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Periodic points of solenoidal automorphisms in terms of inverse limits

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Abstract

In this paper, we describe the periodic points of automorphisms of a one dimensional solenoid, considering it as the inverse limit, $\varprojlim_{k}(S^1, \gamma_k)$ *of a sequence* (γ_k) *of maps on the circle* S^1*. The periodic points are discussed for a class of automorphisms on some higher dimensional solenoids also.*

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1. Introduction

We consider a dynamical system of the form (G, f), where G is a compact group and f is an automorphism of G. The study of dynamics involves the eventual behaviour of trajectories of its points i.e., the sequences $(f^n(x))_{n=0}^{\infty}$, where $x \in G$ and $f^n = f \circ f \circ ... \circ f$ (n times for $n \in \mathbb{N}$) and f^0 is the identity map on G. A point $x \in G$ is said to be periodic with a period n if there is an $n \in \mathbb{N}$ such that $f^n(x) = x$. The sets of periods and periodic points of a family of dynamical systems are well studied in literature. See for instance, [3], [6], [7], [8], [12], [17] and [18]. In this paper, we give a description of the sets of periodic points of automorphisms of a one dimensional solenoid i.e., we describe the set $P(f) = \{x \in \Sigma : x \text{ is a periodic point of } f\}$, where Σ is a one

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dimensional solenoid and f is an automorphism of Σ. It is then extended to some automorphisms of certain higher dimensional solenoids.

Solenoids are extensively studied in literature. Some of the papers consider solenoids as dual groups of subgroups of $\mathbb{Q}^n, n \in \mathbb{N}$, while others consider them as inverse limits of certain maps on the n-dimensional torus, $\mathbb{T}^n$. In [19], it is shown that an ergodic automorphism of a solenoid is measure theoretically isomorphic to a Bernoulli shift. The papers [1], [5] and [14] discuss about the structure of a solenoid, whereas [13] describes the structure of group of automorphisms of a solenoid. The articles [2] and [11] calculate the entropy and the zeta function respectively, for an automorphism of a solenoid. The papers [9] and [10] consider the flows on higher dimensional solenoids. We use results from [9] to describe the sets of periodic points of some automorphisms on certain higher dimensional solenoids.

There are articles on counting the number of periodic points of a dynamical system; this forms a crucial part in defining the zeta function. The number of periodic points of any given period for some continuous homomorphisms of a one dimensional solenoid was discussed in [15]. We show that our description of periodic points of one dimensional solenoidal automorphisms is in accordance with this result.

A characterization of sets of periodic points for automorphisms of a one dimensional solenoid was given in [12], where it was described in terms of adeles, a number theoretic concept. It was based on a description of subgroups of $\mathbb{Q}$. It may however be noted that this characterization for one dimensional solenoids may not be extended to higher dimensions with similar ideas, as there is no neat description of subgroups of $\mathbb{Q}^n$ for $n > 1$. We now describe the sets of periodic points, again for automorphisms of one dimensional solenoids in a different manner, namely in terms of inverse limits and then use it to describe the periodic points of some automorphisms on certain higher dimensional solenoids that are inverse limits of sequences of maps on $\mathbb{T}^n, n > 1$. In all these cases, we show that the set of periodic points of a given period is the inverse limit of the same maps (that define the solenoid) restricted to a subgroup of $\mathbb{T}^n$. This may help in giving a characterization of periodic points for automorphisms of other higher dimensional solenoids also.

1.1. **Definitions and Notations.** A solenoid is a compact connected finite dimensional abelian group. An equivalent interpretation is that, a topological group Σ is an $n-$dimensional solenoid if and only if its Pontryagin dual $\widehat{\Sigma}$ is (isomorphic to) a subgroup of the discrete additive group $\mathbb{Q}^n$ and contains $\mathbb{Z}^n$ (see [16]). Thus, a one dimensional solenoid is a topological group whose dual is a subgroup of $\mathbb{Q}$ and contains $\mathbb{Z}$. Definition 1.2 gives an equivalent description of a one dimensional solenoid Σ as the inverse limit of a sequence of maps on the circle S^1, where $\mathbb{Z} \subsetneqq \widehat{\Sigma} \subseteq \mathbb{Q}$. In section 3, we consider n-dimensional solenoids, for $n > 1$, which are inverse limits of sequences of maps on $\mathbb{T}^n$. We use the notations $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$ and P to denote the sets of integers, positive integers, non-negative integers and the prime numbers respectively.

For $a, b \in \mathbb{Z}$, we use the notations $a \mid b$, if a divides b; else, we write $a \nmid b$. To represent a sequence $(a_1, a_2, \cdots)$ of positive integers, besides the customary notations $(a_k)_{k=1}^{\infty}$ and (a_k), we also use a single capital letter. For instance, we write $A = (a_k) = (a_k)_{k=1}^{\infty} = (a_1, a_2, \cdots)$. We write $diag\,[m_1, m_2, \cdots, m_n]$ for an $n \times n$ diagonal matrix with $m_1, m_2, \cdots, m_n$ on the principal diagonal.

Definition 1.1. Let X_k be a topological space for each $k \in \mathbb{N}_0$ and $f_k : X_k \to X_{k-1}$ be a continuous map for each $k \in \mathbb{N}$. Then the subspace of $\prod_{k=0}^{\infty} X_k$ defined as $\varprojlim_k (X_k, f_k) = \{(x_k) \in \prod_{k=0}^{\infty} X_k : x_{k-1} = f_k(x_k), \forall k \in \mathbb{N}\}$ is called the inverse limit of the sequence of maps (f_k).

Definition 1.2. Let $A = (a_1, a_2, \cdots)$ be a sequence of integers such that $a_k \geq 2$ for every $k \in \mathbb{N}$. The solenoid corresponding to the sequence A, denoted by Σ_A, is defined as $\Sigma_A = \{(x_k) \in (S^1)^{(\mathbb{N}_0)} : x_{k-1} = a_k x_k \,(mod\, 1)$ for every $k \in \mathbb{N}\}$.

In other words, the one dimensional solenoid Σ_A is the inverse limit, $\varprojlim_k (S^1, \gamma_k)$, where $\gamma_k : S^1 \to S^1$ is defined as $\gamma_k(x) = a_k x \,(mod\, 1)$.

2. One Dimensional Solenoids

The descriptions of a one dimensional solenoid as an inverse limit and as the dual group of a subgroup of $\mathbb{Q}$ are very closely related. The dual of a one dimensional solenoid Σ_A, where $A = (a_k)$ is isomorphic to the subgroup of $\mathbb{Q}$ generated by $\{\frac{1}{a_1 a_2 \cdots a_k} : k \in \mathbb{N}\}$. Now, a subgroup of $\mathbb{Q}$ is characterized by a sequence, called the *height sequence*, indexed by prime numbers and with values in $\mathbb{N}_0 \cup \{\infty\}$. We will now discuss about this sequence and establish a relation between the terms of this sequence and the integers a_k's. One may refer to [4] for more details about the structure of subgroups of $\mathbb{Q}$.

Let $S \subseteq \mathbb{Q}$ and $x \in S$. For a $p \in P$, the $p-$height of x with respect to S, denoted by $h_p^{(S)}(x)$ is defined as the largest non-negative integer n, if it exists, such that $\frac{x}{p^n} \in S$; otherwise, define $h_p^{(S)}(x) = \infty$. Thus, we have a sequence $(h_p^{(S)}(x))$, p ranging over prime numbers in the usual order, with values in $\mathbb{N}_0 \cup \{\infty\}$. We call such sequences as *height sequences*. If (u_p) and (v_p) are two height sequences such that $u_p = v_p$ for all but finitely many primes and $u_p = \infty \Leftrightarrow v_p = \infty$, then they are said to be equivalent. If S is a subgroup of $\mathbb{Q}$, then there is a unique height sequence (up to equivalence) associated to all non-zero elements of S. Also, two subgroups of $\mathbb{Q}$ are isomorphic if and only if their associated height sequences are equivalent.

Given a subgroup S of $\mathbb{Q}$, for every $p \in P$, we assign an element $n_p^{(S)}$ of $\mathbb{N}_0 \cup \{\infty\}$ as follows. Let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the field of $p-$adic numbers and the ring of $p-$adic integers respectively and $|u|_p$ denote the $p-$adic norm of $u \in \mathbb{Q}_p$. Then define $n_p^{(S)} = sup\{h_p^{(S)}(x) : x \in S \cap \mathbb{Z}_p^*\}$, where $\mathbb{Z}_p^*$ is the multiplicative group $\{x \in \mathbb{Z}_p : |x|_p = 1\}$. Now, the information whether $n_p^{(S)}$

is finite or not, for a given p, is going to play a crucial role in our discussion. So, we define $D_\infty^{(S)} = \{p \in P : n_p^{(S)} = \infty\}$. We will use the notations $n_p^{(S)}$ and $D_\infty^{(S)}$, as defined here, throughout this paper. We now have the following relation between the sequences $(n_p^{(S)})$ and A, where S is the dual of Σ_A.

Proposition 2.1. *Let Σ_A be a one dimensional solenoid and $S = \widehat{\Sigma}_A$, where $A = (a_k)$. Let $p \in P$ and $n_p = n_p^{(S)}$. Then,*

(1) *$p \in D_\infty^{(S)}$ if and only if for every $j \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $p^j | a_1 a_2 \cdots a_k$.*

(2) *If $p \notin D_\infty^{(S)}$, then n_p is the largest non-negative integer such that $p^{n_p} | a_1 a_2 \cdots a_k$ for some k.*

Proof. (1) Suppose $p \in D_\infty^{(S)}$. Since $n_p = \infty$, for any $j \in \mathbb{N}$, there exists an $x \in S \cap \mathbb{Z}_p^*$ with $h_p^{(S)}(x) > j$. Now, $x \in \mathbb{Z}_p^*$ implies that $x = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $(a,p) = (b,p) = 1$. Also, $h_p^{(S)}(x) > j$ implies that $\frac{x}{p^j} = \frac{a}{p^j b} \in S$. But, $S = \left\{ \frac{i}{a_1 a_2 \cdots a_k} : i \in \mathbb{Z}, k \in \mathbb{N} \right\}$. Thus, $\frac{a}{p^j b} = \frac{i}{a_1 a_2 \cdots a_k}$ for some $i \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then, we have $a a_1 a_2 \cdots a_k = i p^j b$ implying that $p^j | a_1 a_2 \cdots a_k$.

For the converse, let $j \in \mathbb{N}$. Then, there exists a $k \in \mathbb{N}$ such that $a_1 a_2 \cdots a_k = p^j i$ for some $i \in \mathbb{N}$. This implies that $\frac{1}{p^j} = \frac{i}{a_1 a_2 \cdots a_k} \in S$ and thus $h_p^{(S)}(1) \geq j$. Since j is chosen arbitrarily and $1 \in S \cap \mathbb{Z}_p^*$, we get, $n_p = \infty$ i.e., $p \in D_\infty^{(S)}$.

(2) Suppose $p \notin D_\infty^{(S)}$. Then, $n_p = max\{h_p^{(S)}(x) : x \in S \cap \mathbb{Z}_p^*\}$. Say $n_p = h_p^{(S)}(x_0)$ for some $x_0 \in S \cap \mathbb{Z}_p^*$ i.e., $\frac{x_0}{p^{n_p}} \in S$. Let $x_0 = \frac{u_0}{v_0}$, for some $u_0, v_0 \in \mathbb{Z}$. Then $(u_0, p) = (v_0, p) = 1$. Now, $\frac{x_0}{p^{n_p}} \in S$ implies that $\frac{u_0}{p^{n_p} v_0} = \frac{i}{a_1 a_2 \cdots a_k}$ for some $i \in \mathbb{Z}$ and $k \in \mathbb{N}$ i.e, $u_0 a_1 a_2 \cdots a_k = i p^{n_p} v_0$ and hence $p^{n_p} | a_1 a_2 \cdots a_k$.

If possible, let $l > n_p$ such that $p^l | a_1 a_2 \cdots a_j$ for some j. But then, $a_1 a_2 \cdots a_j = p^l i'$ for some $i' \in \mathbb{N}$ implying that $\frac{1}{p^l} = \frac{i'}{a_1 a_2 \cdots a_j} \in S$ and thus $h_p^{(S)}(1) \geq l > n_p$ which is a contradiction. Therefore, n_p is the largest integer such that $p^{n_p} | a_1 a_2 \cdots a_k$ for some k.

□

The following corollary follows from the above proposition.

Corollary 2.2. *Let Σ_A, S and $D_\infty^{(S)}$ be defined as above. Then, for a $p \in P$, $p \in D_\infty^{(S)}$ if and only if p divides infinitely many a_k's.*

If f is an automorphism of a one dimensional solenoid Σ, then its dual is an automorphism of a subgroup of $\mathbb{Q}$ and thus, it is multiplication by a non-zero rational number, say $\frac{\alpha}{\beta}$ and for any $(x_k) \in \Sigma$, $f((x_k)) = (\frac{\alpha}{\beta} x_k (mod\, 1))$. We say that f is induced by $\frac{\alpha}{\beta}$. It is known that f is ergodic if and only if $\frac{\alpha}{\beta} \neq \pm 1$. Further, we can assume that $A = (\beta b_k)$, where each b_k is a positive integer coprime to β. In this case, we can write $f((x_k)) = (\alpha b_1 x_1, \alpha b_2 x_2, ...)$ for each $(x_k) \in \Sigma_{(\beta b_k)}$. See [19] for all these details about automorphisms.

We now state and prove our main results, namely the description of periodic points (Theorem 2.5) and the number of periodic points (Theorem 2.7) of an automorphism of a one dimensional solenoid. Before that, the following proposition describes the elements of a one dimensional solenoid with rational coordinates, in terms of the prime factors of a_k's and the succeeding proposition shows that a periodic point should have only rational coordinates.

Proposition 2.3. *Let Σ_A be a one dimensional solenoid where $A = (a_k)$ and $(x_k) = (\frac{u_k}{v_k}) \in \Sigma_A \cap \mathbb{Q}^{\mathbb{N}_0}$, where $u_k, v_k \in \mathbb{Z}$ such that $(u_k, v_k) = 1$. For a $p \in P$, denote $|v_k|_p = \frac{1}{p^{c_k}}$, for every $k \geq 0$ and let $|a_k|_p = \frac{1}{p^{d_k}}$, for every $k \geq 1$. If h is the least integer such that $c_h > 0$, then $c_k = c_h + d_{h+1} + d_{h+2} + \cdots + d_k$, for every $k > h$.*

Proof. It follows from the definition of a one dimensional solenoid that $\frac{u_h}{v_h} = a_{h+1}a_{h+2}\cdots a_k \frac{u_k}{v_k} + j$ for some $j \in \mathbb{Z}$. Since $c_h > 0$, it follows that $(u_h, p) = (u_k, p) = 1$. Then, we can find positive integers $a'_{h+1}, a'_{h+2}, \cdots, a'_k, v'_k$ and v'_h, each of which is coprime to p, such that

$$\frac{u_h}{p^{c_h} v'_h} = \frac{p^{d_{h+1}+d_{h+2}+\cdots d_k} a'_{h+1} a'_{h+2} \cdots a'_k u_k}{p^{c_k} v'_k} + j$$

$$\Rightarrow p^{c_k} v'_k u_h = p^{c_h + d_{h+1} + \cdots d_k} v'_h a'_{h+1} \cdots a'_k u_k + j p^{c_k + c_h} v'_k v'_h$$

$$\Rightarrow p^{c_k} \left(v'_k u_h - p^{c_h} j v'_k v'_h \right) = p^{c_h + d_{h+1} + \cdots d_k} v'_h a'_{h+1} \cdots a'_k u_k$$

Now, since $c_h > 0$, p does not divide $\left(v'_k u_h - p^{c_h} j v'_k v'_h \right)$. Thus, $c_k = c_h + d_{h+1} + \cdots d_k$ for every $k > h$. □

Proposition 2.4. *Let Σ_A be a one dimensional solenoid and $S = \widehat{\Sigma}_A$, where $A = (a_k)$. If (x_k) is periodic in (Σ_A, ϕ), where ϕ is an automorphism of Σ_A induced by $\frac{\alpha}{\beta}$, then $x_k \in \mathbb{Q}$ for every $k \in \mathbb{N}_0$. Further, for any $p \in D_\infty^{(S)}$, we have $|x_k|_p \leq 1$ for every $k \in \mathbb{N}_0$.*

Proof. Say $\phi^l((x_k)) = (x_k)$ for some $l \in \mathbb{N}$. Then, for any $k \in \mathbb{N}_0$, $\frac{\alpha^l}{\beta^l} x_k = x_k + j_k$ for some $j_k \in \mathbb{Z}$ and thus $x_k \in \mathbb{Q}$. Let $x_k = \frac{u_k}{v_k}$, where $u_k,\ v_k \in \mathbb{Z}$ and $(u_k, v_k) = 1$. Then, $(\alpha^l - \beta^l) u_k = \beta^l v_k j_k$ for every $k \geq 0$. For a prime number p, let us now denote $|v_k|_p = \frac{1}{p^{c_k}}$, for every $k \geq 0$ and $|a_k|_p = \frac{1}{p^{d_k}}$, for every $k \geq 1$.

Let $p \in D_\infty^{(S)}$. Then, by Corollary 2.2, $p | a_k$ for infinitely many k and thus $d_k > 0$ for infinitely many k. Suppose there exists an $r \in \mathbb{N}_0$ such that $p | v_r$. Then, $c_r > 0$ and $(\alpha^l - \beta^l) u_r = \beta^l v_r j_r$ implies that $p^{c_r} | (\alpha^l - \beta^l)$. Now from Proposition 2.3, $c_{r+k} = c_h + d_{h+1} + \cdots + d_r + d_{r+1} + \cdots + d_{r+k}$, where h is the least integer such that $c_h > 0$. Then, $h \leq r$ and $c_{r+k} = c_r + d_{r+1} + d_{r+2} + \cdots + d_{r+k}$. Again, since $(\alpha^l - \beta^l) u_{r+k} = \alpha^l v_{r+k} j_{r+k}$ for every $k \geq 0$, we get $p^{c_r + d_{r+1} + \cdots + d_{r+k}} |$

$(\alpha^l - \beta^l)$. This is a contradiction, as infinitely many of $d_{r+1}, d_{r+2}, \cdots$ are non-zero. Hence, $p \nmid v_k$ for any k. Therefore, $|x_k|_p \leq 1$ for every $k \geq 0$.

□

In the following theorem about the set of periodic points of the dynamical system$(\Sigma_{(a_k)}, \frac{\alpha}{\beta})$, we assume that $a_k = \beta b_k$, where each b_k is a positive integer coprime to β. As noted already, there is no loss of generality in assuming this (see [19]).

Theorem 2.5. *Let ϕ be an automorphism of a one dimensional solenoid Σ_A induced by $\frac{\alpha}{\beta}$, where $A = (\beta b_k)$, each b_k being co-prime to β. For each $l \in \mathbb{N}$, define $U_l = \bigcap_{p \in P} \left(\frac{1}{p^{e_{p,l}}} \mathbb{Z}_p \cap \mathbb{Q} \cap S^1 \right)$, where $p^{e_{p,l}} = \frac{1}{|\alpha^l - \beta^l|_p}$. If $\gamma_{k,l} : U_l \to U_l$ is the map defined as $\gamma_{k,l}(x) = \beta b_k x (mod\ 1)$ for each $k \in \mathbb{N}$ and $l \in \mathbb{N}$, then $P(\phi) = \bigcup_{l=1}^{\infty} \varprojlim_k (U_l, \gamma_{k,l})$.*

Proof. Let (x_k) be a periodic point with a period l. Then, $x_k \in \mathbb{Q}$ for every $k \geq 0$; say $x_k = \frac{u_k}{v_k}$, where $u_k,\ v_k \in \mathbb{Z}$ such that $(u_k, v_k) = 1$. Again, for every prime p, let $|v_k|_p = \frac{1}{p^{c_k}}$, for every $k \geq 0$. Now, $\phi^l((x_k)) = (x_k)$ implies that $(\alpha^l - \beta^l) u_k = \beta^l v_k j_k$ for some $j_k \in \mathbb{Z}$. Since $p^{c_k} | v_k$, it follows that $p^{c_k} | (\alpha^l - \beta^l)$ and thus $c_k \leq e_{p,l}$. We can now write $x_k = \frac{1}{p^{e_{p,l}}} . \frac{p^{e_{p,l} - c_k} . u_k}{v_k^{'}}$, for some $v_k^{'} \in \mathbb{Z}$ such that $(v_k^{'}, p) = 1$. It then follows that $x_k \in \frac{1}{p^{e_{p,l}}} \mathbb{Z}_p$, because $|\frac{p^{e_{p,l} - c_k} . u_k}{v_k^{'}}|_p \leq \frac{1}{p^{e_{p,l} - c_k}} \leq 1$. Since p was chosen arbitrarily, we conclude that $x_k \in U_l$, for every $k \geq 0$.

On the other hand, let $(x_k) \in \varprojlim_k (U_l, \gamma_{k,l})$ for some $l \in \mathbb{N}$. Say $x_k = \frac{u_k}{v_k}$, where $u_k,\ v_k \in \mathbb{Z}$ such that $(u_k, v_k) = 1$. Write $v_k = \prod_{p | v_k} p^{c_p}$, for some $c_p \in \mathbb{N}$. Then, for any $p | v_k$, $|x_k|_p = p^{c_p}$. Also, $|x_k|_p \leq p^{e_{p,l}}$, for any $p \in P$. Thus, $c_p \leq e_{p,l}$ and hence $v_k | (\alpha^l - \beta^l)$. Therefore, $\frac{\alpha^l - \beta^l}{v_k} \in \mathbb{Z}$, for every k. Then, $\phi^l((x_k)) - (x_k) = \left((\alpha^l - \beta^l) b_{k+1} b_{k+2} ... b_{k+l} x_{k+l} \right) = (0)$ implying that (x_k) is periodic. □

Remark 2.6. The set of periodic points of period l is equal to $\varprojlim_k (U_l, \gamma_{k,l})$. Here U_l is a subgroup of S^1 and the map $\gamma_{k,l}$ is the restriction of γ_k to U_l, where γ_k is a map on S^1 such that $\Sigma_{(\beta b_k)} = \varprojlim_k (S^1, \gamma_k)$.

The following theorem about the number of periodic points, which follows from the above description, is in accordance with a similar result in [15].

Theorem 2.7. *Let ϕ be an automorphism of a one dimensional solenoid Σ_A induced by $\frac{\alpha}{\beta}$ and for every $l \in \mathbb{N}$, let $e_{p,l} = \frac{1}{|\alpha^l - \beta^l|_p}$. Then the number of periodic points of ϕ with a period l is $\prod_{p \notin D_{\infty}^{(S)}} p^{e_{p,l}}$.*

Proof. Since $\alpha^l - \beta^l \in \mathbb{Z}$, $e_{p,l}$ is positive only for finitely many primes. Thus, there is a finite subset F of $P \setminus D_\infty^{(S)}$ such that for a $p \notin D_\infty^{(S)}$, $e_{p,l} \neq 0$ if and only if $p \in F$.
Therefore $\prod_{p \notin D_\infty^{(S)}} p^{e_{p,l}} = \prod_{p \in F} p^{e_{p,l}}$.

We first claim that (x_k) is periodic with a period l if and only if for every $k \in \mathbb{N}_0$, $x_k = \frac{u_k}{v_k}$, where $u_k,\ v_k \in \mathbb{Z}$, $0 \leq u_k < v_k$ and $v_k = \prod_{p \in F} p^{f_{p,k}}$ with $0 \leq f_{p,k} \leq e_{p,l}$.

If $\phi^l((x_k)) = (x_k)$, then for every $k \in \mathbb{N}_0$, $x_k \in \frac{1}{p^{e_{p,l}}}\mathbb{Z}_p \cap \mathbb{Q}$, for every $p \in P$. Let $x_k = \frac{u_k}{v_k}$ for some $u_k,\ v_k \in \mathbb{Z}$ such that $(u_k, v_k) = 1$. Now, $x_k \in \frac{1}{p^{e_{p,l}}}\mathbb{Z}_p$ implies that $|x_k|_p \leq p^{e_{p,l}}$, for every p. From Proposition 2.4, if $p \in D_\infty^{(S)}$, then $p \nmid v_k$. Also, for a prime p not in F, $e_{p,l} = 0$ implies that $p \nmid v_k$. Thus, the prime factorisation of $v_k = \prod_{p \in F} p^{f_{p,k}}$ for some $0 \leq f_{p,k} \leq e_{p,l}$. Since $x_k \in [0, 1)$, we conclude that $0 \leq u_k < v_k$.

Conversely, if $x_k = \frac{u_k}{v_k}$, where u_k and v_k satisfy the given conditions, then $|x_k|_p \leq 1$, for $p \notin F$ and $|x_k|_p \leq p^{f_{p,k}}$ for $p \in F$. In any case $|x_k|_p \leq p^{e_{p,l}}$ and thus $x_k \in U_l$. Hence the claim follows.

For a $p \in F$, let $|a_k|_p = \frac{1}{p^{d_k}}$, for every $k \in \mathbb{N}$. As this d_k depends on p we will denote $d_k = d_k^{(p)}$. Again, there are at most finitely many $k \in \mathbb{N}$ for which $d_k^{(p)} > 0$, as $F \subseteq P \setminus D_\infty^{(S)}$; let these positive integers be denoted by $d_{k_1}^{(p)}, d_{k_2}^{(p)}, ..., d_{k_{\alpha(p)}}^{(p)}$, where $\alpha(p) \in \mathbb{N}_0$. Further, assume that $k_1 < k_2 < ... < k_{\alpha(p)}$. Let $K = max\{k_{\alpha(p)} : p \in F\}$, if $k_{\alpha(p)} > 0$ for at least some $p \in F$; otherwise, define $K = 0$. Then, $d_k^{(p)} = 0$ for every $k > K$ and for every $p \in F$.

Let $(x_k) \in \Sigma_A$ be periodic; say $x_k = \frac{u_k}{v_k}$, where $u_k,\ v_k \in \mathbb{Z}$ such that $(u_k, v_k) = 1$. We have $x_K = \frac{u_K}{v_K}$, where $0 \leq u_K < v_K$ and $v_K = \prod_{p \in F} p^{f_{p,K}}$ with $0 \leq f_{p,K} \leq e_{p,l}$. For any $k < K$, the value of x_k is uniquely determined by x_K, as $x_k = a_{k+1}a_{k+2}...a_K x_K \,(mod\ 1)$.

Now, let $k > K$. It follows from Proposition 2.3 that $v_k = v_K$. Also, $x_K = a_{K+1}...a_k x_k \,(mod\ 1)$ i.e., $\frac{u_K}{v_K} = a_{K+1}...a_k \frac{u_k}{v_k} + j$ for some $j \in \mathbb{Z}$. By denoting $a_{K+1}...a_k = g_k$ and using the fact that $v_k = v_K$, we have $\frac{u_K}{v_K} = g_k \frac{u_k}{v_K} + j$. Since $d_k^{(p)} = 0$ for any $k > K$ and every $p \in F$, it follows that $p \nmid g_k$ for any $p \in F$. Having defined $\frac{u_K}{v_K}$, the distinct possible values for $\frac{u_k}{v_K}$ are $\frac{u_k}{v_K} = \frac{u_K}{g_k v_K} - \frac{j}{g_k}$, where $j \in \{0, 1, ..., g_k - 1\}$. Consider two such values, say $\frac{u_k^{(1)}}{v_K} = \frac{u_K}{g_k v_K} - \frac{j_1}{g_k}$ and $\frac{u_k^{(2)}}{v_K} = \frac{u_K}{g_k v_K} - \frac{j_2}{g_k}$ for some $j_1,\ j_2 \in \{0, 1, ...g_k - 1\}$. Then, $\frac{u_k^{(1)} - u_k^{(2)}}{v_K} = \frac{j_2 - j_1}{g_k}$ and thus $g_k\left(u_k^{(1)} - u_k^{(2)}\right) = v_K(j_2 - j_1)$. Now, if $j_1 \neq j_2$, then $|j_2 - j_1| < g_k$ and thus $g_k \nmid (j_2 - j_1)$. But then, there will be a prime p such that $p \mid g_k$ and $p \mid v_K$. On one hand, $p \mid g_k$ implies that $p \notin F$. On the other hand, $p \mid v_K$ implies that $p \mid \alpha^l - \beta^l$ and also $p \notin D_\infty^{(S)}$, which means that $p \in F$ leading to a contradiction. Hence, $j_1 = j_2$ i.e., $u_k^{(1)} = u_k^{(2)}$. Thus, there is only one possible value for x_k. Thus, a periodic point (x_k) is uniquely determined by the coordinate x_K. Now, since $0 \leq f_{p,K} \leq e_{p,l}$, the possible values of x_K are $\frac{i}{\prod\limits_{p \in F} p^{e_{p,l}}}$, where $0 \leq i < \prod\limits_{p \in F} p^{e_{p,l}}$. Thus, the theorem follows. □

3. n-DIMENSIONAL SOLENOIDS

We now extend our result about periodic points to some automorphisms of certain higher dimensional solenoids. Though this seems to be a small class, the reason for considering it is that the result follows immediately from what we have shown for one dimensional case. The higher dimensional solenoids that we are going to consider are isomorphic to products of one dimensional solenoids, as described in [9]. We mention here some notations, definitions and results from this paper that are needed to discuss our result.

For a positive integer $n > 1$, let $\pi^n : \mathbb{R}^n \to \mathbb{T}^n$ be the homomorphism defined as $\pi^n((x_1, x_2, ..., x_n)) = (x_1 \ (mod\ 1), x_2 \ (mod\ 1), ..., x_n \ (mod\ 1))$. Let $\overline{M} = (M_k)_{k=1}^{\infty} = (M_1, M_2, ...)$ be a sequence of $n \times n$ matrices with integer entries and non-zero determinant. Then, the $n-$dimensional solenoid $\sum_{\overline{M}}$ is defined as $\sum_{\overline{M}} = \{(\mathbf{x}_k) \in (\mathbb{T}^n)^{\mathbb{N}_0} : \pi^n(M_k\mathbf{x}_k) = \mathbf{x}_{k-1}$ for every $k \in \mathbb{N}\}$. In other words, $\sum_{\overline{M}} = \varprojlim\limits_{k} (\mathbb{T}^n, \delta_k)$, where $\delta_k : \mathbb{T}^n \to \mathbb{T}^n$ is defined as $\delta_k(\mathbf{x}) = \pi^n(M_k\mathbf{x})$

If ϕ is an automorphism of $\sum_{\overline{M}}$, then there is a matrix $L \in GL(n, \mathbb{Q})$ such that $\phi((\mathbf{x}_k)) = (\pi^n(L\mathbf{x}_k))$. We say that ϕ is induced by the matrix L. Now, consider n sequences of positive integers $A_1 = (a_1^1, a_2^1, ...)$, $A_2 = (a_1^2, a_2^2, ...)$, ... $A_n = (a_1^n, a_2^n, ...)$. Then define the sequence $\overline{M} = (M_k)$ of matrices as $M_k = diag[a_k^1, a_k^2, ..., a_k^n]$. These sequences of positive integers and matrices give rise to n one-dimensional solenoids and an $n-$dimensional solenoid. The following lemma from [9] gives a connection between these.

Lemma 3.1. *The map* $\eta : \prod_{i=1}^n \sum_{A_i} \to \sum_{\overline{M}}$ *given by* $\eta((x_k^1)_{k=1}^{\infty}, (x_k^2)_{k=1}^{\infty}, \ldots$ $\ldots, (x_k^n)_{k=1}^{\infty}) = ((x_1^1, x_1^2, \ldots, x_1^n), (x_2^1, x_2^2, \ldots, x_2^n), \ldots, (x_k^1, x_k^2, \ldots, x_k^n), \ldots)$ *is a topological isomorphism.*

We reserve these symbols A_i, $i = 1, 2, ..., n$ for the sequences of positive integers and M_k, $k \in \mathbb{N}$ for the corresponding diagonal matrices as described above. Now, let ϕ be an automorphism of $\sum_{\overline{M}}$ induced by a diagonal matrix, say $D = diag[\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, ..., \frac{\alpha_n}{\beta_n}]$. Then for each i, $\frac{\alpha_i}{\beta_i}$ induces an automorphism of the one dimensional solenoid $\sum_{A_i}$, say ψ_i. Again, by following [19], we assume that $A_i = (\beta_i b_k^i)$ for some suitable sequence (b_k^i) of positive integers. Then, the map $\psi : ((x_k^1)_{k=1}^{\infty}.(x_k^2)_{k=1}^{\infty}, \ldots, (x_k^n)_{k=1}^{\infty}) \mapsto (\psi_1((x_k^1)_{k=1}^{\infty}), \psi_2((x_k^2)_{k=1}^{\infty}), \ldots, \psi_n((x_k^n)_{k=1}^{\infty}))$ is an automorphism of $\prod_{i=1}^n \sum_{A_i}$. It is easy to see that $\eta \circ \psi = \phi \circ \eta$. Thus, we have the following proposition.

Proposition 3.2. $(\prod_{i=1}^n \sum_{A_i}, \psi)$ *is conjugate to* $(\sum_{\overline{M}}, \phi)$.

We now state and prove a theorem regarding the periodic points.

Theorem 3.3. *For each $l \in \mathbb{N}$, define $V_l = \prod_{i=1}^{n} \left(\bigcap_{p \in P} \left(\frac{1}{p^{e_{p,l,i}}} \mathbb{Z}_p \cap \mathbb{Q} \cap S^1 \right) \right)$, where $p^{e_{p,l,i}} = \frac{1}{|\alpha_i^l - \beta_i^l|_p}$. If $\delta_{k,l} : V_l \to V_l$ is the map defined as $\delta_{k,l}(\mathbf{x}) = \pi^n(M_k \mathbf{x})$ for each $k \in \mathbb{N}$ and $l \in \mathbb{N}$, then $P(\phi) = \bigcup_{l=1}^{\infty} \varprojlim_k (V_l, \delta_{k,l})$.*

Proof. Let $P_l(\phi)$ and $P_l(\psi)$ be the sets of periodic points of ϕ and ψ respectively, with a period $l \in \mathbb{N}$. Since η is a conjugacy from $(\prod_{i=1}^{n} \sum_{A_i}, \psi)$ to $(\sum_{\overline{M}}, \phi)$, it follows that $P_l(\phi) = \eta\,(P_l(\psi))$. But $P_l(\psi) = \prod_{i=1}^{n} P_l(\psi_i)$, where ψ_i is the automorphism of $\sum_{A_i}$ induced by $\frac{\alpha_i}{\beta_i}$. Thus by Theorem 2.5, $P_l(\psi) = \prod_{i=1}^{n} \left\{ (x_k^i)_{k=1}^{\infty} \in \sum_{A_i} : x_k^i \in \mathbb{Q} \text{ and } |x_k^i|_p \leq \frac{1}{p^{e_{p,l,i}}} \text{ for every } p \in P \right\}$. Then, $P_l(\phi) = \left\{ \left((x_k^1, x_k^2, \cdots x_k^n) \right)_{k=1}^{\infty} \in \sum_{\overline{M}} : x_k^i \in \mathbb{Q} \text{ and } |x_k^i|_p \leq \frac{1}{p^{e_{p,l,i}}} \text{ for every } p \in P \right\} = \varprojlim_k (V_l, \delta_{k,l})$. Thus, $P(\phi) = \bigcup_{l=1}^{\infty} \varprojlim_k (V_l, \delta_{k,l})$. □

Remark 3.4. The set of periodic points of ϕ with a period l is equal to $\varprojlim_k (V_l, \delta_{k,l})$. Here, V_l is a subgroup of $\mathbb{T}^n$ and $\delta_{k,l}$ is the restriction of δ_k to V_l, where each δ_k is a map on $\mathbb{T}^n$ such that $\sum_{\overline{M}} = \varprojlim_k (\mathbb{T}^n, \delta_k)$.

4. Conclusion

The periodic points of an automorphism of a one dimensional solenoid are described here. There are papers that discuss the number of periodic points or in general the zeta function of such automorphisms, whereas this paper gives an explicit description of these points. The paper [12], on the other hand, describes the sets of periodic points using adeles, but these ideas may not be useful for higher dimensional solenoids. Here, we have extended this result to certain automorphisms of higher dimensional solenoids also. Hence, the present description in terms of inverse limits may be helpful in more general cases.

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