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad +4\|u\|t^3x^6\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\leq (1+r)A\|u\|, \quad t\geq 0, \quad x\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial}{\partial x}F_1u(t,x)\right| &\leq 4\int_0^t\int_0^x (t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2)\beta(t_2)\\
&\quad \times(u(t_2,x_2))^2dx_2dt_2dx_1dt_1\\
&\quad +4\int_0^t\int_0^x (t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)\\
&\quad \times|u(t_1,x_2)|dx_2dx_1dt_1\\
&\leq 4r\|u\|\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad +4\|u\|\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)^3 g(t_1,x_1)dx_1dt_1\\
&\leq 4r\|u\|t^4x^5\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad +4\|u\|t^4x^5\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\leq (1+r)A\|u\|, \quad t\geq 0, \quad x\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial x^2}F_1u(t,x)\right| &\leq 12\int_0^t\int_0^x (t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2)\beta(t_2)\\
&\quad \times(u(t_2,x_2))^2dx_2dt_2dx_1dt_1\\
&\quad +12\int_0^t\int_0^x (t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)\\
&\quad \times|u(t_1,x_2)|dx_2dx_1dt_1
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad 12r\|u\|\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad +12\|u\|\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)^2 g(t_1,x_1)dx_1dt_1\\
&\leq \quad 12r\|u\|t^4x^4\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad +12\|u\|t^4x^4\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\leq \quad (1+r)A\|u\|, \quad t\geq 0, \quad x\geq 0,
\end{aligned}
$$

Consequently

$$
\|F_1u\|\leq (1+r)A\|u\|.
$$

Next,

$$
\begin{aligned}
|F_2u(t,x)| \quad \leq \quad & \int_0^t\int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2)\beta(t_2)\\
&\times|u(t_2,x_2)|dx_2dt_2dx_1dt_1\\
&-\frac{1}{2}\int_0^t\int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}\alpha(t_2)(u(t_2,x_2))^2\\
&\times dx_2dt_2dx_1dt_1\\
&+\int_0^t\int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}|u(t_2,x_1)|dt_2dx_1dt_1\\
&+\int_0^t\int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)\\
&\times u_0(x_2)dx_2dx_1dt_1
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad r\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -\frac{1}{2}r^2\int_0^t\int_0^x x_1(t-t_1)^4(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +r\int_0^t\int_0^x t_1(t-t_1)^4(x-x_1)^4 g(t_1,x_1)dx_1dt_1\\
&\quad +r\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)^4 g(t_1,x_1)dx_1dt_1\\
&\leq \quad rt^4x^6\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -\frac{1}{2}r^2t^4x^5\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +rt^5x^4\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\quad +rt^4x^6\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\leq \quad \left(3+\frac{r}{2}\right)rA,\quad t\geq 0,\quad x\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial}{\partial t}F_2u(t,x)\right| \quad &\leq \quad 4\int_0^t\int_0^x (t-t_1)^3(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2)\beta(t_2)\\
&\quad \times|u(t_2,x_2)|dx_2dt_2dx_1dt_1\\
&\quad -2\int_0^t\int_0^x (t-t_1)^3(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}\alpha(t_2)(u(t_2,x_2))^2\\
&\quad \times dx_2dt_2dx_1dt_1\\
&\quad +4\int_0^t\int_0^x (t-t_1)^3(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}|u(t_2,x_1)|dt_2dx_1dt_1\\
&\quad +4\int_0^t\int_0^x (t-t_1)^3(x-x_1)^4 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)\\
&\quad \times u_0(x_2)dx_2dx_1dt_1
\end{aligned}
$$

$$\begin{aligned}
&\leq \quad 4r\int_0^t\int_0^x x_1^2(t-t_1)^3(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -2r^2\int_0^t\int_0^x x_1(t-t_1)^3(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +4r\int_0^t\int_0^x t_1(t-t_1)^3(x-x_1)^4 g(t_1,x_1)dx_1dt_1\\
&\quad +4r\int_0^t\int_0^x x_1^2(t-t_1)^3(x-x_1)^4 g(t_1,x_1)dx_1dt_1\\
&\leq \quad 4rt^3x^6\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -2r^2t^3x^5\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +4rt^4x^4\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\quad +4rt^3x^6\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\leq \quad \left(3+\frac{r}{2}\right)rA, \quad t\geq 0, \quad x\geq 0,
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial^2}{\partial t^2}F_2u(t,x)\right| \quad &\leq \quad 12\int_0^t\int_0^x (t-t_1)^2(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2)\beta(t_2)\\
&\quad \times|u(t_2,x_2)|dx_2dt_2dx_1dt_1\\
&\quad -6\int_0^t\int_0^x (t-t_1)^2(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}\alpha(t_2)(u(t_2,x_2))^2\\
&\quad \times dx_2dt_2dx_1dt_1\\
&\quad +12\int_0^t\int_0^x (t-t_1)^2(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}|u(t_2,x_1)|dt_2dx_1dt_1\\
&\quad +12\int_0^t\int_0^x (t-t_1)^2(x-x_1)^4 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)\\
&\quad \times u_0(x_2)dx_2dx_1dt_1
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad 12r\int_0^t\int_0^x x_1^2(t-t_1)^2(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -6r^2\int_0^t\int_0^x x_1(t-t_1)^2(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +12r\int_0^t\int_0^x t_1(t-t_1)^2(x-x_1)^4 g(t_1,x_1)dx_1dt_1\\
&\quad +12r\int_0^t\int_0^x x_1^2(t-t_1)^2(x-x_1)^4 g(t_1,x_1)dx_1dt_1\\
&\leq \quad 12rt^2x^6\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -6r^2t^2x^5\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +12rt^3x^4\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\quad +12rt^2x^6\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\leq \quad \left(3+\frac{r}{2}\right)rA,\quad t\geq 0,\quad x\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial}{\partial x}F_2u(t,x)\right| \quad &\leq \quad 4\int_0^t\int_0^x (t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2)\beta(t_2)\\
&\quad \times|u(t_2,x_2)|dx_2dt_2dx_1dt_1\\
&\quad -2\int_0^t\int_0^x (t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}\alpha(t_2)(u(t_2,x_2))^2\\
&\quad \times dx_2dt_2dx_1dt_1\\
&\quad +4\int_0^t\int_0^x (t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{t_1}|u(t_2,x_1)|dt_2dx_1dt_1\\
&\quad +4\int_0^t\int_0^x (t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)\\
&\quad \times u_0(x_2)dx_2dx_1dt_1
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad 4r\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -2r^2\int_0^t\int_0^x x_1(t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +4r\int_0^t\int_0^x t_1(t-t_1)^4(x-x_1)^3 g(t_1,x_1)dx_1dt_1\\
&\quad +4r\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)^3 g(t_1,x_1)dx_1dt_1\\
&\leq \quad 4rt^4x^5\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -2r^2t^4x^4\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +4rt^5x^3\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\quad +4rt^4x^5\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\leq \quad \left(3+\frac{r}{2}\right)rA, \quad t\geq 0, \quad x\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial x^2}F_2u(t,x)\right| \quad &\leq \quad 12\int_0^t\int_0^x (t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2)\beta(t_2)\\
&\quad \times|u(t_2,x_2)|dx_2dt_2dx_1dt_1\\
&\quad -6\int_0^t\int_0^x (t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}\alpha(t_2)(u(t_2,x_2))^2\\
&\quad \times dx_2dt_2dx_1dt_1\\
&\quad +12\int_0^t\int_0^x (t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{t_1}|u(t_2,x_1)|dt_2dx_1dt_1\\
&\quad +12\int_0^t\int_0^x (t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)\\
&\quad \times u_0(x_2)dx_2dx_1dt_1
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad 12r\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -6r^2\int_0^t\int_0^x x_1(t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +12r\int_0^t\int_0^x t_1(t-t_1)^4(x-x_1)^2 g(t_1,x_1)dx_1dt_1\\
&\quad +12r\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)^2 g(t_1,x_1)dx_1dt_1\\
&\leq \quad 12rt^4x^4\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -6r^2t^4x^3\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +12rt^5x^2\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\quad +12rt^4x^4\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\leq \quad \left(3+\frac{r}{2}\right)rA, \quad t\geq 0, \quad x\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial^3}{\partial x^3}F_2u(t,x)\right| \quad &\leq \quad 24\int_0^t\int_0^x(t-t_1)^4(x-x_1)g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2)\beta(t_2)\\
&\quad \times|u(t_2,x_2)|dx_2dt_2dx_1dt_1\\
&\quad -12\int_0^t\int_0^x(t-t_1)^4(x-x_1)g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}\alpha(t_2)(u(t_2,x_2))^2\\
&\quad \times dx_2dt_2dx_1dt_1\\
&\quad +24\int_0^t\int_0^x(t-t_1)^4(x-x_1)g(t_1,x_1)\int_0^{t_1}|u(t_2,x_1)|dt_2dx_1dt_1\\
&\quad +24\int_0^t\int_0^x(t-t_1)^4(x-x_1)g(t_1,x_1)\int_0^{x_1}(x_1-x_2)\\
&\quad \times u_0(x_2)dx_2dx_1dt_1
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad 24r\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -12r^2\int_0^t\int_0^x x_1(t-t_1)^4(x-x_1)g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +24r\int_0^t\int_0^x t_1(t-t_1)^4(x-x_1)g(t_1,x_1)dx_1dt_1\\
&\quad +24r\int_0^t\int_0^x x_1^2(t-t_1)^4(x-x_1)g(t_1,x_1)dx_1dt_1\\
&\leq \quad 24rt^4x^3\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dx_1dt_1\\
&\quad -12r^2t^4x^2\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\alpha(t_2)dt_2dx_1dt_1\\
&\quad +24rt^5x\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\quad +24rt^4x^3\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\\
&\leq \quad \left(3+\frac{r}{2}\right)rA, \quad t\geq 0, \quad x\geq 0.
\end{aligned}
$$

Consequently

$$
\|F_2u\|\leq\left(3+\frac{r}{2}\right)rA, \quad \left\|\frac{\partial^2}{\partial t^2}F_2u\right\|_{\mathcal{C}^0}\leq\left(3+\frac{r}{2}\right)rA, \quad \left\|\frac{\partial^3}{\partial x^3}F_2u\right\|_{\mathcal{C}^0}\leq\left(3+\frac{r}{2}\right)rA.
$$

By the Arzela-Ascoli theorem, it follows that the operator $F_2:\{u\in E:\|u\|\leq r\}\to E$ is a completely continuous operator. Let now, $u_1,u_2\in\{u\in E:\|u\|\leq r\}$. Then

$$
\begin{aligned}
|F_1u_1(t,x)-F_1u_2(t,x)| \quad &\leq \quad \Bigg(\int_0^t\int_0^x(t-t_1)^4(x-x_1)^4g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2)\\
&\qquad \times\beta(t_2)\left(|u_1(t_2,x_2)|+|u_2(t_2,x_2)|\right)dx_2dt_2dx_1dt_1
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)dx_2dx_1dt_1\Big)\|u_1-u_2\| \\
\leq\;& \Big(2rx^6t^4\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dt_1 \\
&+x^6t^4\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\Big)\|u_1-u_2\| \\
\leq\;& (2r+1)A\|u_1-u_2\|,\quad t\geq 0,\quad x\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial}{\partial t}F_1u_1(t,x)-\frac{\partial}{\partial t}F_1u_2(t,x)\right| \leq\;& \Big(4\int_0^t\int_0^x (t-t_1)^3(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2) \\
&\times\beta(t_2)\left(|u_1(t_2,x_2)|+|u_2(t_2,x_2)|\right)dx_2dt_2dx_1dt_1 \\
&+4\int_0^t\int_0^x (t-t_1)^3(x-x_1)^4 g(t_1,x_1)\int_0^{x_1}(x_1-x_2) \\
&\times dx_2dx_1dt_1\Big)\|u_1-u_2\| \\
\leq\;& \Big(8rx^6t^3\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dt_1 \\
&+4x^6t^3\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\Big)\|u_1-u_2\| \\
\leq\;& (2r+1)A\|u_1-u_2\|,\quad t\geq 0,\quad x\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial}{\partial x}F_1u_1(t,x)-\frac{\partial}{\partial x}F_1u_2(t,x)\right| \leq\;& \Big(4\int_0^t\int_0^x (t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2) \\
&\times\beta(t_2)\left(|u_1(t_2,x_2)|+|u_2(t_2,x_2)|\right)dx_2dt_2dx_1dt_1
\end{aligned}
$$

$$
\begin{aligned}
&+4\int_0^t\int_0^x (t-t_1)^4(x-x_1)^3 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)dx_2dx_1dt_1\Big)\|u_1-u_2\| \\
\leq\ & \Big(8rx^5t^4\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dt_1 \\
&+4x^5t^4\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\Big)\|u_1-u_2\| \\
\leq\ & (2r+1)A\|u_1-u_2\|, \quad t\geq 0, \quad x\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial x^2}F_1u_1(t,x)-\frac{\partial^2}{\partial x^2}F_1u_2(t,x)\right| \leq\ & \Big(12\int_0^t\int_0^x (t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2) \\
&\times\beta(t_2)\left(|u_1(t_2,x_2)|+|u_2(t_2,x_2)|\right)dx_2dt_2dx_1dt_1 \\
&+12\int_0^t\int_0^x (t-t_1)^4(x-x_1)^2 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)dx_2dx_1dt_1\Big)\|u_1-u_2\| \\
\leq\ & \Big(24rx^4t^4\int_0^t\int_0^x g(t_1,x_1)\int_0^{t_1}\beta(t_2)dt_2dt_1 \\
&+12x^4t^4\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\Big)\|u_1-u_2\| \\
\leq\ & (2r+1)A\|u_1-u_2\|, \quad t\geq 0, \quad x\geq 0.
\end{aligned}
$$

Therefore

$$\|F_1u_1-F_1u_2\|\leq (2r+1)A\|u_1-u_2\|.$$

This completes the proof. □

Theorem 6.3. *Suppose (A1) and (A2). Then the IVP (6.1)-(6.2) has at least one non-negative solution $u\in\mathcal{C}^1([0,\infty),\mathcal{C}^2([0,\infty)))$.*

Proof. Set

$$
\begin{aligned}
\mathcal{P} &= \{u\in E: u(t,x)\geq 0, \quad t\geq 0, \quad x\geq 0\}, \\
\Omega &= \{u\in\mathcal{P}: \|u\|\leq r_1, \quad u(t,x)\leq u_0(x), \quad t\geq 0, \quad x\geq 0\}, \\
U &= \{u\in\mathcal{P}: \|u\|\leq r_1, \quad \frac{1}{2}u_0(x)\leq u(t,x)\leq u_0(x), \quad t\geq 0, \quad x\geq 0\}.
\end{aligned}
$$

For $u\in E$, define the operators

$$
\begin{aligned}
Tu(t,x) &= -F_1u(t,x), \\
Su(t,x) &= F_2u(t,x), \quad t\geq 0, \quad x\geq 0.
\end{aligned}
$$

(1) Let $u, v \in \Omega$. Then $(I-T)(u-v) = (I+F_1)(u-v)$ and using Lemma 6.2, we get

$$\begin{aligned}\|(I-T)(u-v)\| &\geq \|u-v\| - \|F_1(u-v)\| \\ &\geq (1-(1+2r_1)A)\,\|u-v\|.\end{aligned}$$

Thus, $I-T : \Omega \to E$ is Lipschitz invertible with $\gamma = \frac{1}{1-(1+2r_1)A}$.

(2) By Lemma 6.2, we have that $S : \overline{U} \to E$ is a completely continuous operator. Therefore $S : \overline{U} \to E$ is 0-set contraction.

(3) Let $v \in \overline{U}$ be arbitrarily chosen. For $u \in \Omega$, we have

$$\begin{aligned}&-F_1u(t,x) + F_2v(t,x) \\ &\geq -\int_0^t \int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \int_0^{x_1} (x_1-x_2)\beta(t_2) \\ &\quad \times (u(t_2,x_2))^2 dx_2 dt_2 dx_1 dt_1 \\ &\quad - \int_0^t \int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1) \int_0^{x_1} (x_1-x_2) \\ &\quad \times u(t_1,x_2) dx_2 dx_1 dt_1 \\ &\quad + \int_0^t \int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1) \int_0^{t_1} \int_0^{x_1} (x_1-x_2)\beta(t_2) \\ &\quad \times v(t_2,x_2) dx_2 dt_2 dx_1 dt_1 \\ &\quad + \int_0^t \int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1) \int_0^{x_1} (x_1-x_2) \\ &\quad \times u_0(x_2) dx_2 dx_1 dt_1\end{aligned}$$

$$\begin{aligned}
&\geq \left(\frac{r_1}{2}-r_1^2\right)\int_0^t\int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1)\int_0^{t_1}\int_0^{x_1}(x_1-x_2)\beta(t_2)\\
&\quad \times dx_2dt_2dx_1dt_1\\
&\quad +\int_0^t\int_0^x (t-t_1)^4(x-x_1)^4 g(t_1,x_1)\int_0^{x_1}(x_1-x_2)\\
&\quad \times(u_0(x_2)-u(t_1,x_2))dx_2dx_1dt_1\\
&\geq 0,\quad t\geq 0,\quad x\geq 0,
\end{aligned}$$

and

$$\begin{aligned}
-F_1u(t,x)+F_2v(t,x) &\leq \|F_1u\|+\|F_2v\|\\
&\leq (1+r_1)r_1A+\left(3+\frac{r_1}{2}\right)r_1A\\
&= \left(4+\frac{3}{2}r_1\right)r_1A\\
&< \frac{r_1}{2}\\
&\leq u_0(x),\quad t\geq 0,\quad x\geq 0.
\end{aligned}$$

For $u\in\Omega$, define the operator

$$Lu(t,x)=-F_1u(t,x)+F_2v(t,x),\quad t\geq 0,\quad x\geq 0.$$

Then, using Lemma 6.2, we get

$$\begin{aligned}
\|Lu\| &\leq \|F_1u\|+\|F_2v\|\\
&\leq r_1(1+r_1)A+\left(3+\frac{r_1}{2}\right)r_1A\\
&= \left(4+\frac{3}{2}r_1\right)r_1A\\
&\leq \frac{r_1}{2}.
\end{aligned}$$

Consequently $L:\Omega\to\Omega$. Again, applying Lemma 6.2, we obtain

$$\|Lu_1-Lu_2\|\leq(2r_1+1)A\|u_1-u_2\|.$$

Therefore $L:\Omega\to\Omega$ is a contraction operator and there exists a unique $u\in\Omega$ so that $u=Lu$ or $(I-T)u=Sv$. Then $S(\overline{U})\subset(I-T)(\Omega)$.

(4) Assume that there are an $u \in \partial U$ and $\lambda \geq 1$ so that

$$Su = (I - T)(\lambda u) \quad \text{and} \quad \lambda u \in \Omega.$$

Then

$$Su = (I + F_1)(\lambda u)$$

and applying Lemma 6.2, we obtain

$$\begin{aligned} \left(3 + \frac{r_1}{2}\right) r_1 A &\geq \|Su\| \\ &\geq \lambda\|u\| - \|F_1(\lambda u)\| \\ &\geq \lambda\|u\| - (1 + r_1)A\|\lambda u\| \\ &= (1 - (1 + r_1)A)\,\lambda\|u\| \\ &\geq (1 - (1 + r_1)A)\|u\| \\ &= r_1(1 - (1 + r_1)A), \end{aligned}$$

whereupon

$$\left(3 + \frac{r_1}{2}\right) A \geq 1 - (1 + r_1)A \quad \text{or} \quad \left(4 + \frac{3}{2} r_1\right) A \geq 1,$$

which is a contradiction.

Hence and Proposition 3.4, it follows that the operator $T + S$ has at least one fixed point in $U \bigcap \Omega$, which is a nontrivial nonnegative solution of the IVP (6.1)-(6.2). This completes the proof.

6.1. **Example.** Below, we will illustrate our main result. Let

$$h(x) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}. \end{aligned}$$

Therefore

$$\begin{aligned} -\infty &< \lim_{s \to \pm\infty} (1 + s + \cdots + s^9)h(s) < \infty, \\ -\infty &< \lim_{s \to \pm\infty} (1 + s + \cdots + s^9)l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_1 so that

$$(1+s+\cdots+s^9)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \le C_1,$$

$$(1+s+\cdot+s^9)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \le C_1,$$

$s\in[0,\infty)$. Note that by [10](pp. 707, Integral 79), we have

$$\int\frac{dz}{1+z^4}=\frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2}+\frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s)=\frac{s^{10}}{(1+s^{44})\left(1+(1+s+\cdots+s^9)^2\right)^{28}},\quad s\in[0,\infty),$$

and

$$g_1(t,x)=Q(t)Q(x),\quad t,x\in[0,\infty).$$

Then there exists a positive constant A_1 such that

$$720(1+t+\cdots+t^6)(1+x+\cdots+x^6)\int_0^t\int_0^x g_1(t_1,x_1)dx_1dt_1\le A_1,$$

$t,x\ge 0$. Take $g(t,x)=\frac{g_1(t,x)}{280A_1}$, $A=\frac{1}{50}$, $r_1=\frac{1}{4}$. Consider the IVP

$$u_t-u_{xx}-uu_x=u(1-u),\quad t>0,\quad x\ge 0,$$

$$u(0,x)=\frac{1}{8}+\frac{1}{8(1+x^2)},\quad x\ge 0.$$

Here $\alpha=-1$, $\beta=1$ on $[0,\infty)$,

$$\frac{r_1}{2}\le u_0(x)=\frac{1}{8}+\frac{1}{8(1+x^2)}\le r_1,\quad x\ge 0,$$

$$1-(1+r_1)A=\frac{39}{40}>0,\quad \left(4+\frac{3}{2}r_1\right)A=\frac{7}{80}<\frac{1}{2},$$

and

$$120\left(1+t+t^2+t^3+t^4\right)\left(1+x+x^2+x^3+x^4+x^5+x^6\right)$$

$$
\begin{aligned}
& \times \int_0^t \int_0^x g(t_1,x_1)\left(1+\int_0^{t_1}(\beta(t_2)-\alpha(t_2))dt_2\right)dx_1dt_1 \\
\leq\ & 240(1+t)\left(1+t+t^2+t^3+t^4\right)\left(1+x+x^2+x^3+x^4+x^5+x^6\right) \\
& \times \int_0^t \int_0^x g(t_1,x_1)dx_1dt_1 \\
\leq\ & \frac{720}{280A_1}\left(1+t+t^2+t^3+t^4+t^5\right)\left(1+x+x^2+x^3+x^4+x^5+x^6\right) \\
& \times \int_0^t \int_0^x g_1(t_1,x_1)dx_1dt_1 \\
\leq\ & \frac{1}{280} \\
\leq\ & A.
\end{aligned}
$$

Therefore the considered IVP has at least one non-negative solution $u \in \mathcal{C}^1([0,\infty), \mathcal{C}^2([0,\infty)))$. □

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