APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 22, no. 2 (2021), 447-459
doi:10.4995/agt.2021.15566

Index boundedness and uniform connectedness of space of the G-permutation degree

R. B. Beshimov [a], D. N. Georgiou [b] and R. M. Zhuraev [a]

[a] National University of Uzbekistan named after Mirzo Ulugbek, str. University, 100174 Tashkent, Uzbeksitan (rbeshimov@mail.ru,rmjurayev@mail.ru)
[b] University of Patras, Department of mathematics, 26504 Patras, Greece (georgiou@math.upatras.gr)

Communicated by S. Romaguera

Abstract

In this paper the properties of space of the G-permutation degree, like: weight, uniform connectedness and index boundedness are studied. It is proved that:
(1) If $(X, \mathcal{U})$ is a uniform space, then the mapping $\pi^s_{n,G} : (X^n, \mathcal{U}^n) \to (SP^n_G X, SP^n_G \mathcal{U})$ is uniformly continuous and uniformly open, moreover $w(\mathcal{U}) = w(SP^n_G \mathcal{U})$;
(2) If the mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a uniformly continuous (open), then the mapping $SP^n_G f : (SP^n_G X, SP^n_G \mathcal{U}) \to (SP^n_G Y, SP^n_G \mathcal{V})$ is also uniformly continuous (open);
(3) If the uniform space $(X, \mathcal{U})$ is uniformly connected, then the uniform space $(SP^n_G X, SP^n_G \mathcal{U})$ is also uniformly connected.

2010 MSC: *54A05; 54E15; 55S15.*

Keywords: *G-permutation degree space; uniform space; uniform connectedness; index boundedness of uniform space; uniform continuity.*

1. Introduction

In [19], a functor $O : Comp \to Comp$ of weakly additive functionals acting in the category of compact and its continuous mappings is defined. It was proved that the functor $O : Comp \to Comp$ satisfies the normality conditions, except the preimage preservation condition. In [6], categorical and cardinal

Received 05 May 2021 – Accepted 08 June 2021

properties of hyperspaces with finite number of components are investigated. It was proved that the functor $C_n : Comp \to Comp$ is not normal, i.e., it does not preserve epimorphisms of continuous mappings. The authors of this paper also discussed the properties of density, caliber and Shanin number for the space $C_n(X)$. This space $C_n(X)$ is of great interest for researchers, since it contains the hyperspaces $\exp_n X$ of closed sets with cardinalities not greater than n elements. It was proved in [4] that the Radon functor satisfies all the normality conditions.

In [7], the topological properties of topological groups were studied. In [5], categorical and topological properties of the functor OS_τ of semiadditive τ-smooth functionals in the category $Tych$ of Tychonoff spaces and their continuous mappings, which extends the functors OS of semiadditive functionals in the category $Comp$ of compact and their continuous mappings, were investigated.

In [3], some properties of the functor $O_\beta : Tych \to Tych$ were considered, where β is the Čech-Stone compact extension in the category of Tychonoff spaces and their continuous mappings. This functor is regarded as an extension of the functor $O : Comp \to Comp$. The author in [3] proved that the space $O_\beta(X)$ is a convex subset of the space $C_p(C_b(X))$, where

$$C_b(X) = \{f \in C(X) | \, f : X \to R \text{ is a bounded function}\}$$

and $C_p(X)$ is the space of pointwise convergence. It was proved in [2] that if a covariant functor $\mathcal{F} : Comp \to Comp$ is weakly normal, then $\mathcal{F}^\beta : Tych \to Tych$ does not increase the density and weak density for any infinite Tychonoff space.

In [11], it was proved that the functor SP^n_G preserves the property of the fibers of the map to be a compact Q-manifold. In [9] some classes of uniform spaces are considered. In particular, the uniformly continuous mappings and absolutes, generalizations of metrics, normed, uniform unitary spaces, topological and uniform groups, its completions and spectral characterizations are studied. In addition, the properties of uniformly continuous and uniformly open mappings between uniform spaces are studied, too. But, it should be noted here that the class of uniformly continuous and uniformly open maps itself was introduced by Michael in [18].

In what follows, we present the basic notions that will be used in the rest of this article.

It is known that a *permutation group* is the group of all permutations, that is one-to-one mappings $X \to X$. A permutation group of a set X is usually denoted by $S(X)$. Especially, if $X = \{1, 2, \ldots, n\}$, then $S(X)$ is denoted by S_n.

Let X^n be the n-th power of a compact space X. The permutation group S_n of all permutations acts on the n-th power X^n as permutation of coordinates. The set of all orbits of this action with quotient topology is denoted by SP^nX. Thus, points of the space SP^nX are finite subsets (equivalence classes) of the product X^n. Two points $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$

are considered to be *equivalent* if there exists a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$. The space SP^nX is called the *n-permutation degree of the space X*. Equivalent relation by which we obtain space SP^nX is called the *symmetric equivalence relation*. The n-th permutation degree is a quotient of X^n. Therefore, the quotient map is denoted by $\pi_n^s : X^n \to SP^nX$, where for every $x = (x_1, x_2, \ldots, x_n) \in X^n$,

$$\pi_n^s((x_1, x_2, \ldots, x_n)) = [(x_1, x_2, \ldots, x_n)]$$

is an orbit of the point $x = (x_1, x_2, \ldots, x_n) \in X^n$ [20].

Let G be a subgroup of the permutation group S_n and let X be a compact space. The group G acts also on the n-th power of the space X as permutation of coordinates. The set of all orbits of this action with quotient topology is denoted by SP^n_GX. The space SP^n_GX is called *G-permutation degree* of the space X [13]. Actually, it is the quotient space of the product of X^n under the G-symmetric equivalence relation.

An operation SP^n is the covariant functor in the category of compacts and it is said to be a *functor of G-permutation degree*. If $G = S_n$, then $SP^n_G = SP^n$ and if the group G consists of the unique element only, then $SP^n = X^n$.

Let T be a set and let A and B be subsets of $T \times T$, i.e., relations on the set T. The inverse relation of A will be denoted by A^{-1}, that is,

$$A^{-1} = \{(x,\, y) :\ (y,\, x) \in A\}.$$

The composition of A and B will be denoted by AB; thus we have

$$AB = \{(x,\, z) :\ \text{there exists a } y \in T \text{ such that } (x,\, y) \in A \text{ and } (y, z) \in B\}.$$

For an arbitrary relation $A \subset T \times T$ and for a positive integer n the relation $A^n \subset T \times T$ is defined inductively by the formulas:

$$A^1 = A \text{ and } A^n = A^{n-1}A.$$

Every set $V \subset T \times T$ that contains the diagonal $\Delta_T = \{(x, x) :\ x \in T\}$ of T is called an *entourage of the diagonal*.

Definition 1.1. Let T be a non-empty set. A family $\mathcal{U}$ of subsets of $T \times T$ is called a *uniformity on T*, if this family satisfies the following conditions:

(U1) Each $U \in \mathcal{U}$ contains the diagonal $\Delta_T = \{(x,\, x) : x \in T\}$ of T;
(U2) If $V_1,\, V_2 \in \mathcal{U}$, then $V_1 \cap V_2 \in \mathcal{U}$;
(U3) If $U \in \mathcal{U}$ and $U \subset V$, then $V \in \mathcal{U}$;
(U4) For each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V^2 \subset U$;
(U5) For each $U \in \mathcal{U}$ we have $U^{-1} \in \mathcal{U}$.

The pair $(T,\mathcal{U})$ is called *uniform space* [17]. Also, the elements of the uniformity $\mathcal{U}$ are called *entourages*. For an entourage $U \in \mathcal{U}$ and a point $x \in T$ the set

$$U(x) = \{y \in T :\ (x, y) \in U\}$$

is called the *U-ball* centered at x. For a subset $A \subset T$ the set $U(A) = \bigcup_{a \in A} U(a)$ is called the *U-neighborhood* of A [1].

A family $\mathcal{B}$ is called a *base for the uniformity* $\mathcal{U}$, if for any $V \in \mathcal{U}$ there exists a $W \in \mathcal{B}$ with $W \subset U$. The smallest cardinal number of the form $|\mathcal{B}|$, where $\mathcal{B}$ is a base for $\mathcal{U}$, is called the *weight of the uniformity* $\mathcal{U}$ and is denoted by $\omega(\mathcal{U})$.

Every base $\mathcal{B}$ for a uniformity on T has the following properties:
(BU1) For every $V_1, V_2 \in \mathcal{B}$ *there exists a* $V \in \mathcal{B}$ *such that* $V \subset V_1 \cap V_2$*;*
(BU2) For every $V \in \mathcal{B}$ *there exists a* $W \in \mathcal{B}$ *such that* $W^2 \subset V$*.*

Proposition 1.2 ([15])**.** *Suppose that a non-empty set* X *is given. Consider a family* $\mathcal{B}$ *of entourages of the diagonal, which has the properties (BU1)–(BU2) and* $\mathcal{B}^{-1} = \mathcal{B}$*. A family* $\mathcal{U}$ *is a uniformity on* X*, if it consists of all entourages which contain a member of* $\mathcal{B}$*. The family* $\mathcal{B}$ *is a base for* $\mathcal{U}$*. The uniformity* $\mathcal{U}$ *is called* the uniformity generated by the base $\mathcal{B}$.

Let $\{(X_s, U_s) : s \in S\}$ be a family of uniform spaces. A family $\mathcal{B}$ of all entourages of the diagonal, which has the form

$$\{(\{x_s\}, \{y_s\}) : (x_{s_i}, y_{s_i}) \in V_{s_i} \text{ for } s_1, s_2, \ldots, s_k \in S,\ V_{s_i} \in \mathcal{U}_{s_i},\ i = 1, 2, \ldots, k\},$$

generates a uniformity on the set $\prod_{s\in S} X_s$. This uniformity is called a *Cartesian product* of the uniformities $\{U_s : s \in S\}$ and is denoted by $\prod_{s\in S} \mathcal{U}_s$. If all the uniformities $\mathcal{U}_s$ are equal to each other, i.e., if $X_s = X$ and $\mathcal{U}_s = \mathcal{U}$ for $s \in S$, then the Cartesian product $\prod_{s\in S} \mathcal{U}_s$ is also denoted by $\mathcal{U}^\tau$, where $\tau = |S|$ [10].

Definition 1.3. A function $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is called uniformly continuous, if for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that

$$(f \times f)(U) = \{(f(x_1), f(x_2)) : (x_1, x_2) \in U\} \subset V.$$

Note that the condition $(f \times f)(U) \subset V$ is equivalent to the condition

$$f(U(x)) \subset V(f(x)) \text{ or } (f \times f)^{-1}(V) \in \mathcal{U} \text{ [14]}.$$

A uniformly continuous function $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly open, if for any $U \in \mathcal{U}$ there exists a $V \in \mathcal{V}$ such that

$$V(f(x)) \subset f(U(x))$$

for all $x \in X$ [12].

Let $\exp_c X$ and $\exp_c Y$ be the hyperspaces of X and Y, consisting of all nonempty compact subsets equipped with the Hausdorff uniformity. In [12], it was proved that if a (continuous) surjection $f : X \to Y$ between uniform spaces X and Y is perfect, then f is uniformly open if and only if $\exp_c f : \exp_c X \to \exp_c Y$ is uniformly open.

Remark 1.4. The uniform continuity of the mapping f does not always imply uniform openness, i.e., there is a mapping f that can be uniformly continuous, but cannot be uniformly open.

As an example, consider the mapping $f : (\mathbb{R},\mathcal{U}) \to (\mathbb{R}, \mathcal{V})$, where $f(x) = x$, $x \in X$, $\mathcal{V} = \{\Delta, \mathbb{R} \times \mathbb{R}\}$, $\mathcal{U} = \{U \subset \mathbb{R} \times \mathbb{R} : \Delta = \{(x, x) : x \in \mathbb{R}\} \subset U\}$ and

$\mathbb{R}$ is the set of all real numbers. This mapping f is uniformly continuous, but not uniformly open.

Recall that a bijective mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$, acting from the uniform space $(X, \mathcal{U})$ to the uniform space $(Y, \mathcal{V})$, is called *a uniform isomorphism* if the mappings $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ and $f^{-1} : (Y, \mathcal{V}) \to (X, \mathcal{U})$ are uniformly continuous [8].

Let $(X, \mathcal{U})$ be a uniform space and $D \in \mathcal{U}$. A pair of points x, y of the uniform space $(X,\mathcal{U})$ is said to be *related by a D-chain*, if there exists an integer k such that $(x, y) \in D^k$. The uniform space X is called *uniformly connected*, if every entourage D of X and every pair of points of X are related by a D-chain [16].

In our paper we use the following theorem, which have been proved in [15].

Theorem 1.5. *The uniformly continuous image of a uniformly connected space is uniformly connected.*

The smallest cardinal number τ is called an *index boundedness* of a uniform space $(X, \mathcal{U})$, if the uniformity $\mathcal{U}$ has a base $\mathcal{B}$ consisting of entourages of cardinality $\leq \tau$. The index boundedness is denoted by $l(\mathcal{U})$. The uniform space $(X, \mathcal{U})$ is called τ-bounded, if $l(\mathcal{U}) \leq \tau$ [8].

2. Uniformly open and uniformly continuous mappings

Theorem 2.1. *Let $(X, \mathcal{U})$ be a uniform space. A family $\mathcal{B}$ of all subsets of $SP^n_G X \times SP^n_G X$ of the form*

$$O\big[U_1, U_2, \ldots, U_n\big] = \{([x], [y]) : \textit{there exist permutations}$$
$$\sigma, \delta \in G \textit{ such that } \big(x_i, y_{\sigma(i)}\big) \in U_{\delta(i)},\ i = 1, 2, \ldots, n\},$$

where $\big\{U_1, U_2, \ldots, U_n\big\} \subset \mathcal{U}$, has the properties (BU1)–(BU2) and generates some uniformity on $SP^n_G X$.

Proof. First, we show that every set of the form $O\big[U_1, U_2, \ldots, U_n\big]$ is an entourage of the diagonal, where $\big\{U_1, U_2, \ldots, U_n\big\} \subset \mathcal{U}$. Take an arbitrary point $[x] = [(x_1, x_2, \ldots, x_n)] \in SP^n_G X$. Then $\big(x_i, x_i\big) \in U_i$ for all $i = 1, 2, \ldots, n$. In this case, we have that $\sigma = \delta = e$ is the unit element of the group G. Therefore,

$$([x], [x]) \in O\big[U_1, U_2, \ldots, U_n\big]$$

and

$$\Delta = \{([x], [x]) : [x] \in SP^n_G X\} \subset O\big[U_1, U_2, \ldots, U_n\big].$$

Choose any two entourages $O\big[U_1, U_2, \ldots, U_n\big]$ and $O\big[V_1, V_2, \ldots, V_n\big]$ of the family $\mathcal{B}$. It is clear, that $U_i \cap V_i \in \mathcal{U}$ for each $i = 1, 2, \ldots, n$. So, it is enough to show the following relation:

$$O\big[U_1 \cap V_1, U_2 \cap V_2, \ldots, U_n \cap V_n\big] \subset O\big[U_1, U_2, \ldots, U_n\big] \cap O\big[V_1, V_2, \ldots, V_n\big].$$

Let

$$([x], [y]) \in O\big[U_1 \cap V_1, U_2 \cap V_2, \ldots, U_n \cap V_n\big].$$

Then there exist permutations $\theta,\, \phi \in G$ such that $\big(x_i,\, y_{\theta(i)}\big) \in \big(U_{\phi(i)} \cap V_{\phi(i)}\big)$ for all $i = 1, 2, \ldots, n$. Put $\big(x_i,\, y_{\theta(i)}\big) \in U_{\phi(i)}$ and $\big(x_i,\, y_{\theta(i)}\big) \in V_{\phi(i)}$ for all $i = 1, 2, \ldots, n$. It means that

$$([x], [y]) \in O\big[U_1,\, U_2, \ldots,\, U_n\big] \cap O\big[V_1,\, V_2, \ldots,\, V_n\big].$$

For any entourage $O\big[U_1,\, U_2, \ldots,\, U_n\big] \in \mathcal{B}$ there is a $W_i \in \mathcal{U}$ such that $W_i^2 \subset U_i$ for each $i = 1, 2, \ldots, n$. Put $W = \bigcap\limits_{i=1}^{n} W_i$. We prove that

$$O\big[W_1',\, W_2', \ldots,\, W_n'\big]^2 \subset O\big[U_1,\, U_2, \ldots,\, U_n\big],$$

where $W_i' = W$ for every $i = 1, 2, \ldots, n$.
Let

$$([x], [y]) \in O\big[W_1',\, W_2', \ldots,\, W_n'\big]^2.$$

Then there exists an orbit $[z] \in SP^n_G X$ such that $([x], [z]) \in O\big[W_1',\, W_2', \ldots,\, W_n'\big]$ and $([z], [y]) \in O\big[W_1',\, W_2', \ldots,\, W_n'\big]$. Since $([x], [z]) \in O\big[W_1',\, W_2', \ldots,\, W_n'\big]$ there are permutations $\sigma_1,\, \delta_1 \in G$ such that $\big(x_i,\, z_{\sigma_1(i)}\big) \in W'_{\delta_1(i)} = W$ for all $i = 1, 2, \ldots, n$. If $([z], [y]) \in O\big[W_1',\, W_2', \ldots,\, W_n'\big]$, then there exist permutations $\varphi, \gamma \in G$ such that $\big(z_j,\, y_{\varphi(j)}\big) \in W'_{\gamma(j)} = W$ for all $j = 1, 2, \ldots, n$. Put $j = \sigma_1(i)$ and we obtain $\big(x_i,\, z_{\sigma_1(i)}\big) \in W$ and $\big(z_{\sigma_1(i)},\, y_{\varphi\sigma_1(i)}\big) \in W$. Thus, $\big(x_i,\, y_{\varphi\sigma_1(i)}\big) \in W^2 \subset W_i^2 \subset U_i$ for each $i = 1, 2, \ldots, n$. Consequently $([x], [y]) \in O[U_1,\, U_2, \ldots,\, U_n]$, i.e. $O[W_1',\, W_2', \ldots,\, W_n']^2 \subset O\big[U_1,\, U_2, \ldots,\, U_n\big]$.

Now we prove that

$$O\big[U_1,\, U_2, \ldots,\, U_n\big]^{-1} = O\big[U_1^{-1},\, U_2^{-1}, \ldots,\, U_n^{-1}\big].$$

Indeed, let $([x], [y]) \in O\big[U_1,\, U_2, \ldots,\, U_n\big]^{-1}$. Then $([y], [x]) \in O\big[U_1,\, U_2, \ldots,\, U_n\big]$ and there are permutations $\sigma_2,\, \delta_2 \in G$ such that $\big(y_i,\, x_{\sigma_2(i)}\big) \in U_{\delta_2(i)}$ for every $i = 1, 2, \ldots, n$. Put $j = \sigma_2(i)$. This implies that $i = \sigma_2^{-1}(j)$. The relation $\big(y_{\sigma_2^{-1}(j)},\, x_j\big) \in U_{\delta_2\sigma_2^{-1}(j)}$ implies that $\big(x_j,\, y_{\sigma_2^{-1}(j)}\big) \in U^{-1}_{\delta_2\sigma_2^{-1}(j)}$ for all $j = 1, 2, \ldots, n$. Therefore, $([x], [y]) \in O\big[U_1^{-1},\, U_2^{-1}, \ldots,\, U_n^{-1}\big]$. We have $O\big[U_1,\, U_2, \ldots,\, U_n\big]^{-1} \subset O\big[U_1^{-1},\, U_2^{-1}, \ldots,\, U_n^{-1}\big]$. The reverse inclusion is similarly.

By Proposition 1.2, the family $\mathcal{B}$ generates some uniformity $SP^n_G\mathcal{U}$ on the set $SP^n_G X$. Theorem 2.1 is proved. $\square$

Consider a mapping $\pi^s_{n,\,G} : (X^n, \mathcal{U}^n) \to \big(SP^n_G X,\, SP^n_G\mathcal{U}\big)$ defining as follows:

$$\pi^s_{n,\,G}\big(x_1,\, x_2, \ldots,\, x_n\big) = \big[(x_1,\, x_2, \ldots,\, x_n)\big]_G$$

for each $\big(x_1,\, x_2, \ldots,\, x_n\big) \in X^n$.

Theorem 2.2. *Let $(X, \mathcal{U})$ be a uniform space. Then the mapping*

$$\pi^s_{n,\,G} : \big(X^n,\, \mathcal{U}^n\big) \to \big(SP^n_G X,\, SP^n_G\mathcal{U}\big)$$

is uniformly continuous.

Proof. Let $O\big[U_1, U_2, \ldots, U_n\big]$ be any entourage in $\big(SP^n_G X,\, SP^n_G \mathcal{U}\big)$. Consider an entourage

$$U = \big\{(a,\, b) :\, \big(a_i,\, b_i\big) \in U_i,\ i = 1, 2, \ldots, n\big\}$$

in $\big(X^n, \mathcal{U}^n\big)$, where $a = \big(a_1,\, a_2, \ldots,\, a_n\big)$ and $b = \big(b_1,\, b_2, \ldots,\, b_n\big)$ are points of X^n. We prove that for all $x = \big(x_1,\, x_2, \ldots,\, x_n\big) \in X^n$,

$$\pi^s_{n,\,G}(U(x)) \subset O\big[U_1,\, U_2, \ldots,\, U_n\big]([x]).$$

Indeed, if $y = \big(y_1,\, y_2, \ldots,\, y_n\big) \in U(x)$, then $\big(x_i,\, y_i\big) \in U_i$ for each $i = 1, 2, \ldots, n$. From $\big(x_i,\, y_i\big) \in U_i$ we have $\big(x_i,\, y_{\sigma(i)}\big) \in U_{\delta(i)}$, where $\sigma = \delta = e \in G$. In this case, we have that $[y] \in O\big[U_1,\, U_2, \ldots,\, U_n\big]([x])$. Thus, $\pi^s_{n,\,G}(U(x)) \subset O\big[U_1,\, U_2, \ldots,\, U_n\big]([x])$. Theorem 2.2 is proved. $\square$

Theorem 2.3. *For a uniform space* $(X, \mathcal{U})$ *the mapping*

$$\pi^s_{n,\,G} : \big(X^n, \mathcal{U}^n\big) \to \big(SP^n_G X,\, SP^n_G \mathcal{U}\big)$$

is uniformly open.

Proof. By Theorem 2.2 the mapping $\pi^s_{n,\,G}$ is uniformly continuous. Let U be an arbitrary entourage in $\mathcal{U}^n$. Then there is a trace $\{U_1,\, U_2, \ldots,\, U_n\} \subset \mathcal{U}$ such that

$$\big\{(a, b) :\, \big(a_i,\, b_i\big) \in U_i,\ i = 1, 2, \ldots, n\big\} \subset U,$$

where $a = \big(a_1,\, a_2, \ldots,\, a_n\big)$ and $b = \big(b_1,\, b_2, \ldots,\, b_n\big)$ are points of X^n. We show that for any point $x = \big(x_1,\, x_2, \ldots,\, x_n\big) \in X^n$ we have

$$O\big[U'_1,\, U'_2, \ldots,\, U'_n\big]([x]) \subset \pi^s_{n,\,G}(U(x)),$$

where $U'_k = \bigcap\limits_{i=1}^{n} U_i$ for $k = 1, 2, \ldots, n$.

Indeed, if $[y] \in O\big[U'_1,\, U'_2, \ldots,\, U'_n\big]([x])$, then there exist permutations $\sigma,\, \delta \in G$ such that $\big(x_i,\, y_{\sigma(i)}\big) \in U'_{\delta(i)}$ for all $i = 1, 2, \ldots, n$. In particular, $\big(x_i,\, y_{\sigma(i)}\big) \in U_i$ for all $i = 1, 2, \ldots, n$, i.e.,

$$(x,\, y_\sigma) \in \big\{(a,\, b) :\, \big(a_i,\, b_i\big) \in U_i,\ i = 1, 2, \ldots, n\big\} \tag{2.1}$$

where $y_\sigma = \big(y_{\sigma(1)},\, y_{\sigma(2)}, \ldots,\, y_{\sigma(n)}\big)$. From (2.1) it follows that $y_\sigma \in U(x)$ and $[y] \in \pi^s_{n,\,G}(U(x))$. Therefore, $O\big[U'_1,\, U'_2, \ldots,\, U'_n\big]([x]) \subset \pi^s_{n,\,G}(U(x))$ for a point $x \in X^n$. Theorem 2.3 is proved. $\square$

Proposition 2.4. *Let* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *be a uniformly open mapping and* $f(X) = Y$. *Then* $w(\mathcal{V}) \leq w(\mathcal{U})$.

Proof. Let $w(\mathcal{U}) = \tau \geq \aleph_0$. Then there is a base $\mathcal{B} = \{U_\alpha : \alpha \in M\}$ of uniformity $\mathcal{U}$ such that $|M| = \tau$. We shall prove that the family

$$(f \times f)(\mathcal{B}) = \{(f \times f)(U_\alpha) :\, \alpha \in M\}$$

is a base of uniformity $\mathcal{V}$. Since the map f is uniformly open, we have that $(f \times f)(U_\alpha) \in \mathcal{V}$ for each $\alpha \in M$. For any entourage $V \in \mathcal{V}$ the relation $(f \times f)^{-1}(V) \in \mathcal{U}$ is true. In this case, there exists an index $\alpha_0 \in M$ such that $U_{\alpha_0} \subset (f \times f)^{-1}(V)$, i.e. $(f \times f)(U_{\alpha_0}) \subset V$. It means that the family $(f \times f)(\mathcal{B})$ is a base of uniformity $\mathcal{V}$. Proposition 2.4 is proved. $\square$

For a uniform space $(X, \mathcal{U})$ we define a mapping $\lambda : X \to SP^n_G X$, where $\lambda(x) = [(x, x, \dots, x)],\ x \in X$.

Proposition 2.5. *For a uniform space* $(X, \mathcal{U})$ *the mapping*

$$\lambda : (X, \mathcal{U}) \to \big(SP^n_G X,\, SP^n_G \mathcal{U}\big)$$

is a uniform embedding.

Proof. Let $\lambda|_\Delta : X \to \Delta$ be the restriction of the map $\lambda : X \to SP^n_G X$, where $\Delta = \{[(x, x, \dots, x)] : x \in X\}$. It is known that it is bijective. Let us show that the map $\lambda|_\Delta$ is uniformly continuous. Choose an arbitrary entourage $O\big[U_1, U_2, \dots, U_n\big] \in SP^n_G \mathcal{U}$. Put $U = \bigcap\limits_{i=1}^{n} U_i$. By the definition of uniformity we have $U \in \mathcal{U}$. It suffices to show that

$$\lambda|_\Delta(U(x)) \subset \big(O\big[U_1, U_2, \dots, U_n\big] \cap (\Delta \times \Delta)\big)\big(\lambda|_\Delta(x)\big)$$

for all $x \in X$. Clearly, $\lambda|_\Delta(x) = \lambda(x)$ and

$$\big(O\big[U_1, U_2, \dots, U_n\big] \cap (\Delta \times \Delta)\big)(\lambda|_\Delta(x)) = O\big[U_1, U_2, \dots, U_n\big](\lambda(x)) \cap \Delta$$

for $x \in X$. Let $y \in U(x)$. Then $(x, y) \in U \subset U_i$ for every $i = 1, 2, \dots, n$. In this case, we have $(\lambda(x), \lambda(y)) \in O\big[U_1, U_2, \dots, U_n\big]$, i.e.,

$$\lambda(y) \in O\big[U_1, U_2, \dots, U_n\big](\lambda(x)) \cap \Delta.$$

Now we show that the mapping $(\lambda|_\Delta)^{-1} : \Delta \to X$ is uniformly continuous. Take an arbitrary entourage $V \in \mathcal{U}$. The following relation holds:

$$(\lambda|_\Delta)^{-1}\big(O\big[V_1', V_2', \dots, V_n'\big](\lambda(x)) \cap \Delta\big) \subset V(x),\ x \in X,$$

where $V_k' = \bigcap\limits_{i=1}^{n} V_i$ for each $k = 1, 2, \dots, n$. It means that the mapping $(\lambda|_\Delta)^{-1}$ is uniformly continuous. Proposition 2.5 is proved. $\square$

Lemma 2.6 ([8])**.** *Let* $(X, \mathcal{U})$ *be a uniform space and* $(Y, \mathcal{U}|_Y)$ *be its subspace, where* $\mathcal{U}|_Y = \{U \cap (Y \times Y) : U \in \mathcal{U}\}$. *Then* $w(\mathcal{U}|_Y) \leq w(\mathcal{U})$.

Theorem 2.7. *Let* $(X, \mathcal{U})$ *be a uniform space. Then the equality* $w(\mathcal{U}) = w(SP^n_G \mathcal{U})$ *holds.*

Proof. Let $(X, \mathcal{U})$ be a uniform space. By Proposition 2.5 and Lemma 2.6 it follows that $w(\mathcal{U}) \leq w(SP^n_G \mathcal{U})$. By the definition of uniformity $SP^n_G \mathcal{U}$ on the set $SP^n_G X$ we have $w(SP^n_G \mathcal{U}) \leq w(\mathcal{U})$. Thus, we directly obtain $w(\mathcal{U}) = w(SP^n_G \mathcal{U})$. Theorem 2.7 is proved. $\square$

Consider an arbitrary mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$, where $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces. For an equivalence class $[(x_1, x_2, \dots, x_n)] \in SP^n_G X$, put

$$SP^n_G f\big[(x_1, x_2, \dots, x_n)\big]_G = \big[\big(f(x_1), f(x_2), \dots, f(x_n)\big)\big]_G.$$

The following mapping is defined

$$SP^n_G f : \big(SP^n_G X,\, SP^n_G \mathcal{U}\big) \to \big(SP^n_G Y,\, SP^n_G \mathcal{V}\big).$$

We obtained the following result.

Theorem 2.8. *Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniformly continuous mapping. Then the mapping $SP^n_G f : \big(SP^n_G X,\, SP^n_G \mathcal{U}\big) \to \big(SP^n_G Y,\, SP^n_G \mathcal{V}\big)$ is also uniformly continuous.*

Proof. Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniformly continuous mapping. Take an arbitrary entourage $O\big[V_1, V_2, \ldots, V_n\big] \in SP^n_G \mathcal{V}$. Then there is an entourage $U_i \in \mathcal{U}$ such that $f\big(U_i(a)\big) \subset V_i\big(f(a)\big)$ for all $a \in X$ and $i = 1, 2, \ldots, n$. We show that

$$SP^n_G f\big(O\big[U_1, U_2, \ldots, U_n\big]([x])\big) \subset O\big[V_1, V_2, \ldots, V_n\big]\big(SP^n_G f([x])\big)$$

for a point $[x] \in SP^n_G X$. Choose an orbit $[y] \in O\big[U_1, U_2, \ldots, U_n\big]([x])$. Then there exist permutations $\sigma, \delta \in G$ such that $\big(x_i, y_{\sigma(i)}\big) \in U_{\delta(i)}$ for all $i = 1, 2, \ldots, n$. We have $y_{\sigma(i)} \in U_{\delta(i)}\big(x_i\big)$ for any $i = 1, 2, \ldots, n$. Therefore,

$$f\big(y_{\sigma(i)}\big) \in f\big(U_{\delta(i)}\big(x_i\big)\big) \subset V_{\delta(i)}\big(f\big(x_i\big)\big).$$

It means that

$$\big(f\big(x_i\big),\, f\big(y_{\sigma(i)}\big)\big) \in V_{\delta(i)} \tag{2.2}$$

for all $i = 1, 2, \ldots, n$. Put

$$SP^n_G f([x]) = \big[\big(f\big(x_1\big),\, f\big(x_2\big), \ldots,\, f\big(x_n\big)\big)\big]_G$$

and

$$SP^n_G f([y]) = \big[\big(f(y_1),\, f\big(y_2\big), \ldots,\, f\big(y_n\big)\big)\big]_G.$$

From (2.2) we obtain

$$\big(SP^n_G f([x]),\, SP^n_G f([y])\big) \in O\big[V_1, V_2, \ldots, V_n\big].$$

Hence, $SP^n_G f([y]) \in O\big[V_1, V_2, \ldots, V_n\big]\big(SP^n_G f([x])\big)$. Theorem 2.8 is proved. □

Theorem 2.9. *Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniformly open mapping. Then the mapping $SP^n_G f : \big(SP^n_G X,\, SP^n_G \mathcal{U}\big) \to \big(SP^n_G Y,\, SP^n_G \mathcal{V}\big)$ is also uniformly open.*

Proof. Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniformly open mapping. Take an arbitrary entourage $O\big[U_1, U_2, \ldots, U_n\big] \in SP^n_G \mathcal{U}$. In this case there exists entourage $V_i \in \mathcal{V}$ such that $V_i(f(a)) \subset f(U_i(a))$ for each $a \in X$ and $i = 1, 2, \ldots, n$. It suffices to show that

$$O\big[V_1, V_2, \ldots, V_n\big]\big(SP^n_G f([x])\big) \subset SP^n_G f\big(O\big[U_1, U_2, \ldots, U_n\big]([x])\big).$$

Choose an arbitrary point $[y] \in O\big[V_1, V_2, \ldots, V_n\big]\big(SP^n_G f([x])\big)$. Then there are permutations $\sigma, \delta \in G$ such that $\big(f\big(x_i\big),\, y_{\sigma(i)}\big) \in V_{\delta(i)}$ for all $i = 1, 2, \ldots, n$. Moreover, $y_{\sigma(i)} \in V_{\delta(i)}(f(x_i)) \subset f(U_{\delta(i)}(x_i))$ for each $i = 1, 2, \ldots, n$. Since

$y_{\sigma(i)} \in f\big(U_{\delta(i)}(x_i)\big)$, there exists a point $z_i \in U_{\delta(i)}(x_i)$ such that $y_{\sigma(i)} = f(z_i)$ for all $i = 1, 2, \ldots, n$. Put $z = (z_1, z_2, \ldots, z_n) \in X^n$ and we have

$$[y] = SP^n_G f([z]). \tag{2.3}$$

The relation $z_i \in U_{\delta(i)}(x_i)$ implies

$$[z] \in O\big[U_1, U_2, \ldots, U_n\big]([x]) \tag{2.4}$$

for $i = 1, 2, \ldots, n$. By relations (2.3) and (2.4) it follows that

$$[y] \in SP^n_G f\big(O\big[U_1, U_2, \ldots, U_n\big]([x])\big).$$

Hence

$$O\big[V_1, V_2, \ldots, V_n\big]\big(SP^n_G f([x])\big) \subset SP^n_G f\big(O\big[U_1, U_2, \ldots, U_n\big]([x])\big).$$

Theorem 2.9 is proved. □

3. Uniformly connected spaces and index boundedness

Theorem 3.1. *If a uniform space $(X,\mathcal{U})$ is uniformly connected, then the uniform space $\big(SP^n_G X, SP^n_G \mathcal{U}\big)$ is also uniformly connected.*

Proof. Let $x, y \in X^n$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Take an arbitrary entourage

$$U = \{(a, b) : (a_i, b_i) \in U_i,\ i = 1, 2, \ldots, n\} \in \mathcal{U}^n$$

on X^n, where $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, ..., b_n)$. Since X is uniformly connected, there is a $k_i \in Z$ such that $(x_i, y_i) \in U_i^{k_i}$ for every $i = 1, 2, \ldots, n$. Put $k = \max\{k_i : i = 1, 2, \ldots, n\}$. Therefore, $(x_i, y_i) \in U_i^k$ for each $i = 1, 2, \ldots, n$. In this case there are points $z_i^1, z_i^2, \ldots, z_i^{k-1}$ such that

$$\begin{cases} (x_i, z_i^1) \in U_i, \\ (z_i^1, z_i^2) \in U_i, \\ \quad \ldots . \\ (z_i^{k-1}, y_i) \in U_i. \end{cases}$$

for every $i = 1, 2, ..., n$. Consider $k-1$ points of X^n;

$z^1 = (z_1^1, z_2^1, \ldots, z_n^1)$, $z^2 = (z_1^2, z_2^2, \ldots, z_n^2), \ldots, z^{k-1} = (z_1^{k-1}, z_2^{k-1}, \ldots, z_n^{k-1})$.

By definition of entourage U, we have

$$(x, z^1) \in U,\ (z^1, z^2) \in U, \ldots, (z^{k-1}, y) \in U.$$

It means that $(x, y) \in U^k$. Hence, $(X^n, \mathcal{U}^n)$ is uniformly connected space. By Theorem 1.1 [15] and Theorem 2.2 the space $\big(SP^n_G X, SP^n_G \mathcal{U}\big)$ is uniformly connected. Theorem 3.1 is proved. □

We say that a uniform space $(X, \mathcal{U})$ is *discrete*, if $\Delta_X \in \mathcal{U}$ [15].

Example 3.2. Any discrete uniform space is not uniformly connected.

Indeed, take points $x, y \in X$ with $x \neq y$. Then for any integer number k we have $(x, y) \notin \Delta_X = \Delta_X^k$.

Theorem 3.3 ([15]). *For every uniformity U on a set X the family*

$$\tau_{\mathcal{U}} = \{G \subset X : \text{ for every } x \in G \text{ there exists a } V \in \mathcal{U} \text{ such that } V(x) \subset G\}$$

is a topology on the set X, which is called the topology induced by the uniformity $\mathcal{U}$.

Remark 3.4. The uniformly connectedness of the space $(X, \mathcal{U})$ does not imply connectedness with respect to the topology induced by the uniformity $\mathcal{U}$, in general.

Consider the family $\mathcal{B} = \{U_\varepsilon : \varepsilon > 0\}$ of subsets of $\mathbb{R} \times \mathbb{R}$, where $U_\varepsilon = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < \varepsilon\}$ and $\mathbb{R}$ is the real line. The family $\mathcal{B}$ has the properties (BU1)–(BU2) and generates some uniformity on the real line $\mathbb{R}$. This uniformity is called natural uniformity on $\mathbb{R}$.

The family $\mathcal{B}_{\mathbb{Q}} = \{U_\varepsilon \cap (\mathbb{Q} \times \mathbb{Q}) : \varepsilon > 0\}$ is a base of a uniformity on $\mathbb{Q}$ (the set of all rational numbers) and generates some uniformity $\mathcal{U}_{\mathbb{Q}}$ on $\mathbb{Q}$. For any $\varepsilon > 0$ and $r_1, r_2 \in \mathbb{Q}$ $(r_1 < r_2)$ we have $k = [\frac{r_2 - r_1}{\varepsilon}] + 1$. Consider a sequence of points $\{a_i\}_{i=1}^{k-1}$ defined by the formula $a_i = r_1 + \frac{r_2 - r_1}{k} i$ with $i = 1, 2, \ldots, k-1$. It is clear that $a_i \in \mathbb{Q}$ for all $i = 1, 2, \ldots, k-1$. In this case we have

$$\begin{cases} \big(r_1, a_1\big) \in V_\varepsilon, \\ \big(a_1, a_2\big) \in V_\varepsilon, \\ \quad \cdots\,\cdot \\ \big(a_{k-1}, r_2\big) \in V_\varepsilon, \end{cases}$$

where $V_\varepsilon = U_\varepsilon \cap (\mathbb{Q} \times \mathbb{Q})$. Thus, $\big(r_1, r_2\big) \in V_\varepsilon^k$. Therefore, the uniform space $\big(\mathbb{Q}, \mathcal{U}_{\mathbb{Q}}\big)$ is uniformly connected, but not connected, since $\mathbb{Q} = ((-\infty, \sqrt{2}) \cap \mathbb{Q}) \cup ((\sqrt{2}, \infty) \cap \mathbb{Q})$ and $((-\infty, \sqrt{2}) \cap \mathbb{Q}) \cap ((\sqrt{2}, \infty) \cap \mathbb{Q}) = \varnothing$.

Theorem 3.5. *Let $(X, \mathcal{U})$ be a uniform space. Then the equality $l(\mathcal{U}) = l\big(SP_G^n \mathcal{U}\big)$ holds.*

Proof. First, we show the inequality $l\big(SP_G^n \mathcal{U}\big) \le l\big(\mathcal{U}^n\big)$. Let $l\big(\mathcal{U}^n\big) = \tau \ge \aleph_0$. Then there is a base $\mathcal{B}^n$ of uniformity $\mathcal{U}^n$ such that $|V| \le \tau$ for any $V \in \mathcal{B}^n$. We consider the family $\pi_{n,G}^s\big(\mathcal{B}^n\big) = \{\pi_{n,G}^s(V) : V \in \mathcal{B}^n\}$ and prove that the family $\pi_{n,G}^s\big(\mathcal{B}^n\big)$ is the base of the uniformity $SP_G^n \mathcal{U}$. Since $\pi_{n,G}^s$ is a uniformly open mapping, $\pi_{n,G}^s(V) \in SP_G^n \mathcal{U}$ for any entourage $V \in \mathcal{B}^n$. Consider an arbitrary entourage

$$O\big[U_1, U_2, \ldots, U_n\big] \in SP_G^n \mathcal{U}.$$

For an entourage $\big(\pi_{n,G}^s\big)^{-1}\big(O\big[U_1, U_2, \ldots, U_n\big]\big) \in \mathcal{U}^n$ there exists an entourage $V \in \mathcal{B}^n$ such that

$$V \subset \big(\pi_{n,G}^s\big)^{-1}\big(O\big[U_1, U_2, \ldots, U_n\big]\big).$$

Hence, we obtain

$$\pi_{n,G}^s(V) \subset O\big[U_1, U_2, \ldots, U_n\big],$$

i.e., we have that $l\big(SP_G^n \mathcal{U}\big) \le \tau$.

Now we show the inverse inequality $l\big(\mathcal{U}^n\big) \le l(\mathcal{U})$. Let $l(\mathcal{U}) = \kappa \ge \aleph_0$ and let $\mathcal{B}$ be a base for the uniformity $\mathcal{U}$, consisting of entourages of cardinality $\le \kappa$.

We denote by $\mathcal{B}'$ the family of all entourages of the form $\bigcap\limits_{i=1}^{n} pr_i^{-1}\big(U_i\big)$, where $U_i \in \mathcal{B}$ and pr_i is the projection of X^n onto $X_i = X$ for each $i = 1, 2, \dots, n$. Then by definition of Cartesian product of the uniform spaces, the family $\mathcal{B}'$ is a base for the uniformity $\mathcal{U}^n$. Since

$$|\bigcap_{i=1}^{n} pr_i^{-1}(U_i)| \leq \max\{|U_i| : i = 1, 2, \dots, n\} \leq \kappa$$

it follows that $l\big(\mathcal{U}^n\big) \leq l(\mathcal{U})$.

Consequently, we have $l\big(SP^n_G\mathcal{U}\big) \leq l(\mathcal{U})$. Note that a uniform subspace of a τ-bounded space is also τ-bounded. By Proposition 2.5, $l(\mathcal{U}) \leq l(SP^n_G\mathcal{U})$. Hence, we get the equality $l(\mathcal{U}) = l\big(SP^n_G\mathcal{U}\big)$. □

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