# ORDER SPECTRUM OF THE CESÀRO OPERATOR IN BANACH LATTICE SEQUENCE SPACES

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Abstract. The discrete Cesàro operator $C$ acts continuously in various classical Banach sequence spaces within $\mathbb{C}^{\mathbb{N}}$. For the coordinatewise order, many such sequence spaces $X$ are also complex Banach lattices (eg. $c_0, \ell^p$ for $1 < p \leq \infty$, and $\mathrm{ces}(p)$ for $p \in \{0\} \cup (1, \infty)$). In such Banach lattice sequence spaces, $C$ is always a positive operator. Hence, its order spectrum is well defined within the Banach algebra of all regular operators on $X$. The purpose of this note is to show, for every $X$ belonging to the above list of Banach lattice sequence spaces, that the order spectrum $\sigma_{\mathrm{o}}(C)$ of $C$ *coincides* with its usual spectrum $\sigma(C)$ when $C$ is considered as a continuous linear operator on the Banach space $X$.

## 1. Introduction

Let $E$ be a complex Banach lattice and $\mathcal{L}(E)$ denote the unital Banach algebra of all continuous linear operators from $E$ into itself, equipped with the operator norm $\|\cdot\|_{\mathrm{op}}$. The unit is the identity operator $I : E \longrightarrow E$. Associated with each $T \in \mathcal{L}(E)$ is its *spectrum*

$$\sigma(T) := \{\lambda \in \mathbb{C} : (\lambda I - T) \text{ is not invertible in } \mathcal{L}(E)\}$$

and its resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$. An operator $T \in \mathcal{L}(E)$ is called *regular* if it is a finite linear combination of *positive operators*. The complex vector space of all regular operators is denoted by $\mathcal{L}^r(E)$; it is also a unital Banach algebra for the norm

$$\|T\|_r := \inf\{\|S\|_{\mathrm{op}} : S \in \mathcal{L}(E), S \geq 0, |T(z)| \leq S(|z|)\ \forall\, z \in E\}, \quad T \in \mathcal{L}^r(E). \tag{1.1}$$

Again $I : E \longrightarrow E$ is the unit. Moreover, $\|T\|_{\mathrm{op}} \leq \|T\|_r$ for $T \in \mathcal{L}^r(E)$, with equality whenever $T \geq 0$ (i.e., if $T$ is a positive operator). The spectrum of $T \in \mathcal{L}^r(E)$, considered as an element of the Banach algebra $\mathcal{L}^r(E)$, is denoted by $\sigma_{\mathrm{o}}(T)$ and is called its *order spectrum*. Then $\rho_{\mathrm{o}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{o}}(T)$ is the *order resolvent* of $T$. Clearly

$$\sigma(T) \subseteq \sigma_{\mathrm{o}}(T), \quad T \in \mathcal{L}^r(E). \tag{1.2}$$

From the usual formula for the spectral radius, [5, Ch.I, §2, Proposition 8], it follows that the spectral radii for $T \in \mathcal{L}^r(E)$ satisfy $r(T) = r_{\mathrm{o}}(T)$ whenever $T \geq 0$. Standard references for the above concepts and facts are [3], [17], [18], for example.

It is clear from (1.2) that $r(T) \leq r_{\mathrm{o}}(T)$ for $T \in \mathcal{L}^r(E)$. So, if $r(T) < r_{\mathrm{o}}(T)$, then (1.2) cannot be an equality. This is the strategy applied in [18, pp.79-80] to exhibit a regular operator for which $\sigma(T) \subsetneqq \sigma_{\mathrm{o}}(T)$. For an example of a *positive operator* $T$ satisfying $\sigma(T) \subsetneqq \sigma_{\mathrm{o}}(T)$, see [3, pp.283-284]. In the contrary direction, a rich supply of classical operators $T$ for which the equality

$$\sigma(T) = \sigma_{\mathrm{o}}(T) \tag{1.3}$$

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is satisfied arise in harmonic analysis, [3, Theorem 3.4].

The aim of this note is to contribute two further classes of operators $T$ which satisfy (1.3). In Section 2 it is shown that in any *Banach function space* $E$, all multiplication operators $T$ by $L^\infty$-functions are regular operators and satisfy (1.3). This is a consequence of the fact that the algebra of such multiplication operators is maximal commutative. Let $\mathbb{N} := \{1, 2, \ldots\}$. The remaining three sections deal with the classical *Cesàro operator* $C : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ defined by

$$C(x) := \left(\tfrac{1}{n}\sum_{k=1}^n x_k\right)_{n=1}^\infty \quad x = (x_n)_{n=1}^\infty \in \mathbb{C}^\mathbb{N}, \tag{1.4}$$

which is clearly a *positive operator* for the coordinatewise order in the positive cone of $\mathbb{C}^\mathbb{N} = \mathbb{R}^\mathbb{N} \oplus i\mathbb{R}^\mathbb{N}$. Section 3 establishes some general results for determining the regularity of linear operators in *Banach lattice sequence spaces.* These results are designed to apply to the particular operators $(C - \lambda I)^{-1}$, where $C$ is given in (1.4). In Section 4 we will consider the restriction of $C$ to the Banach lattice sequence spaces $c_0$ and $\ell^p, 1 < p \le \infty$, and show that (1.3) is satisfied in all cases (with $C$ in place of $T$). Section 5 is devoted to proving the same fact, but now when $C$ acts in the discrete Cesàro spaces $\mathrm{ces}(p), 1 < p < \infty$, and in $\mathrm{ces}(0)$.

## 2. Multiplication operators

Let $(\Omega, \Sigma, \mu)$ be a *localizable measure space* (in the sense of [10, 64A]), that is, the associated measure algebra is a complete Boolean algebra and, for every measurable set $A \in \Sigma$ with $\mu(A) > 0$ there exists $B \in \Sigma$ such that $B \subseteq A$ and $0 < \mu(B) < \infty$ (i.e., $\mu$ has the finite subset property). All $\sigma$-finite measures are localizable, [10, 64H Proposition]. Every Banach function space $E$ (of $\mathbb{C}$-valued functions) over $(\Omega, \Sigma, \mu)$ is a complex Banach lattice for the pointwise $\mu$-a.e. order. Given any $\varphi \in L^\infty(\mu)$, the multiplication operator $M_\varphi : E \longrightarrow E$ defined by $f \longmapsto \varphi f$, for $f \in E$, belongs to $\mathcal{L}(E)$ and satisfies $\|M_\varphi\|_{\mathrm{op}} = \|\varphi\|_\infty$. Define a unital, commutative subalgebra of $\mathcal{L}(E)$ by

$$\mathcal{M}_E(L^\infty(\mu)) := \{M_\varphi : \varphi \in L^\infty(\mu)\};$$

the unit is the identity operator $I = M_{\mathbf{1}}$ where $\mathbf{1}$ is the constant function 1 on $\Omega$. Recall that the *commutant* of $\mathcal{M}_E(L^\infty(\mu))$ is defined by

$$\mathcal{M}_E(L^\infty(\mu))^c := \{A \in \mathcal{L}(E) : AM_\varphi = M_\varphi A \ \ \forall\, \varphi \in L^\infty(\mu)\} \subseteq \mathcal{L}(E).$$

It is known that $\mathcal{M}_E(L^\infty(\mu))$ is a *maximal commutatitive*, unital subalgebra of $\mathcal{L}(E)$, that is, $\mathcal{M}_E(L^\infty(\mu)) = \mathcal{M}_E(L^\infty(\mu))^c$, [9, Proposition 2.2]. Moreover, also the *bicommutant* $\mathcal{M}_E(L^\infty(\mu))^{cc} = \mathcal{M}_E(L^\infty(\mu))$.

**Proposition 2.1.** *Let $(\Omega, \Sigma, \mu)$ be a localizable measure space and $E$ be a Banach function space over $(\Omega, \Sigma, \mu)$.*

(i) $\mathcal{M}_E(L^\infty(\mu)) \subseteq \mathcal{L}^r(E)$.

(ii) *$\mathcal{M}_E(L^\infty(\mu))$ is inverse closed in $\mathcal{L}(E)$. That is, if $T \in \mathcal{M}_E(L^\infty(\mu))$ is invertible in $\mathcal{L}(E)$ (i.e., there exists $S \in \mathcal{L}(E)$ satisfying $ST = I = TS$), then necessarily $S \in \mathcal{M}_E(L^\infty(\mu))$.*

(iii) *For every $T \in \mathcal{M}_E(L^\infty(\mu))$ we have $\sigma_{\mathrm{o}}(T) = \sigma(T)$.*

*Proof.* (i) Let $\varphi \in L^\infty(\mu)$. Then $\varphi = [(\mathrm{Re}\,\varphi)^+ - (\mathrm{Re}\,\varphi)^-] + i[(\mathrm{Im}\,\varphi)^+ - (\mathrm{Im}\,\varphi)^-]$ with all four functions $(\mathrm{Re}\,\varphi)^+, (\mathrm{Re}\,\varphi)^-, (\mathrm{Im}\,\varphi)^+, (\mathrm{Im}\,\varphi)^-$ belonging to the positive cone $L^\infty(\mu)^+$ of $L^\infty(\mu)$. Since $M_\varphi = [M_{(\mathrm{Re}\,\varphi)^+} - M_{(\mathrm{Re}\,\varphi)^-}] + i[M_{(\mathrm{Im}\,\varphi)^+} - M_{(\mathrm{Im}\,\varphi)^-}]$ is a linear combination of positive operators, it is clear that $M_\varphi \in \mathcal{L}^r(E)$.

(ii) Since $\mathcal{M}_E(L^\infty(\mu))$ is maximal commutative in $\mathcal{L}(E)$, it follows that $\mathcal{M}_E(L^\infty(\mu))$ is inverse closed in $\mathcal{L}(E)$, [5, Ch.II, §15, Theorem 4].

(iii) In view of (1.1) it suffices to show that $\rho(T) \subseteq \rho_{\mathrm{o}}(T)$. Suppose that $T = M_\varphi$ with $\varphi \in L^\infty(\mu)$. Fix $\lambda \in \rho(T)$. Then $\lambda I - T = M_{(\lambda\mathbf{1}-\varphi)}$ belongs to $\mathcal{M}_E(L^\infty(\mu))$ because $(\lambda\mathbf{1} - \varphi) \in L^\infty(\mu)$. Since $M_{(\lambda\mathbf{1}-\varphi)}$ is invertible in $\mathcal{L}(E)$, it follows from part (ii) that actually $(\lambda I - T)^{-1} \in \mathcal{M}_E(L^\infty(\mu))$ and hence, by part (i), that also $(\lambda I - T)^{-1} \in \mathcal{L}^r(E)$. □

**Remark 2.2.** We point out that $\|T\|_{\mathrm{op}} = \|T\|_r$ for each $T \in \mathcal{M}_E(L^\infty(\mu))$. Indeed, let $\varphi \in L^\infty(\mu)$ satisfy $T = M_\varphi$, in which case $\|M_\varphi\|_{\mathrm{op}} = \|\varphi\|_\infty$. Define $S := \|\varphi\|_\infty I$ and note that $S \geq 0$ with $\|S\|_{\mathrm{op}} = \|\varphi\|_\infty$. Moreover,

$$|M_\varphi(f)| = |\varphi f| \leq \|\varphi\|_\infty |f| = S(|f|), \quad f \in E,$$

and so $\|T\|_r \leq \|S\|_{\mathrm{op}} = \|\varphi\|_\infty = \|T\|_{\mathrm{op}}$; see (1.1). The reverse inequality $\|T\|_{\mathrm{op}} \leq \|T\|_r$ always holds.

## 3. The Cesàro operator in Banach sequence spaces

We begin with some preliminaries. Equipped with the topology of pointwise convergence $\mathbb{C}^\mathbb{N}$ is a locally convex Fréchet space. Let $A = (a_{nm})_{n,m=1}^\infty$ be any lower triangular (infinite) matrix, i.e., $a_{nm} = 0$ whenever $m > n$. Then $A$ induces the continuous linear operator $T_A : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ defined by

$$T_A(x) := \left(\sum_{m=1}^\infty a_{nm} x_m\right)_{n=1}^\infty, \quad x \in \mathbb{C}^\mathbb{N}. \tag{3.1}$$

For $x \in \mathbb{C}^\mathbb{N}$ define $|x| := (|x_n|)_{n=1}^\infty$. Then also $|x| \in \mathbb{C}^\mathbb{N}$. A vector subspace $X \subseteq \mathbb{C}^\mathbb{N}$ is called *solid* (or an *ideal*) if $y \in X$ whenever $x \in X$ and $y \in \mathbb{C}^\mathbb{N}$ satisfy $|y| \leq |x|$. It is always assumed that $X$ contains the vector space consisting of all elements of $\mathbb{C}^\mathbb{N}$ which have only finitely many non-zero coordinates. In addition, it is assumed that $X$ has a norm $\|\cdot\|_X$ with respect to which it is a complex *Banach lattice* for the *coordinatewise order* and such that the natural inclusion $X \subseteq \mathbb{C}^\mathbb{N}$ is continuous. Under the previous requirements $X$ is called a *Banach lattice sequence space*.

**Lemma 3.1.** *Let $A = (a_{nm})_{n,m=1}^\infty$ be a lower triangular matrix with all entries non-negative real numbers and $X \subseteq \mathbb{C}^\mathbb{N}$ be a Banach lattice sequence space such that $T_A(X) \subseteq X$. Let $B = (b_{nm})_{n,m=1}^\infty$ be any matrix such that*

$$|b_{nm}| \leq a_{nm}, \quad n, m \in \mathbb{N}. \tag{3.2}$$

*Then the restricted operator $T_A : X \longrightarrow X$ belongs to $\mathcal{L}(X)$. Moreover, $T_B : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ satisfies $T_B(X) \subseteq X$ and the restricted operator $T_B : X \longrightarrow X$ also belongs to $\mathcal{L}(X)$. In addition, $\|T_B\|_{\mathrm{op}} \leq \|T_A\|_{\mathrm{op}}$.*

*Proof.* Condition (3.2) implies that $B$ is also a lower triangular matrix. Moreover, the continuity of both $T_A : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ and of the inclusion map $X \subseteq \mathbb{C}^\mathbb{N}$ imply, via the Closed Graph Theorem in the Banach space $X$, that the restricted operator $T_A \in \mathcal{L}(X)$.

Given $x \in X$ we have for each $n \in \mathbb{N}$, via (3.2), that

$$(T_B(x))_n = \left|\sum_{m=1}^\infty b_{nm} x_m\right| \leq \sum_{m=1}^\infty |b_{nm}| \cdot |x_m| \leq \sum_{m=1}^\infty a_{nm} |x_m| = (T_A(|x|))_n.$$

Since $X$ is solid and $T_A(|x|) \in X$, these inequalities and (3.1) imply that $T_B(x) \in X$. Moreover, as $\|\cdot\|_X$ is a lattice norm it follows that

$$\begin{aligned} \|T_B(x)\|_X &= \left\|\left(\sum_{m=1}^\infty b_{nm} x_m\right)_{n=1}^\infty\right\|_X \leq \left\|\left(\sum_{m=1}^\infty a_{nm} |x_m|\right)_{n=1}^\infty\right\|_X \\ &= \|T_A(|x|)\|_X \leq \|T_A\|_{\mathrm{op}} \|x\|_X, \end{aligned}$$

for each $x \in X$, where the stated series are actually finite sums. Hence, $\|T_B\|_{\mathrm{op}} \le \|T_A\|_{\mathrm{op}}$ and the proof is complete. □

Since the operator $T_A$ as given in Lemma 3.1 satisfies $T_A \ge 0$, it is clearly regular.

**Corollary 3.2.** *Let $A = (a_{nm})_{n,m=1}^\infty$ be a lower triangular matrix with non-negative real entries and $X \subseteq \mathbb{C}^{\mathbb{N}}$ be a Banach lattice sequence space such that $T_A(X) \subseteq X$. Let $B = (b_{nm})_{n,m=1}^\infty$ be any matrix satisfying* (3.2). *Then the operator $T_B \in \mathcal{L}(X)$ is necessarily regular, that is, $T_B \in \mathcal{L}^r(X)$.*

*Proof.* Define the non-negative real numbers $s_{nm} := (\operatorname{Re} b_{nm})^+, u_{nm} := (\operatorname{Re} b_{nm})^-, v_{nm} := (\operatorname{Im} b_{nm})^+$ and $w_{nm} := (\operatorname{Im} b_{nm})^-$ for each $n, m \in \mathbb{N}$. Then $b_{nm} = (s_{nm} - u_{nm}) + i(v_{nm} - w_{nm})$ and $\{s_{nm}, u_{nm}, v_{nm}, w_{nm}\} \subseteq [0, a_{nm}]$ for $n, m \in \mathbb{N}$. Setting $S := (s_{nm})_{n,m=1}^\infty, U := (u_{nm})_{n,m=1}^\infty, V := (v_{nm})_{n,m=1}^\infty$ and $W := (w_{nm})_{n,m=1}^\infty$ it is clear from the definition (3.1) that each operator $T_S \ge 0, T_U \ge 0, T_V \ge 0$ and $T_W \ge 0$ (in $X$) belongs to $\mathcal{L}(X)$; see Lemma 3.1. Since $T_B = (T_S - T_U) + i(T_V - T_W)$, it follows that $T_B \in \mathcal{L}^r(X)$. □

Together with appropriate estimates, Corollary 3.2 will be the main ingredient required to establish (1.3) for $C$ (in place of $T$) when it acts in various classical Banach lattice sequence spaces $X$.

Let $\Sigma_0 := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. We recall the formula for the inverses $(C - \lambda I)^{-1} : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ whenever $\lambda \in \mathbb{C} \setminus \Sigma_0$, [14, p.266]. Namely, for $n \in \mathbb{N}$ the $n$-th row of the lower triangular matrix determining $(C - \lambda I)^{-1}$ has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^n (1 - \frac{1}{k\lambda})}, \quad 1 \le m < n, \quad \text{and} \quad \frac{n}{1 - n\lambda} = \frac{1}{(\frac{1}{n} - \lambda)}, \quad m = n, \tag{3.3}$$

with all other entries in row $n$ being 0. We write

$$(C - \lambda I)^{-1} = T_{D_\lambda} - \frac{1}{\lambda^2} T_{E_\lambda}, \tag{3.4}$$

where the diagonal matrix $D_\lambda = (d_{nm}(\lambda))_{n,m=1}^\infty$ is given by

$$d_{nn}(\lambda) := \frac{1}{(\frac{1}{n} - \lambda)} \quad \text{and} \quad d_{nm}(\lambda) := 0 \quad \text{if } n \ne m. \tag{3.5}$$

Setting $\gamma[\lambda] := \operatorname{dist}(\lambda, \Sigma_0) > 0$ it is routine to check that

$$|d_{nn}(\lambda)| \le \frac{1}{\gamma[\lambda]}, \quad n \in \mathbb{N}, \quad \lambda \in \mathbb{C} \setminus \Sigma_0. \tag{3.6}$$

Moreover, $E_\lambda = (e_{nm}(\lambda))_{n,m=1}^\infty$ is the lower triangular matrix given by $e_{1m}(\lambda) = 0$, for $m \in \mathbb{N}$, and for all $n \ge 2$ by

$$e_{nm}(\lambda) := \begin{cases} \frac{1}{n \prod_{k=m}^n (1 - \frac{1}{k\lambda})} & \text{if} \quad 1 \le m < n \\ 0 & \text{if} \quad m \ge n. \end{cases} \tag{3.7}$$

**Lemma 3.3.** *Let $X \subseteq \mathbb{C}^{\mathbb{N}}$ be any Banach lattice sequence space. For each $\lambda \in \mathbb{C} \setminus \Sigma_0$ the diagonal operator $T_{D_\lambda}$, with $D_\lambda = (d_{nm}(\lambda))_{n,m=1}^\infty$ given by* (3.5), *is regular in $X$, that is, $T_{D_\lambda} \in \mathcal{L}^r(X)$.*

*Proof.* Fix $\lambda \notin \Sigma_0$ and let $A := \frac{1}{\gamma[\lambda]} I$, where $I$ is the identity matrix in $\mathbb{C}^{\mathbb{N}}$, in which case $T_A(X) \subseteq X$ is clear. It follows from (3.6) that the matrix $B := D_\lambda$ satisfies (3.2). Hence, the regularity of $T_{D_\lambda}$ in $X$ follows from Corollary 3.2. □

**Remark 3.4.** (i) Since any Banach lattice sequence space $X \subseteq \mathbb{C}^{\mathbb{N}}$ is a Banach function space over the $\sigma$-finite measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, relative to counting measure $\mu$, and the function $n \longmapsto d_{nn}(\lambda)$ on $\mathbb{N}$ belongs to $L^\infty(\mu)$ by (3.6), the regularity of $T_{D_\lambda} \in \mathcal{L}(X)$ also follows from Proposition 2.1(i).

(ii) For appropriate $X$ and $\lambda \notin \Sigma_0$, it is clear from (3.4) and Lemma 3.3 that the regularity of $(C-\lambda I)^{-1} \in \mathcal{L}(X)$ is completely determined by the matrix $E_\lambda$.

The following inequalities will be needed in the sequel. For $\alpha < 1$ we refer to [14, Lemma 7] and for general $\alpha \in \mathbb{R}$ to [1, Lemma 3.2(i)].

**Lemma 3.5.** *Let $\lambda \in \mathbb{C}\setminus\Sigma_0$ and set $\alpha := Re(\frac{1}{\lambda})$. Then there exist positive constants $P(\alpha)$ and $Q(\alpha)$ such that*

$$\frac{P(\alpha)}{n^\alpha} \leq \prod_{k=1}^n |1-\frac{1}{k\lambda}| \leq \frac{Q(\alpha)}{n^\alpha}, \quad n \in \mathbb{N}. \tag{3.8}$$

## 4. The classical spaces $\ell^p, 1 < p \leq \infty$, and $c_0$

For each $1 < p \leq \infty$ let $C_p \in \mathcal{L}(\ell^p)$ denote the Cesàro operator as given by (1.4) when it is restricted to $\ell^p$. As a consequence of Hardy's inequality, [11, Theorem 326], it is known that $\|C_p\|_{\text{op}} = p'$, where $\frac{1}{p}+\frac{1}{p'} = 1$ (with $p' := 1$ when $p = \infty$). Concerning the spectrum of $C_p$ we have

$$\sigma(C_p) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| \leq \tfrac{p'}{2}\}, \quad 1 < p \leq \infty. \tag{4.1}$$

Various proofs of (4.1) are known for $1 < p < \infty$, [6], [12], [13], [15], [16]; see the discussion on p.268 of [6]. For the case $p = \infty$ we refer to [12, Theorem 4], for example.

**Remark 4.1.** For each $\lambda \neq 0$ set $\alpha := \text{Re}(\frac{1}{\lambda})$. Then, for any $b > 0$ we have

$$\alpha < \tfrac{1}{b} \text{ and only if } |\lambda - \tfrac{b}{2}| > \tfrac{b}{2}.$$

The corresponding results for $\alpha > \frac{1}{b}$ and $\alpha = \frac{1}{b}$ also hold.

**Proposition 4.2.** *For each $1 < p < \infty$ the order spectrum of the positive operator $C_p \in \mathcal{L}(\ell^p)$ satisfies*

$$\sigma_{\text{o}}(C_p) = \sigma(C_p). \tag{4.2}$$

*Proof.* Via (1.2) it suffices to verify that $\rho(C_p) \subseteq \rho_{\text{o}}(C_p)$.

With the notation of (3.4) and (3.7) it is shown on p.269 of [6], as a consequence of (3.8) in Lemma 3.5 above, that for every $\lambda \neq 0$ satisfying $\alpha := \text{Re}(\frac{1}{\lambda}) < 1$ there exists a constant $\beta(\lambda) > 0$ such that

$$|e_{nm}(\lambda)| \leq \frac{\beta(\lambda)}{n^{1-\alpha}m^\alpha}, \quad 1 \leq m \leq n, \quad n \in \mathbb{N}. \tag{4.3}$$

Set $B := E_\lambda$ and let $A$ be the lower triangular matrix whose entries $a_{nm}(\lambda) \geq 0$ are given by the right-side of (4.3) for each $n \in \mathbb{N}$ and $1 \leq m \leq n$ (and 0 otherwise). According to (4.3) the matrices $A$ and $B$ satisfy (3.2). Let $X := \ell^p$ for $p \in (1,\infty)$ fixed. Then Corollary 3.2 implies that $E_\lambda$ will be regular (i.e., $T_{E_\lambda} \in \mathcal{L}^r(\ell^p)$) whenever $T_A(\ell^p) \subseteq \ell^p$. Note that $T_A \in \mathcal{L}(\mathbb{C}^\mathbb{N})$ is given by

$$x \longmapsto \beta(\lambda)\left(\frac{1}{n^{1-\alpha}}\sum_{m=1}^n \frac{x_m}{m^\alpha}\right)_{n=1}^\infty := \beta(\lambda)G_\lambda(x), \quad x \in \mathbb{C}^\mathbb{N}. \tag{4.4}$$

So, if $\text{Re}(\frac{1}{\lambda}) < 1$, then (4.4) implies that $T_A \in \mathcal{L}(\ell^p)$ whenever $G_\lambda : \ell^p \longrightarrow \ell^p$ is continuous.

Let now $\lambda \in \rho(C_p)$, that is, $|\lambda - \frac{p'}{2}| > \frac{p'}{2}$. Then $\alpha := \text{Re}(\frac{1}{\lambda}) < \frac{1}{p'}$, because of Remark 4.1, and hence, $(1-\alpha)p > 1$. Then the Proposition on p.269 of [6] yields that indeed $G_\lambda \in \mathcal{L}(\ell^p)$. As noted above, this implies that $T_{E_\lambda} \in \mathcal{L}^r(\ell^p)$. Combined with (3.4) and Lemma 3.3 it follows that $(C_p - \lambda I)^{-1} \in \mathcal{L}^r(\ell^p)$, that is, $\lambda \in \rho_{\text{o}}(C_p)$. This completes the proof of (4.2). ☐

Recall that $\|C_\infty\|_{\rm op} = 1$ and, from (4.1) for $p = \infty$, that

$$\sigma(C_\infty) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| \leq \tfrac{1}{2}\} \tag{4.5}$$

**Proposition 4.3.** *The order spectrum of the positive operator $C_\infty \in \mathcal{L}(\ell^\infty)$ satisfies*

$$\sigma_{\rm o}(C_\infty) = \sigma(C_\infty).$$

*Proof.* Again by (1.2) it suffices to prove that $\rho(C_\infty) \subseteq \rho_{\rm o}(C_\infty)$.

Fix $\lambda \in \rho(C_\infty)$. According to (4.5), for $b = 1$ the condition in Remark 4.1 is satisfied with $\alpha := {\rm Re}(\frac{1}{\lambda})$. Hence, the inequalities (4.3) are valid and so $A := (a_{nm}(\lambda))_{n,m=1}^\infty \geq 0$ and $B := E_\lambda$ can again be defined exactly as in the proof of Proposition 4.2. Then (3.2) is satisfied with $X := \ell^\infty$. Arguing as in the proof of Proposition 4.2 (via Corollary 3.2) it remains to verify that $T_A : \ell^\infty \longrightarrow \ell^\infty$ is continuous, where $T_A$ is given by (4.4). To this effect, since $(1 - \alpha) > 0$ by Remark 4.1, it follows that

$$\sup_{n\in\mathbb{N}} \sum_{m=1}^\infty |a_{nm}(\lambda)| = \beta(\lambda) \sup_{n\in\mathbb{N}} \frac{1}{n^{1-\alpha}} \sum_{m=1}^\infty \frac{1}{m^\alpha} < \infty; \tag{4.6}$$

this has been verified on p.778 of [2] (put $w(n) = 1$ there for all $n \in \mathbb{N}$) by considering each of the cases $\alpha < 0$, $\alpha = 0$ and $0 < \alpha < 1$ separately. But, condition (4.6) is known to imply that $T_A \in \mathcal{L}(\ell^\infty)$, [19, Ex.2, p.220]. The proof that $\lambda \in \rho_{\rm o}(C_\infty)$ is thereby complete. □

To conclude this section we consider the Cesàro operator $C$, as given by (1.4), when it is restricted to $c_0$; denote this operator by $C_0$. It is shown in [12, Theorem 3], [14], that $\|C_0\|_{\rm op} = 1$ and

$$\sigma(C_0) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| \leq \tfrac{1}{2}\}. \tag{4.7}$$

**Proposition 4.4.** *The order spectrum of the positive operator $C_0 \in \mathcal{L}(c_0)$ satisfies*

$$\sigma_{\rm o}(C_0) = \sigma(C_0).$$

*Proof.* Since (4.7) shows that $\sigma(C_0) = \sigma(C_\infty)$, the entire proof of Proposition 4.3 can be easily adapted (now for $X := c_0$ and fixed $\lambda \in \rho(C_0)$), using the same notation, *up to the stage* where (4.6) is shown to be valid. In *addition* to the validity of (4.6) it is also true that

$$\lim_{n\to\infty} a_{nm}(\lambda) = \frac{\beta(\lambda)}{m^\alpha} \lim_{n\to\infty} \frac{1}{n^{1-\alpha}} = 0, \quad m \in \mathbb{N}, \tag{4.8}$$

because $\alpha := {\rm Re}(\frac{1}{\lambda})$ satisfies $(1 - \alpha) > 0$. The two conditions (4.6) and (4.8) together are known to imply that $T_A \in \mathcal{L}(c_0)$, [19, Theorem 4.51-C]. Again via Corollary 3.2 and Lemma 3.3 we can conclude that $T_{E_\lambda} \in \mathcal{L}^r(c_0)$ and hence, also $(C_0 - \lambda I)^{-1}$ is regular on $c_0$. □

## 5. The discrete Cesàro spaces ces$(p), 1 < p < \infty$, and ces$(0)$

For $1 < p < \infty$ the discrete Cesàro spaces are defined by

$${\rm ces}(p) := \{x \in \mathbb{C}^\mathbb{N} : \|x\|_{{\rm ces}(p)} := \big(\sum_{n=1}^\infty (\tfrac{1}{n} \sum_{k=1}^n |x_k|)^p\big)^{1/p} < \infty\}.$$

In view of (1.4) we see that $\|x\|_{{\rm ces}(p)} = \|C(|x|)\|_{\ell^p}$ for $x \in {\rm ces}(p)$. It is known that each space ${\rm ces}(p), 1 < p < \infty$, is a reflexive Banach lattice sequence space for the norm $\|\cdot\|_{{\rm ces}(p)}$ and the coordinatewise order. The spaces ${\rm ces}(p)$ have been thoroughly treated in [4]. According to Theorem 5.1 of [8] the restriction of $C$ (see (1.4)) to $\mathrm{ces}(p)$, denoted here by $C_{(p)}$, is continuous with $\|C_{(p)}\|_{\mathrm{op}} = p'$ and

$$\sigma(C_{(p)}) = \left\{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| \le \tfrac{p'}{2}\right\}, \quad 1 < p < \infty. \tag{5.1}$$

**Proposition 5.1.** *For each $1 < p < \infty$ the order spectrum of the positive operator $C_{(p)} \in \mathcal{L}(\mathrm{ces(p)})$ satisfies*

$$\sigma_{\mathrm{o}}(C_{(p)}) = \sigma(C_{(p)}). \tag{5.2}$$

*Proof.* In view of (1.2) it suffices to verify that $\rho(C_{(p)}) \subseteq \rho_{\mathrm{o}}(C_{(p)})$.

We decompose the set $\rho(C_{(p)})$ into two disjoint parts, namely the set

$$\rho_1 := \{\lambda \in \mathbb{C}\setminus\{0\} : \ \mathrm{Re}\left(\tfrac{1}{\lambda}\right) \le 0\} = \{u \in \mathbb{C}\setminus\{0\} : \ \mathrm{Re}(u) \le 0\} \tag{5.3}$$

and its complement $\rho_2 := \rho(C_{(p)})\setminus\rho_1$.

First fix $\lambda \in \rho_1$. Then $\lambda \notin \Sigma_0$ and so we may consider $E_\lambda = (e_{nm}(\lambda))_{n,m=1}^\infty$ and $D_\lambda = (d_{nm}(\lambda))_{n,m=1}^\infty$ as specified by (3.7) and (3.6), respectively. It is shown on p.72 of [8] that

$$|e_{nm}(\lambda)| \le \tfrac{1}{n}, \quad 1 \le m < n, \quad n \in \mathbb{N}. \tag{5.4}$$

*Warning*: In [8] the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ is used rather than $\mathbb{N} = \{1, 2, 3, \ldots\}$ which is used here and so the inequalities from [8] are slightly different when they are stated here. Back to our proof, it is clear from (1.4) that the matrix $A = (c_{nm})_{n,m=1}^\infty$ for the Cesàro operator $C$ is lower triangular with its $n$-th row, for each $n \in \mathbb{N}$, given by $c_{nm} := \frac{1}{n}$ for $1 \le m \le n$ and $c_{nm} := 0$ for $m > n$. Setting $B := E_\lambda$ it is clear from (5.4) that (3.2) is satisfied for the pair $A, B$ in the space $X := \mathrm{ces}(p)$. Since $C_{(p)} = T_A : \mathrm{ces}(p) \longrightarrow \mathrm{ces}(p)$ is continuous, it follows from Corollary 3.2 that $T_{E_\lambda} \in \mathcal{L}^r(\mathrm{ces}(p))$ and hence, via Lemma 3.3 and (3.4), that also $(C_{(p)} - \lambda I)^{-1} \in \mathcal{L}^r(\mathrm{ces}(p))$.

Consider now the set $\rho_2$. From (5.1) it is routine to establish that a non-zero point $z \in \mathbb{C}$ belongs to $\sigma(C_{(p)})$ if and only if $\mathrm{Re}(\frac{1}{z}) \ge \frac{1}{p'}$. From the case of equality in Remark 4.1, it follows that $\rho_2 = \bigcup_{0<\alpha<1/p'} \Gamma_\alpha$, where

$$\Gamma_\alpha := \left\{z \in \mathbb{C}\setminus\{0\} : \mathrm{Re}\left(\tfrac{1}{z}\right) = \alpha\right\} = \left\{z \in \mathbb{C}\setminus\{0\} : \left|z - \tfrac{1}{2\alpha}\right| = \tfrac{1}{2\alpha}\right\}. \tag{5.5}$$

Fix a point $\lambda \in \rho_2$. Then there exists a unique number $\alpha \in (0, \frac{1}{p'})$ such that $\lambda \in \Gamma_\alpha$, namely $\alpha := \mathrm{Re}(\frac{1}{\lambda})$. In the notation of (3.7) it is shown on p.72 of [8] that

$$|e_{nm}(\lambda)| \le e_{nm}\left(\tfrac{1}{\alpha}\right), \quad n, m \in \mathbb{N}. \tag{5.6}$$

Note that $e_{nm}(\frac{1}{\alpha}) \ge 0$ for all $n, m \in \mathbb{N}$ follows from (3.7) as $0 < \alpha < \frac{1}{p'}$ implies that $1 - \frac{1}{k(1/\alpha)} = (1 - \frac{\alpha}{k}) > 0$ for $m \le k \le n$. Setting $\widetilde{A} := E_{1/\alpha}$ and $\widetilde{B} := E_\lambda$ it is clear from (5.6) that (3.2) is satisfied for the pair $\widetilde{A}, \widetilde{B}$ in place of $A, B$. Moreover, $\frac{1}{\alpha} > p'$ implies that $\frac{1}{\alpha} \in \rho(C_{(p)})$, that is, $(C_{(p)} - \frac{1}{\alpha}I)^{-1} \in \mathcal{L}(\mathrm{ces}(p))$. Since $T_{D_{1/\alpha}} \in \mathcal{L}(\mathrm{ces}(p))$ by Lemma 3.3 (with $\frac{1}{\alpha}$ in place of $\lambda$), the identity $T_{E_{1/\alpha}} = \alpha^2(T_{D_{1/\alpha}} - (C_{(p)} - \frac{1}{\alpha}I)^{-1})$ shows that $T_{\widetilde{A}} \in \mathcal{L}(\mathrm{ces}(p))$. Hence, Corollary 3.2 can be applied to conclude that $T_{\widetilde{B}} = T_{E_\lambda} \in \mathcal{L}^r(\mathrm{ces}(p))$. It then follows from (3.4) and Lemma 3.3 that $(C_{(p)} - \lambda I)^{-1} \in \mathcal{L}^r(\mathrm{ces}(p))$. ☐

The remaining space to consider is $\mathrm{ces}(0) := \{x \in \mathbb{C}^{\mathbb{N}} : C(|x|) \in c_0\}$ equipped with the norm

$$\|x\|_{\mathrm{ces}(0)} := \|C(|x|)\|_{c_0} = \sup_{n\in\mathbb{N}} \tfrac{1}{n} \textstyle\sum_{k=1}^{n} |x_k|, \quad x \in \mathrm{ces}(0).$$

It is a Banach lattice sequence space for the norm $\|\cdot\|_{\mathrm{ces}(0)}$ and the coordinatewise order. According to [8, Theorem 6.4], the restriction of $C$ (see ((1.4)) to ces(0), denoted here by $C_{(0)}$, is continuous with $\|C_{(0)}\|_{\mathrm{op}} = 1$ and

$$\sigma(C_{(0)}) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| \leq \tfrac{1}{2}\}. \tag{5.7}$$

**Proposition 5.2.** *The order spectrum of the positive operator* $C_{(0)} \in \mathcal{L}(\mathrm{ces}(0))$ *satisfies*

$$\sigma_{\mathrm{o}}(C_{(0)}) = \sigma(C_{(0)}).$$

*Proof.* As usual it suffices to show that $\rho(C_{(0)}) \subseteq \rho_{\mathrm{o}}(C_{(0)})$.

Let the set $\rho_1$ be as in (5.3). For each $\alpha \in (0,1)$ let $\Gamma_\alpha$ be given by (5.5). Then (5.7) ensures that we have the disjoint partition $\rho(C_{(0)}) = \rho_1 \cup \rho_2$ with $\rho_2 := \bigcup_{0<\alpha<1} \Gamma_\alpha$.

For any given point $\lambda \in \rho_1$ the estimates (5.4) are again valid (see [8, p.72]) and so the argument in the proof of Proposition 5.1 can be easily adapted ( now for $X := \mathrm{ces}(0)$) to again show that $(C_{(0)} - \lambda I)^{-1} \in \mathcal{L}^r(\mathrm{ces}(0))$.

Fix now $\lambda \in \rho_2$. Then there exists a unique $\alpha \in (0,1)$ such that $\lambda \in \Gamma_\alpha$, namely $\alpha := \mathrm{Re}(\frac{1}{\lambda})$. Then $\mathrm{Re}(1 - \frac{1}{k\lambda}) = (1 - \frac{\alpha}{k}) \geq 0$ for $k \in \mathbb{N}$. Arguing as at the bottom of p.396 in [7], now with $x \in \mathrm{ces}(0)$ in place of $a \in \mathrm{ces}(2)$ there, it follows that the 1-st coordinate of $E_\lambda(x)$ is 0 and, for $n \geq 2$, that the $n$-th coordinate of $E_\lambda(x)$ satisfies

$$|(E_\lambda(x))_n| \leq (E_{1/\alpha}(|x|))_n, \quad x \in \mathrm{ces}(0).$$

Substituting $x := (\delta_{rj})_{j=1}^{\infty}$ into the previous estimates, for each $r \in \mathbb{N}$, yields (5.6). Since $0 < \alpha < 1$ implies that $\frac{1}{\alpha} \in \rho(C_{(0)})$, the argument can be completed along the lines given in the proof of Proposition 5.1 to conclude that $(C_{(0)} - \lambda I)^{-1} \in \mathcal{L}^r(\mathrm{ces}(0))$. We again warn the reader that $\mathbb{N} = \{0, 1, 2, \ldots\}$ is used in [7]. □

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