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Intrinsic characterizations of c-realcompact spaces

Sudip Kumar Acharyya, Rakesh Bharati and A. Deb Ray

Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata - 700019, INDIA (sdpacharyya@gmail.com, bharti.rakesh292@gmail.com, debrayatasi@gmail.com)

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Abstract

c-realcompact spaces are introduced by Karamzadeh and Keshtkar in Quaest. Math. 41, no. 8 (2018), 1135–1167. We offer a characterization of these spaces X via c-stable family of closed sets in X by showing that X is c-realcompact if and only if each c-stable family of closed sets in X with finite intersection property has nonempty intersection. This last condition which makes sense for an arbitrary topological space can be taken as an alternative definition of a c-realcompact space. We show that each topological space can be extended as a dense subspace to a c-realcompact space with some desired extension properties. An allied class of spaces viz CP-compact spaces akin to that of c-realcompact spaces are introduced. The paper ends after examining how far a known class of c-realcompact spaces could be realized as CP-compact for appropriately chosen ideal P of closed sets in X.

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1. Introduction

In what follows X stands for a completely regular Hausdorff topological space. As usual $C(X)$ and $C^*(X)$ denote respectively the ring of all real valued continuous functions on X and that of all bounded real valued continuous functions

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on X. Suppose $C_c(X)$ is the subring of $C(X)$ containing those functions f for which $f(X)$ is a countable set and $C_c^*(X) = C_c(X) \cap C^*(X)$. Formal investigations of these two rings vis-a-vis the topological structure of X are being carried on only in the recent times. It turns out that there is an interplay between the topological structure of X and the ring and lattice structure of $C_c(X)$ and $C_c^*(X)$, which incidentally sheds much light on the topology of X. The articles [3], [4], [7], [8], [11] may be referred in this context. The notion of c-realcompact spaces is the fruit of one such endeavours in the study of X versus $C_c(X)$ or $C_c^*(X)$. A space X is declared c-realcompact in [8] if each real maximal ideal M in $C_c(X)$ is fixed in the sense that there exits a point $x \in X$ such that for each $f \in M$, $f(x) = 0$. M is called real when the residue class field $C_c(X)/M$ is isomorphic to the field $\mathbb{R}$. A number of interesting facts concerning these spaces is discovered in [8]. These may be called countable analogues of the corresponding properties of real compact spaces as developed in [6], chapter 8. In the present article we offer a new characterization of c-realcompact spaces on using the notion, c-stable family of closed sets in X. A family $\mathcal{F}$ of subsets of X is called c-stable if given $f \in C(X, \mathbb{Z})$, there exists $F \in \mathcal{F}$ such that f is bounded on F.

We define a topological space X (not necessarily completely regular) to be c_c-realcompact if each c-stable family of closed sets in X with finite intersection property has nonempty intersection. We check that this new notion of c_c-realcompactness agrees with the already introduced notion of c-realcompactness in [8], within the class of zero-dimensional Hausdorff spaces (Theorem 2.3). We re-establish a modified version of a few known properties of c-realcompact spaces using our new definition of c_c-realcompactness (Theorem 2.4). Furthermore we realize that any topological space X can be extended as a dense subspace to a c_c-realcompact space $\upsilon_0 X$ enjoying some desired extension properties (Theorem 2.5). While constructing this extension of X, we follow closely the technique adopted in [9]. The results mentioned above constitute the first technical section viz §2 of this article.

A family $\mathcal{P}$ of closed sets in X is called an ideal of closed sets if $A \in \mathcal{P}$, $B \in \mathcal{P}$ and C is a closed subset of A imply that $A \cup B \in \mathcal{P}$ and $C \in \mathcal{P}$. Let $\Omega(X)$ stand for the aggregate of all ideals of closed sets in X. For any $\mathcal{P} \in \Omega(X)$ let $C_{\mathcal{P}}(X) = \{f \in C(X) : cl_X(X \setminus Z(f)) \in \mathcal{P}\}$, here $Z(f) = \{x \in X : f(x) = 0\}$ is the zero set of f in X. It is well known that $C_{\mathcal{P}}(X)$ is an ideal in the ring $C(X)$, see [1] and [2] for more information on these ideals. With reference to any such $\mathcal{P} \in \Omega(X)$, we call a family $\mathcal{F}$ of subsets of X $c_{\mathcal{P}}$-stable if given $f \in C(X, \mathbb{Z}) \cap C_{\mathcal{P}}(X)$ there exists $F \in \mathcal{F}$ such that f is bounded on F. We define a space X to be $c_{\mathcal{P}}$-compact if any $c_{\mathcal{P}}$-stable family of closed sets in X with finite intersection property has non-empty intersection. It is clear that a zero-dimensional space X is c_c-realcompact if it is already $c_{\mathcal{P}}$-compact.

We have shown that if X is a noncompact zero-dimensional space and $\mathcal{P} \in \Omega(X)$ such that X is $c_{\mathcal{P}}$-compact, then there exists an $\mathcal{R} \in \Omega(X)$ such that $\mathcal{R} \subsetneqq \mathcal{P}$ and X is $c_{\mathcal{R}}$-compact. Thus within the class of zero-dimensional noncompact spaces X, there is no minimal member $\mathcal{P} \in \Omega(X)$ in the set inclusion sense of the term for which X becomes $c_{\mathcal{P}}$-compact (Theorem 3.2). In the concluding portion of §3 of this article we have examined, how far the known classes of c-realcompact spaces could be achieved as $c_{\mathcal{P}}$-compact spaces for appropriately chosen $\mathcal{P} \in \Omega(X)$. For any infinite cardinal number θ, X is called finally θ-compact if each open cover of X has a subcover with cardinality $< \theta$ (see [10]). In this terminology finally ω_1-compact spaces are Lindelöf and finally ω_0-compact spaces are compact. It is realized that a c-realcompact space X is finally θ-compact if and only if it is $c_{\mathcal{Q}}$-compact, where $\mathcal{Q}$ is the ideal of all closed finally θ-compact subsets of X (Theorem 3.4). A special case of this result reads: X is Lindelöf when and only when X is c_α-compact where α is the ideal of all closed Lindelöf subsets of X.

2. Properties of c_c-realcompact spaces and c_c-realcompactifications

Before stating the first technical result of this section, we need to recall a few terminologies and results from [4] and [8]. Our intention is to make the present article self contained as far as possible. An element α on a totally ordered field F is called infinitely large if $\alpha > n$ for each $n \in \mathbb{N}$. It is clear that F is archimedean if and only if it does not contain any infinitely large element. If M is maximal ideal in $C_c(X)$ then the residue class field $C_c(X)/M$ is totally ordered according to the following definition: for $f \in C(X)$, $M(f) \geqq 0$ if and only if there exists $g \in M$ such that $f \geqq 0$ on $Z(g)$. Here $M(f)$ stands for the residue class in $C(X)/M$, which contains the function f.

Theorem 2.1 (Proposition 2.3 in [8])**.** *For a maximal ideal M in $C_c(X)$ and for $f \in C_c(X)$, $|M(f)|$ is infinitely large in $C_c(X)/M$ if and only if f is unbounded on every zero set of $Z_c(M) = \{Z(g) : g \in M\}$.*

It is proved in [4], Remark 3.6 that if X is a zero-dimensional space, then the set of all maximal ideals of $C_c(X)$ equipped with hull-kernel topology, also called the structure space of $C_c(X)$ is homeomorphic to the Banaschewski compactification $\beta_0 X$ of X. Thus the maximal ideals of $C_c(X)$ can be indexed by virtue of the points of $\beta_0 X$. Indeed a complete description of all these maximal ideals is given by the list $\{M_c^p : p \in \beta_0 X\}$, where $M_c^p = \{f \in C_c(X) : p \in cl_{\beta_0 X} Z(f)\}$ with M_c^p is a fixed maximal ideal if and only if $p \in X$ (see Theorem 4.2 in [4]). It is well known that any continuous map $f : X \to Y$, where X and Y are both zero-dimensional spaces with Y compact also, has an extension to a continuous map $\bar{f} : \beta_0 X \to Y$ (we call this property, the C-extension property of $\beta_0 X$) (see Remark 3.6 in [4]). It follows that for a zero-dimensional space X, any continuous map $f : X \to \mathbb{Z}$ (also written as $f \in C(X, \mathbb{Z})$), has an extension to a continuous map $f^* : \beta_0 X \to \mathbb{Z}^* = \mathbb{Z} \cup \{\omega\}$, the one point compactification of $\mathbb{Z}$. We also write $f^* \in C(\beta_0 X, \mathbb{Z}^*)$. A slightly variant form of the next result is proved in [8], Theorem 2.17 and Theorem 2.18.

Theorem 2.2. *Let X be zero-dimensional and $p \in \beta_0 X$, then the maximal ideal M_c^p in $C_c(X)$ is real if and only if for each $f \in C(X, \mathbb{Z})$, $f^*(p) \neq \omega$ if and only if $|M_c^P(f)|$ is not infinitely large in $C_c(X)/M_c^p$.*

Theorem 2.3. *A zero-dimensional space X is c_c-realcompact if and only if it is c-realcompact.*

Proof. Let X be a c-realcompact space and $\mathcal{F}$ be a family of closed subsets of X with finite intersection property but with $\bigcap \mathcal{F}=\varnothing$. To show that X is c_c-realcompact we shall prove that $\mathcal{F}$ is not a c-stable family. Indeed $\{cl_{\beta_0 X}F : F \in \mathcal{F}\}$ is a family of closed subsets of $\beta_0 X$ with finite intersection property. Since $\beta_0 X$ is compact, there exists a point $p \in \bigcap_{F\in\mathcal{F}} cl_{\beta_0 X}F$ and of course $p \in \beta_0 X \setminus X$. Here M_c^p is a free maximal ideal in $C_c(X)$. Since X is c-realcompact this implies that M_c^p is a hyperreal maximal ideal (meaning that it is not a real maximal ideal of $C_c(X)$). It follows from Theorem 2.2 that there exists $f \in C(X,\mathbb{Z})$ with $f^*(p) = \omega$. Since $p \in cl_{\beta_0 X}F$ for each $F \in \mathcal{F}$, it is therefore clear that 'f' is unbounded on each set in the family $\mathcal{F}$. Therefore $\mathcal{F}$ is not a c-stable family.

Conversely let X be not c-realcompact. Then there exists a real maximal ideal M in $C_c(X)$, which is not fixed. This means that there is a point $p \in \beta_0 X \setminus X$ for which $M = M_c^p$. Since $p \in cl_{\beta_0 X}Z(f)$ for each $f \in M_c^p$, it follows that $\{Z(f) : f \in M_c^p\}$ is a family of closed sets in X with finite intersection property but with empty intersection. To show that X is not c_c-realcompact, it suffices to show that $\{Z(f) : f \in M_c^p\}$ is a c-stable family. So let $g \in C(X,\mathbb{Z})$. Since M_c^p is real, this implies in view of Theorem 2.2 that $g^*(p) \neq \omega$ and hence $|M_c^p(g)|$ is not infinitely large. It follows therefore from Theorem 2.1 that g is bounded on some $Z(f)$ for an $f \in M_c^p$. This settles that $\{Z(f) : f \in M_c^p\}$ is a c-stable family. □

By adapting the arguments of Theorem 5.2, Theorem 5.3 and Theorem 5.4 in [9] appropriately, we can establish the following facts about c_c-realcompact spaces without difficulty:

Theorem 2.4.

(1) *A compact space is c_c-realcompact.*
(2) *A pseudocompact c_c-realcompact space is compact.*
(3) *A closed subspace of a c_c-realcompact space is c_c-realcompact.*
(4) *The product of any set of c_c-realcompact spaces is c_c-realcompact.*
(5) *If a topological space $X = E \cup F$ where E is a compact subset of X and F is a $\mathbb{Z}$-embedded c_c-realcompact subset of X, meaning that each function in $C(F,\mathbb{Z})$ can be extended to a function in $C(X,\mathbb{Z})$, then X is c_c-realcompact.*
(6) *A $\mathbb{Z}$-embedded c_c-realcompact subset of a Hausdorff space X is a closed subset of X.*

We now show that any topological space X can be extended to a c_c-realcompact space containing the original space X as a C-embedded dense subspace and enjoying a desirable extension property. The proof can be accomplished by closely following the arguments adopted to prove Theorem 6.1 in [9]. Nevertheless we give a brief outline of the main points of proof in our theorem.

Theorem 2.5. *Every topological space X can be extended to a c_c-realcompact space $\upsilon^c X$ as a dense subspace with the following extension property: each continuous map from X into a regular c_c-realcompact space Y can be extended to a continuous map from $\upsilon^c X$ into Y. X is c_c-realcompact if and only if $X = \upsilon^c X$.*

Proof. For each $x \in X$ let $\mathcal{G}^x$ be the aggregate of all closed sets in X which contain the point x. Then $\mathcal{G}^x$ is a c-stable family of closed sets in X with finite intersection property and with the prime condition: $A \cup B \in \mathcal{G}^x \implies A \in \mathcal{G}^x$ or $B \in \mathcal{G}^x$, $A, B \subseteq X$. We extend the set X to a bigger set $\upsilon^c X$, so that $\upsilon^c X \setminus X$ becomes an index set for the collection of all maximal c-stable families of closed subsets of X with finite intersection property but with empty intersection. For each $p \in \upsilon^c X \setminus X$, let $\mathcal{G}^p$ designate the corresponding maximal c-stable family of closed sets in X with finite intersection property and with empty intersection. For each closed set F in X, we write $\bar{F} = \{p \in \upsilon^c X : F \in \mathcal{G}^p\}$. Then $\{\bar{F}\colon F$is closed in $X\}$ forms a base for closed sets of some topology on $\upsilon^c X$ and in this topology for any closed set F in X $\bar{F} = cl_{\upsilon^c X} F$. Since X belongs to each $\mathcal{G}^p$, it is clear that X is dense in $\upsilon^c X$. Let $t : X \to Y$ be a continuous map with Y, a regular c_c-realcompact space. Choose $p \in \upsilon^c X$. Let $\mathcal{H}^p = \{G \subseteq Y : G$ is closed in Y and $t^{-1}(G) \in \mathcal{G}^p\}$. Then $\mathcal{H}^p$ is a c-stable family of closed sets in Y with finite intersection property. We select a point $y \in \bigcap \mathcal{H}^p$ and we set $t^0(p) = y$ with the aggrement that $t^0(p) = t(p)$ in case $p \in X$. Thus $t^0 : \upsilon^c X \to Y$ is a well defined map which is further continuous. The remaining parts of the theorem can be proved by making arguments closely as in the proof of Theorem 6.1 of [9]. ☐

3. $c_{\mathcal{P}}$-COMPACT SPACES

In this section all the topological spaces X that will appear will be assumed to be zero-dimensional. We define for any $\mathcal{P} \in \Omega(X)$, $\upsilon_0^{\mathcal{P}}(X) = \{p \in \beta_0 X : f^*(p) \neq \omega$ for each $f \in C_{\mathcal{P}}(X) \cap C(X, \mathbb{Z})\}$. It is clear that if $\mathcal{P}$=$\mathcal{E}\equiv$ the ideal of all closed sets in X then $\upsilon_0^{\mathcal{E}}(X) = \upsilon_0 X \equiv \{p \in \beta_0 X : f^*(p) \neq \omega$ for each $f \in C(X, \mathbb{Z})\}$ the set defined in the begining of the proof of Theorem 3.8 in [8]. The next theorem puts Theorem 2.3 in a more general setting.

Theorem 3.1. *For a $\mathcal{P} \in \Omega$, X is $c_{\mathcal{P}}$-compact if and only if $X = \upsilon_0^{\mathcal{P}}(X)$.*

We omit the proof of this theorem because it can be done by making some appropriate modification in the arguments adopted in the proof of Theorem 2.3.

It is clear that if $\mathcal{P}$, $\mathcal{Q} \in \Omega(X)$ with $\mathcal{P} \subset \mathcal{Q}$, then any $c_{\mathcal{Q}}$-stable family of closed sets in X is also $c_{\mathcal{P}}$-stable, consequently if X is $c_{\mathcal{P}}$-compact then X is $c_{\mathcal{Q}}$-compact also. In particular every $c_{\mathcal{P}}$-compact space is c_c-realcompact and hence c-realcompact in view of Theorem 2.3. The following question therefore seems to be natural.

If X is a zero-dimensional non-compat c-realcompact space, then does there exist a minimal ideal $\mathcal{P}$ of closed sets in X (minimal in some sense of the term) for which X becomes $c_{\mathcal{P}}$-compact?

No possible answer to this question is known to us, however the following proposition shows that the answer to this question is in the negative if the phrase 'minimal' is interpreted in the set inclusion sense of the term.

Theorem 3.2. *Let X be a non compact zero-dimensional space. Suppose $\mathcal{P} \in \Omega(X)$ is such that X is $c_{\mathcal{P}}$-compact. Then there exists $\mathcal{R} \in \Omega(X)$ such that $\mathcal{R} \subsetneqq \mathcal{P}$ and X is $c_{\mathcal{R}}$-compact.*

Proof. We get from Theorem 3.1 that $X = \upsilon_0^{\mathcal{P}} X$. As X is non compact we can choose a point $p \in \beta_0 X \setminus X$. Then $p \notin \upsilon_0^{\mathcal{P}} X$. Accordingly there exists $f \in C_{\mathcal{P}}(X) \cap C(X, \mathbb{Z})$ such that $f^*(p) = \omega$. We select a point $x \in X$ such that $f(x) \neq 0$. Set $\mathcal{R} = \{D \in \mathcal{P} : x \notin D\}$. It is easy to check that $\mathcal{R}$ is an ideal of closed sets in X, i.e., $\mathcal{R} \in \Omega(X)$. Furthermore, $cl_X(X - Z(f))$ is a member of $\mathcal{P}$ containing the point x. This implies that $cl_X(X - Z(f)) \notin \mathcal{R}$. Thus $\mathcal{R} \subsetneq \mathcal{P}$. To show that X is $c_{\mathcal{R}}$-compact. We shall show that $X = \upsilon_0^{\mathcal{R}} X$ (see Theorem 3.1). So choose a point $q \in \beta_0 X \setminus X$ then $q \notin \upsilon_0^{\mathcal{P}} X$, consequently there exists $g \in C_{\mathcal{P}}(X) \cap C(X, \mathbb{Z})$ such that $g^*(q) = \omega$. For the distinct points q, x in $\beta_0 X$ there exist disjoint open sets U, V in this space such that $x \in U$, $q \in V$. Since $\beta_0 X$ is zero-dimensional there exists therefore a clopen set W in $\beta_0 X$ such that $q \in W \subset V$. The map $h : \beta_0 X \to \{0, 1\}$ given by $h(W) = \{1\}$ and $h(\beta_0 X \setminus W) = \{0\}$ is continuous. We note that $h(U) = \{0\}$ and $h(q) = 1$. Let $\psi = h|_X$. Then $\psi \in C(X, \mathbb{Z})$. Take $l = g.\psi$. Since $g \in C_{\mathcal{P}}(X)$ and $C_{\mathcal{P}}(X)$ is an ideal of $C(X)$, it follows that $l \in C_{\mathcal{P}}(X)$. Furthermore the fact that g and ψ are both functions in $C(X, \mathbb{Z})$ implies that $l \in C(X, \mathbb{Z})$. Also the function $h \in C(\beta_0 X, \mathbb{Z})$ is the unique continuous extension of $\psi \in C(X, \mathbb{Z})$, hence we can write $h = \psi^*$. This implies that $l^*(q) = g^*(q)\psi^*(q) = g^*(q)h(q) = \omega$, because $g^*(q) = \omega$ and $h(q) \neq 0$.
On the other hand if $y \in U \cap X$ then $h(y) = 0$ and hence $l(y) = 0$. Since $U \cap X$ is an open neighbourhood of x in the space X, this implies that $x \notin cl_X(X \setminus Z(l))$. Since $cl_X(X \setminus Z(l)) \in \mathcal{P}$, already verified, it follows that $cl_X(X \setminus Z(l)) \in \mathcal{R}$. Thus $l \in C_{\mathcal{R}}(X) \cap C(X, \mathbb{Z})$. Since $l^*(q) = \omega$, this further implies that $q \notin \upsilon_0^{\mathcal{R}}(X)$. □

It is trivial that a (zero-dimensional) compact space is c-realcompact. It is also observed that a Lindelöf space is c-realcompact (Corollary 3.6, [8]). But for an infinite cardinal number θ, a finally θ-compact space may not be c-realcompact. Indeed the space $[0, \omega_1)$ of all countable ordinals is a celebrated example of a zero-dimensional space which is not realcompact (see 8.1, [6]). Since a zero-dimensional c-realcompact space is necessarily realcompact (vide proposition 5.8, [8]) it follows therefore that $[0, \omega_1)$ is not a c-realcompact space. But it is easy to show that $[0, \omega_1)$ is finally ω_2-compact. For the same reason, the Tychonoff plank $T \equiv [0, \omega_1) \times [0, \omega_0)$-$\{(\omega_1, \omega_0)\}$ of 8.20 in [6], is finally ω_2-compact without being c-realcompact. It can be easily shown that a closed subset of a finally θ-compact space is finally θ-compact.

Furthermore, the following characterization of finally θ-compactness of a topological space can be established by routine arguments.

Theorem 3.3. *The following two statements are equivalent for an infinite cardinal number θ.*

1. *X is finally θ-compact.*
2. *If $\mathcal{B}$ is a family of closed sets in X, such that for any subfamily $\mathcal{B}_0$ of $\mathcal{B}$ with $|\mathcal{B}_0| < \theta$, $\bigcap \mathcal{B}_0 \neq \varnothing$, then $\bigcap \mathcal{B} \neq \varnothing$.*

Theorem 3.4. *Let X be c-realcompact and $\mathcal{P}_\theta$ the ideal of all closed finally θ-compact subsets of X. Then X is finally θ-compact if and only if it is $c_{\mathcal{P}_\theta}$-compact.*

Proof. Let X be finally θ-compact and $p \in \beta_0 X \setminus X$. To show that X is $c_{\mathcal{P}_\theta}$-compact, it suffices to show in view of Theorem 3.1 that $p \notin \upsilon_0^{\mathcal{P}_\theta}(X)$. Indeed X is c-realcompact implies that the maximal ideal M_c^p of $C_c(X)$ is not real. Consequently by Theorem 2.2, there exists $f \in C(X, \mathbb{Z})$ such that $f^*(p) = \omega$. Now $cl_X(X \setminus Z(f))$, like any closed subsets of X is finally θ-compact. Thus $f \in C_{\mathcal{P}_\theta}(X) \cap C(X, \mathbb{Z})$, hence $p \notin \upsilon_0^{\mathcal{P}_\theta}(X)$.
To prove the converse, let X be not finally θ-compact. It follows from Theorem 3.3 that there exists a family $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda\}$ of closed sets in X with the following properties: for any subfamily $\mathcal{B}_1$ of $\mathcal{B}$ with $|\mathcal{B}_1| < \theta$, $\bigcap \mathcal{B}_1 \neq \varnothing$ but $\bigcap \mathcal{B} = \varnothing$. Let $\mathcal{D} = \{D_\alpha : \alpha \in \Lambda^*\}$ be the aggregate of all sets $D'_\alpha s$, which are intersections of $< \theta$ many sets in the family $\mathcal{B}$. Then $\mathcal{B} \subseteq \mathcal{D}$ and hence $\bigcap \mathcal{D} = \varnothing$. Also $\mathcal{D}$ has finite intersection property. We shall show that $\mathcal{D}$ is a $c_{\mathcal{P}_\theta}$-stable family and hence X is not $\mathcal{P}_\theta$-compact. Towards such a proof choose $f \in C_{\mathcal{P}_\theta}(X) \cap C(X, \mathbb{Z})$, then $cl_X(X \setminus Z(f)$ is a finally θ-compact subset of X. Since $\{X \setminus B_\alpha : \alpha \in \Lambda\}$ is an open cover of X, there exists a subset Λ_0 of Λ with $|\Lambda_0| < \theta$ such that $cl_X(X \setminus Z(f)) \subseteq \bigcup_{\alpha \in \Lambda_0} (X \setminus B_\alpha)$. This implies that $\bigcap_{\alpha \in \Lambda_0} B_\alpha \subseteq Z(f)$ and we note that $\bigcap_{\alpha \in \Lambda_0} B_\alpha \in \mathcal{D}$. Thus f becomes bounded on a set lying in the family $\mathcal{D}$. Hence $\mathcal{D}$ becomes a $c_{\mathcal{P}_\theta}$-stable family. $\square$

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References

[1] S. K. Acharyya and S. K. Ghosh, A note on functions in $C(X)$ with support lying on an ideal of closed subsets of X, Topology Proc. 40 (2012), 297–301.
[2] S. K. Acharyya and S. K. Ghosh, Functions in $C(X)$ with support lying on a class of subsets of X, Topology Proc. 35 (2010), 127–148.
[3] S. K. Acharyya, R. Bharati and A. Deb Ray, Rings and subrings of continuous functions with countable range, Queast. Math., to appear.
[4] F. Azarpanah, O. A. S. Karamzadeh, Z. Keshtkar and A. R. Olfati, On maximal ideals of $C_c(X)$ and the uniformity of its localizations, Rocky Mountain J. Math. 48, no. 2 (2018), 345–384.

[5] P. Bhattacherjee, M. L. Knox and W. W. Mcgovern, The classical ring of quotients of $C_c(X)$, Appl. Gen. Topol. 15, no. 2 (2014), 147–154.

[6] L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand Reinhold co., New York, 1960.

[7] M. Ghadermazi, O. A. S. Karamzadeh and M. Namdari, On the functionally countable subalgebras of $C(X)$, Rend. Sem. Mat. Univ. Padova. 129 (2013), 47–69.

[8] O. A. S. Karamzadeh and Z. Keshtkar, On c-realcompact spaces, Queast. Math. 41, no. 8 (2018), 1135–1167.

[9] M. Mandelkar, Supports of continuous functions, Trans. Amer. Math. Soc. 156 (1971), 73–83.

[10] R. M. Stephenson Jr, Initially k-compact and related spaces, in: Handbook of Set-Theoretic Topology, ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam, North-Holland, (1984) 603–632.

[11] A. Veisi, e_c-filters and e_c-ideals in the functionally countable subalgebra of $C^*(X)$, Appl. Gen. Topol. 20, no. 2 (2019), 395–405.

