NON-POSITIVE AND NEGATIVE AT INFINITY DIVISORIAL VALUATIONS OF HIRZEBRUCH SURFACES

CARLOS GALINDO, FRANCISCO MONSERRAT, AND CARLOS-JESÚS MORENO-ÁVILA

Abstract. We consider rational surfaces Z defined by divisorial valuations ν of Hirzebruch surfaces. We introduce concepts of non-positivity and negativity at infinity for these valuations and prove that these concepts admit nice local and global equivalent conditions. In particular we prove that, when ν is non-positive at infinity, the extremal rays of the cone of curves of Z can be explicitly given.

1. Introduction

Valuations were introduced by Dedekind and Weber for studying Riemann surfaces but it was KÃ¼rschÃ¡k who gave the first axiomatic definition. In the middle of the past century, Zariski and Abhyankar [1, 2, 32, 33, 34] used the theory of valuations as a main tool to treat resolution of singularities of algebraic varieties, and much more recently, after the proof by Hironaka of resolution in characteristic zero, they are still considered suitable for the positive characteristic case [31].

Valuations are essentially local objects which could be used to prove local uniformization. However, in the last years, they have been used to study global properties. The best known situation corresponds with valuations of the fraction field $K(\mathcal{O}_{\mathbb{P}^2,p})$ of the local ring $\mathcal{O}_{\mathbb{P}^2,p}$ centered at $\mathcal{O}_{\mathbb{P}^2,p}$, $\mathbb{P}^2$ being the projective plane over an algebraically closed field k and p a closed point in $\mathbb{P}^2$. These valuations were classified by Spivakovsky [30] (see also [15, 21]). This classification has five types and works for valuations of the fraction field of any two-dimensional regular local ring R centered at R.

Divisorial and irrational valuations are two of these types and, for them and suitable divisors, it can be defined Seshadri-like constants [10], that is, objects which basically contain the same information for valuations as Seshadri constants for points. Recall that Seshadri constants were used by Demailly [13] for studying the Fujita's conjecture. Even in the most simple case, where the local ring is $\mathcal{O}_{\mathbb{P}^2,p}$ and the divisor is a line, Seshadri-like constants are very difficult to compute; they allow us to establish the concept of minimal valuation and propose a conjecture which implies the Nagata conjecture and is implied by that of Greuel-Lossen-Shustin (see [19] and also [14], where the mentioned Seshadri-like constant is denoted by $\hat{\mu}(\nu)$).

Spivakovsky's classification also contains the so-called exceptional curve valuations. Exceptional curve valuations are given by a pair whose first projection, ν_1, is a divisorial valuation. When the local ring is $\mathcal{O}_{\mathbb{P}^2,p}$, they correspond with flags of the form $\{X = X_r \supset E_r \supset \{q\}\}$, where q is a closed point and X_r is the surface obtained after a finite simple sequence of point blowing-ups starting at p, E_r being the last obtained exceptional divisor which defines ν_1. Newton-Okounkov bodies [25, 28] are the analogue to Seshadri constants for these exceptional curve valuations and, again, are very difficult to explicitly compute [9, 20].

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Recently, in [18], it was considered a class $\mathcal{N}$ of divisorial valuations ν (of $K(\mathcal{O}_{\mathbb{P}^2,p})$ centered at $\mathcal{O}_{\mathbb{P}^2,p}$) which have a similar behaviour as that of curves with only one place at infinity [3]. They were named non-positive at infinity because satisfy $\nu(f) \leq 0$ for every $f \in k[x,y] \setminus \{0\}$, $\{x,y\}$ being affine coordinates in the chart of points which are not in the line at infinity (which contains p). When $\nu(f) < 0$, they are called negative at infinity. These valuations are centered at infinity because the point p is in the line at infinity [5]. Recently, this last class of valuations has been studied and used in different contexts [7, 5, 16, 24, 27]. The set of divisorial valuations ν centered at $\mathcal{O}_{\mathbb{P}^2,p}$ and that of finite simple sequences of point blowing-ups starting with the blowing-up at p are bijective, and each valuation ν determines a rational projective surface X. In [18] we proved that the fact that ν belongs to $\mathcal{N}$ is equivalent to that of the cone of curves $NE(X)$ is regular, and we gave a simple characterization of this fact. Even more, we are able to compute the Seshadri-like constant with respect to a line divisor for valuations in $\mathcal{N}$ [19], and to explicitly give the Newton-Okounkov bodies of flags where the valuation given by the divisor E_r belongs to $\mathcal{N}$ [20].

Since the projective plane and the Hirzebruch surfaces provide the classical minimal models for rational surfaces, we consider divisorial valuations of the fraction field $K(\mathcal{O}_{\mathbb{F}_\delta,p})$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$ (called here divisorial valuations of $\mathbb{F}_\delta$), where $\mathbb{F}_\delta$ is any Hirzebruch surface and $p \in \mathbb{F}_\delta$ is a closed point. Our objective is to find suitable affine charts on a Hirzebruch surface such that, as in the case of the class $\mathcal{N}$, valuations ν which are non positive (or negative) on non-zero regular functions on these charts give rise to surfaces (obtained by the sequence of blowing-ups given by ν which starts at the Hirzebruch surface) with nice geometrical global properties. Notice that, as algebraic objects, valuations of $\mathbb{F}_\delta$ do not differ from valuations centered at regular closed points of other birationally equivalent surfaces; however we desire to relate the mentioned valuations with global geometric aspects of Hirzebruch surfaces. We will show that, in this case, there exist two natural charts "at infinity". On the one hand that given by points which are neither in the fiber F_1 that contains p nor in the special section M_0 of $\mathbb{F}_\delta$, and, on the other hand, by points which are neither in F_1 nor in a particular uniquely defined section $M_1 \neq M_0$. In this paper, we will divide the divisorial valuations of $\mathbb{F}_\delta$ in two classes, special and non-special, according to the chart at infinity to be used for introducing the concepts of non-positive and negative at infinity divisorial valuation of $\mathbb{F}_\delta$. That is, over each point $p \in \mathbb{F}_\delta$, we will consider special or non-special divisorial valuations which will determine the chart to be used to define non positivity or negativity at infinity. We will give several characterizations of those concepts, including one which is very easy to check from the dual graph of the valuation ν (that involves only topological information) and the images by ν of (the germs at p of) the fiber and sections before introduced (see Item (c) in theorems 3.6 and 4.8), and, in the case of negative at infinity valuations, the Iitaka dimension of certain divisor (Item (b) in theorems 3.9 and 4.12).

Each divisorial valuation ν of $\mathbb{F}_\delta$ defines in a unique way a rational surface Z obtained from the simple sequence of point blowing-ups given by ν. A remarkable property of non-positive at infinity valuations is that they determine (in fact, are equivalent to) the surfaces Z as above such that its cone of curves $NE(Z)$ is finite polyhedral and generated either by the classes of the strict transforms of the fiber F_1, the special section M_0 and the exceptional divisors (special valuations) or by the mentioned generators plus the class of the section M_1 (non-special valuations). Since the Hirzebruch surface $\mathbb{F}_1$ can be obtained by blowing-up a point in $\mathbb{P}^2$, our results recover those in [18] concerning the characterization of non-positive and negative at infinity divisorial valuations of $\mathbb{P}^2$, and provide a very simple characterization of the rational surfaces (obtained from a classical minimal model by a finite simple sequence of point blowing-ups) whose cone of curves has the above mentioned generators.

Some complementary results on the effective monoid of surfaces given by blowing-up some very concrete configurations of infinitely near points over Hirzebruch surfaces can be found in [11].

For surfaces defined by non-positive at infinity valuations of $\mathbb{P}^2$, we are able to decide whether their Cox rings are finitely generated [18], and, as mentioned, for these same valuations, we know how to compute their Seshadri-like constants and to explicitly obtain their corresponding Newton-Okounkov bodies. In a forthcoming paper, we will prove that similar properties can be deduced when considering non-positive at infinity valuations of $\mathbb{F}_\delta$.

Section 2 of the paper contains the ingredients we need to develop it. Special and non-special divisorial valuations of $\mathbb{F}_\delta$ are introduced in Definition 3.1. Section 3 studies the special ones and characterizes its non-positivity (respectively, negativity) at infinity in Theorem 3.6 (respectively, Theorem 3.9). The non-special divisorial valuations are considered in Section 4, being theorems 4.8 and 4.12 the main results.

2. Preliminaries

Given a surface Z_0, a (finite) simple sequence of blowing-ups starting at Z_0 is a sequence

$$\pi : Z = Z_n \xrightarrow{\pi_n} Z_{n-1} \to \ldots \to Z_1 \xrightarrow{\pi_1} Z_0, \tag{2.1}$$

of blowing-ups $\pi_i : Z_i \to Z_{i-1}$, $1 \leq i \leq n$, centered at closed points $p_i \in Z_{i-1}$, such that $p_1 = p \in Z_0$ and each p_i, $2 \leq i \leq n$, belongs to the exceptional divisor created by π_{i-1}.

In this paper, we will study some global properties concerning rational surfaces obtained from simple sequences π as above where Z_0 is a Hirzebruch surface. We start by recalling some basic facts about these surfaces (see [22, 4, 29] for additional information).

2.1. **Hirzebruch surfaces.** Let k be an algebraically closed field and $\mathbb{P}^1 = \mathbb{P}^1_k$ the projective line over k. Let δ be a non-negative integer and consider the δth *Hirzebruch surface* $\mathbb{F}_\delta := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\delta))$. Let F and M be two prime divisors providing generators $[F]$ and $[M]$ of $\mathrm{Pic}(\mathbb{F}_\delta)$ such that $F^2 = 0$ and $M^2 = \delta$; the symbol $[\cdot]$ will denote the class in the Picard group, throughout the paper. If $\delta > 0$ we denote by M_0 the $(-\delta)$-curve of $\mathbb{F}_\delta$ and we call it *special section*. If a divisor D is linearly equivalent to $aF + bM$ we will say that D has *degree* (a, b). Notice also that any irreducible curve C in $\mathbb{F}_\delta$ has degree (a, b) satisfying $a \geq 0$ and $b > 0$ [22, V, Proposition 2.20].

From a coordinates point of view (that will be useful for us throughout the paper) Hirzebruch surfaces $\mathbb{F}_\delta$ can be obtained as the quotient of the product of punctured affine planes over k,

$$(\mathbb{A}^2 \setminus \{(0,0)\}) \times (\mathbb{A}^2 \setminus \{(0,0)\}),$$

by an action of the product, $k^* \times k^*$, of multiplicative groups of the field k (see [29, §2.2]). For each $(\lambda, \mu) \in k^* \times k^*$, the action goes as follows:

$$\begin{array}{rccc} (\lambda, 1) : & (X_0, X_1; Y_0, Y_1) & \to & (\lambda X_0, \lambda X_1; Y_0, \lambda^{-\delta} Y_1) \\ (1, \mu) : & (X_0, X_1; Y_0, Y_1) & \to & (X_0, X_1; \mu Y_0, \mu Y_1). \end{array} \tag{2.2}$$

Notice that $\mathbb{F}_\delta$ is equipped with a projection morphism $pr : \mathbb{F}_\delta \to \mathbb{P}^1$ that, in terms of coordinates, is the projection onto the first factor. The class $[M_0]$ (resp., $[M]$) (resp., $[F]$) is the class of the curve with equation $Y_1 = 0$ (resp., $Y_0 = 0$) (resp., any fiber of pr). Notice also that $\mathbb{F}_\delta$ is covered by four affine open sets $U_{ij} := \mathbb{F}_\delta \setminus \mathbf{V}(X_i Y_j)$, $0 \leq i, j \leq 1$.

By convention, if $\delta \geq 1$, a point of $\mathbb{F}_\delta$ will be called *special* or *general* depending on whether it belongs or not to the special fiber M_0. We warn the reader not to get confused with special and non-special valuations (see Definition 3.1), which are different concepts.

2.2. **Divisorial valuations.** A *valuation* of a field K is a surjective map

$$\nu : K \setminus \{0\} \to G,$$

where G is a totally ordered commutative group (the value group of ν), such that, for $f, g \in K \setminus \{0\}$, satisfies:

$$\nu(f+g) \geq \min\{\nu(f), \nu(g)\} \text{ and also } \nu(fg) = \nu(f) + \nu(g).$$

The ring $R_\nu = \{f \in K \setminus \{0\} \mid \nu(f) \geq 0\} \cup \{0\}$ is called the *valuation ring* of ν. It is a local ring whose maximal ideal is $\mathfrak{m}_\nu = \{f \in K \setminus \{0\} \mid \nu(f) > 0\} \cup \{0\}$. When K is the fraction field of a local regular ring $(R, \mathfrak{m})$ and $R \cap \mathfrak{m}_\nu = \mathfrak{m}$, one says that ν is *centered* at R. When $\dim R = 2$, valuations centered at R ($R/\mathfrak{m}$ algebraically closed) are in one-to-one correspondence with (non-necessarily finite) simple sequences of point blowing-ups starting at Spec$R$. Divisorial valuations are those corresponding with finite simple sequences [34, 30].

In this paper p will be a closed point in $\mathbb{F}_\delta$. Denote by K the fraction field of the local ring $R = \mathcal{O}_{\mathbb{F}_\delta, p}$. Then, to give a sequence as (2.1) where $Z_0 = \mathbb{F}_\delta$ is equivalent to give a divisorial valuation ν of K centered at R. The valuation ν is defined by the last exceptional divisor E in the sequence $\pi : Z \to Z_0$ and, frequently and by simplicity, we will say that ν is a valuation of $\mathbb{F}_\delta$. The map π_1 is the blowing-up of Z_0 at $p = p_1$ and π_{i+1}, $1 \leq i \leq n-1$, the blowing-up of Z_i at the unique point p_{i+1} of the exceptional divisor defined by π_i, E_i, such that ν is centered at the local ring $\mathcal{O}_{Z_i, p_{i+1}}$. Write $\mathcal{C}_\nu := \{p_i\}_{i=1}^n$ the sequence (or configuration) of infinitely near points above defined; p_i is said to be *proximate* to p_j, denoted by $p_i \to p_j$, whenever $i > j$ and p_i belongs either to E_j or to the strict transform of E_j on Z_{i-1}. A point p_i is *satellite* whenever there exists $j < i-1$ satisfying $p_i \to p_j$; otherwise, it is called *free*. As in the case of germs of plane curves [6], plane divisorial valuations admit sets of invariants that help to study them. For a valuation ν as above, we will use its sequence of maximal contact values $\{\overline{\beta}_j\}_{j=0}^{g+1}$ [12, (1.5.3)] and its sequence of Pusieux exponents $\{\beta'_j\}_{j=0}^{g+1}$ [12, (1.5.2)]. Notice that both sequences can be obtained one from the other [12, Theorem 1.11]. The continued fraction expansions of the values $\{\beta'_j\}_{j=0}^{g+1}$ determine (and are determined by) the dual graph of ν. The dual graph of a valuation ν as above is a labelled tree, where each vertex represents an exceptional divisor appearing in the sequence of blowing-ups (2.1) and two vertices are joined whenever their corresponding divisor meet. Each vertex is labelled with the number of blowing-ups needed to create the corresponding divisor. The set $\{\overline{\beta}_j\}_{j=0}^{g}$ generates the *semigroup of values* of ν, $S(\nu) = \nu(R \setminus \{0\})$, [30, Remark 6.1], and the sequence of maximal contact values has an extra value $\overline{\beta}_{g+1}$ which coincides with the inverse of the volume of ν, $[\mathrm{vol}(\nu)]^{-1}$. Indeed, by definition,

$$\mathrm{vol}(\nu) = \lim_{\alpha \to \infty} \frac{\dim_k(R/\mathcal{P}_\alpha)}{\alpha^2/2},$$

where $\mathcal{P}_\alpha = \{f \in R \mid \nu(f) \geq \alpha\} \cup \{0\}$; taking into account that the above dimensions depend only on local data, the equality $\overline{\beta}_{g+1} = [\mathrm{vol}(\nu)]^{-1}$ follows as in [20, Remark 2.3].

Along the paper we denote by φ_i, $1 \leq i \leq n$, an analytically irreducible germ of curve at p whose strict transform on Z_i is transversal to E_i at a non-singular point of the exceptional locus. Also, for any curve C on $\mathbb{F}_\delta$, φ_C denotes its germ at p and $(\varphi_i, \varphi_C)_p$ equals 0 (respectively, the intersection multiplicity at p of the germs φ_i and φ_C) if C does not pass through p (respectively, otherwise). Finally, $\mathrm{mult}_{p_j}(\varphi_i)$ (respectively, $\mathrm{mult}_{p_j}(\varphi_C)$), $1 \leq i, j \leq n$, means multiplicity of the strict transform of φ_i (respectively,

φ_C) at p_j. We will use frequently, without any mention, the so-called Noether's formula for valuations, that we recall here for convenience of the reader and whose proof can be found in [8, Theorem 8.1.6]:

Lemma 2.1. *Let ν be a divisorial valuation of K centered at R, with associated configuration $\mathcal{C}_\nu := \{p_i\}_{i=1}^n$, and let C be a curve on $\mathbb{F}_\delta$. Then*

$$\nu(\varphi_C) = (\varphi_n, \varphi_C)_p = \sum_{j=1}^{n} \mathrm{mult}_{p_j}(\varphi_n) \cdot \mathrm{mult}_{p_j}(\varphi_C).$$

2.3. **Non-positive and negative at infinity valuations.** A divisorial valuation of the fraction field of $\mathcal{O}_{\mathbb{P}^2,p}$, centered at $\mathcal{O}_{\mathbb{P}^2,p}$, $p \in \mathbb{P}^2$ being closed point, (or, simply, a divisorial valuation of $\mathbb{P}^2$), is called *non-positive at infinity* when $\nu(h) \leq 0$ for all $h \in \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus L)$, L being a line (the line at infinity) containing p. In case it satisfies $\nu(h) < 0$ for every non-constant function $h \in \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus L)$, ν is named *negative at infinity*. As we mentioned in the introduction, non-positive and negative at infinity divisorial valuations of $\mathbb{P}^2$ have nice global properties involving the surfaces they define.

Afterwards we will introduce the concepts of non-positivity and negativity at infinity for valuations of the fraction field of $\mathcal{O}_{\mathbb{F}_\delta,p}$, centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$, p being a point in $\mathbb{F}_\delta$. As we will see, our definition will depend on the point p and a chart of the Hirzebruch surface which does not contain p. The goal of the paper is to show that the rational surfaces given by these valuations are easy to characterize and also enjoy nice global properties. We focus on the cone of curves and positivity properties of divisors.

3. The sign at infinity of special valuations

We start this section by partitioning the set of divisorial valuations of Hirzebruch surfaces in two subsets because our main results have to do with the concept of "sign at infinity" of valuations which depends on the considered subset. First we need to fix some notations.

Let $\mathbb{F}_\delta$ be a Hirzebruch surface and $p \in \mathbb{F}_\delta$ a point on it. Consider a divisorial valuation ν of the fraction field of $\mathcal{O}_{\mathbb{F}_\delta,p}$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$ and their associated configuration $\mathcal{C}_\nu = \{p_i\}_{i=1}^n$ and composition of blowing-ups $\pi : Z \to Z_0 = \mathbb{F}_\delta$ as in (2.1). Set E_i, $1 \leq i \leq n$, the exceptional divisor produced after blowing-up p_i and, given a divisor C on Z_i, for $0 \leq i \leq n$, we will denote by $\tilde{C}$ and C^* its strict and total transforms C on Z_j, for $j \geq i$. For simplicity of notation, often E_i also means the strict transform of the divisor E_i.

Definition 3.1. A divisorial valuation ν as before is called to be *special* (with respect to $\mathbb{F}_\delta$ and p) when one of the following conditions holds:

(1) $\delta = 0$.
(2) $\delta > 0$ and p is a special point.
(3) $\delta > 0$, p is a general point and there is no integral curve in the complete linear system $|M|$ whose strict transform on Z has negative self-intersection.

The remaining valuations will be called *non-special*.

Remark 3.2. Looking at the local equations of the linear system $|M|$ and taking into account their evolution by blowing-ups, it is not difficult to show that the above condition (3) holds if and only if either p_2 belongs to strict transform of the fiber of pr passing through p on Z_1, or there does not exist $j \geq \delta + 1$ such that the points p_i, $1 \leq i \leq j$, of $\mathcal{C}_\nu$ are free.

Throughout this section ν will be a *special* divisorial valuation of $\mathbb{F}_\delta$. In addition, F_1 will denote the fiber passing through p, and M_0 will denote either the special section (in case $\delta \geq 1$), or the section of degree $(0, 1)$ passing through p (otherwise).

Denote by $\mathrm{Pic}(Z)$ the Picard group of Z and $\cdot$ the intersection pair associated to $\mathrm{Pic}(Z)$. By extension consider the linear space $\mathrm{Pic}_{\mathbb{Q}}(Z) = \mathrm{Pic}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$ and, by abuse of notation, $\cdot$ will denote the corresponding bilinear pairing. Recall that the convex cone of $\mathrm{Pic}_{\mathbb{Q}}(Z)$ generated by the classes of effective divisors (respectively, nef divisors) is called the *cone of curves* (respectively, *nef cone*) and denoted by $NE(Z)$ (respectively, $P(Z)$). Notice that $P(Z)$ is the dual cone of $NE(Z)$. In the following we will denote by $\overline{NE}(Z)$ the closure of $NE(Z)$ in the usual topology.

By [26, Lemma 1.22], the classes $[\tilde{F}_1]$ and $[\tilde{M}_0]$ span extremal rays of both cones $NE(Z)$ and $\overline{NE}(Z)$. For our purposes, it will be useful to consider the strongly convex cone of $\mathrm{Pic}_{\mathbb{Q}}(Z)$, $S_1(Z)$, generated by the set of classes $\{[\tilde{F}_1], [\tilde{M}_0]\} \cup \{[E_i]\}_{i=1}^n$, and also its dual cone

$$S_1^{\vee}(Z) := \{[C] \in \mathrm{Pic}_{\mathbb{Q}}(Z) \mid [C] \cdot [D] \geq 0 \text{ for all } [D] \in S_1(Z)\}.$$

Our next result provides generators for $S_1^{\vee}(Z)$.

Proposition 3.3. *The dual cone $S_1^{\vee}(Z)$ is generated by $[F^*], [M^*]$ and the classes $\{[\Lambda_i]\}_{i=1}^n$ of the divisors*

$$\Lambda_i := a_i F^* + b_i M^* - \sum_{j=1}^{i} \mathrm{mult}_{p_j}(\varphi_i) E_j^*, \tag{3.1}$$

where $a_i := (\varphi_i, \varphi_{M_0})_p$ and $b_i := (\varphi_i, \varphi_{F_1})_p$.

Proof. It is enough to prove that $\{[F^*], [M^*]\} \cup \{[\Lambda_i]\}_{i=1}^n$ is the dual basis of $\{[\tilde{F}_1], [\tilde{M}_0]\} \cup \{[E_i]\}_{i=1}^n$ with respect to the intersection product.

Let $p_{i_{F_1}}$ be the last point in the configuration $\mathcal{C}_\nu$ of the valuation ν giving rise to Z through which the strict transform of F_1 passes. Also, if p belongs to M_0, we define i_{M_0} such that $p_{i_{M_0}}$ is the last point of $\mathcal{C}_\nu$ through which the strict transform of M_0 passes; otherwise we define $i_{M_0} := 0$. Taking into account that φ_i is analytically irreducible, the proximity equalities [8, Theorem 3.5.3] show that $\Lambda_i \cdot E_j = \delta_{ij}$, where δ_{ij} denotes the Kronecker's delta. Also, for each $i \in \{1, 2, \ldots, n\}$, it holds

$$\Lambda_i \cdot \tilde{F}_1 = b_i - \sum_{j=1}^{\min\{i, i_{F_1}\}} \mathrm{mult}_{p_j}(\varphi_i) = 0,$$

and

$$\Lambda_i \cdot \tilde{M}_0 = a_i - \sum_{j=1}^{\min\{i, i_{M_0}\}} \mathrm{mult}_{p_j}(\varphi_i) = 0,$$

where the summations with upper index equal to 0 are defined to be 0. Finally notice that $F^* \cdot \tilde{F}_1 = 0$, $F^* \cdot \tilde{M}_0 = 1$, $M^* \cdot \tilde{F}_1 = 1$, $M^* \cdot \tilde{M}_0 = 0$ and $F^* \cdot E_i = M^* \cdot E_i = 0$ for all $i = 1, 2, \ldots, n$. This concludes the proof. □

Recall that we are considering a special divisorial valuation ν of $\mathbb{F}_\delta$ and the surface Z that ν defines. The divisors Λ_i defined in (3.1) will be useful in this section because, as we are going to prove, they satisfy nice properties.

Lemma 3.4. *Let ν be a special divisorial valuation of $\mathbb{F}_\delta$. Then, with the above notation, it holds that $\Lambda_1^2 \geq 0$, and the inequality $\Lambda_i^2 \geq 0$ for some index $i \in \{2, 3, \ldots, n\}$ implies:*

(a) *$\Lambda_i^2 > 0$, whenever p_i is a satellite point of the configuration $\mathcal{C}_\nu$.*
(b) *$\Lambda_{i-1}^2 \geq 0$ and in case $\Lambda_{i-1}^2 = 0$, the point p_i is satellite and the point p_{i-1} is free.*

Proof. The self-intersection of the divisor Λ_1 satisfies $\Lambda_1^2 = 1 + \delta$ when p_1 is a special point and also when $\delta = 0$. Otherwise, $\Lambda_1^2 = \delta - 1$.

For proving the remaining statements, we can assume, without loss of generality, that $i = n \geq 2$.

We are going to prove the result when p_1 is a special point. Otherwise, the proof is the same after setting $\delta = 0$ or $a_n = 0$.

We start with the proof of Statement (a) for which we will use some properties of the set of maximal contact values of ν, $\{\overline{\beta}_j\}_{j=0}^{g+1}$, (see [30] and [12, Section 1.5]). We divide this proof in two cases.

Case 1(a): $g > 1$. Reasoning by contradiction and taking into account that the point p_n is satellite, we get that

$$0 = \Lambda_n^2 = 2a_nb_n + \delta b_n^2 - e_{g-1}\overline{\beta}_g = e_{g-1}\left[\frac{2a_nb_n + \delta b_n^2}{e_{g-1}} - \overline{\beta}_g\right],$$

where $e_{g-1} := \gcd(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{g-1})$. Since both a_n and b_n are either a multiple of $\overline{\beta}_0$ or $\overline{\beta}_1$, the first addend in the brackets is a multiple of e_{g-1}, which gives a contradiction because $\gcd(e_{g-1}, \overline{\beta}_g) = 1$.

Case 2(a): $g = 1$. We distinguish three sub-cases: *The values a_n and b_n are divisible by $\overline{\beta}_0$.* Then $e_{g-1} = e_0 = \overline{\beta}_0$ and the proof follows as above. *The value a_n satisfies $a_n = \overline{\beta}_1$.* Then $\Lambda_n^2 = \overline{\beta}_0(2\overline{\beta}_1 + \overline{\beta}_0\delta - \overline{\beta}_1) > 0$. *Otherwise.* Then $\Lambda_n^2 = \overline{\beta}_1(2\overline{\beta}_0 + \overline{\beta}_1\delta - \overline{\beta}_0) > 0$, which concludes the proof of Statement (a).

Now we prove Statement (b). Again we can suppose that $i = n$. We also assume that the point p_n is satellite, otherwise $\Lambda_{n-1}^2 > 0$ by Noether's formula. Denote by $\widehat{\nu}$ the divisorial valuation defined by the divisor E_{n-1}. Let $\{\widehat{\overline{\beta}}_j\}_{j=0}^{\widehat{g}+1}$ be the sequence of maximal contact values of $\widehat{\nu}$, set $\widehat{e}_{g-1} := \gcd(\widehat{\overline{\beta}}_0, \widehat{\overline{\beta}}_1, \ldots, \widehat{\overline{\beta}}_{g-1})$ and $e := \widehat{e}_{\widehat{g}-1}/e_{\widehat{g}-1}$. Consider two cases with two sub-cases.

Case 1(b): $g = \hat{g}$. Assume first that $g > 1$. From the following equality, which is proved in [18, Lemma 2],

$$|\widehat{\overline{\beta}}_g - e\overline{\beta}_g| = \frac{1}{e_{g-1}}, \tag{3.2}$$

one can deduce that

$$-\frac{e_{g-1}\widehat{\overline{\beta}}_g}{e} \geq -\frac{1}{e} - e_{g-1}\overline{\beta}_g. \tag{3.3}$$

In this case both valuations ν and $\widehat{\nu}$ are defined by satellite points, therefore $a_{n-1} = ea_n, b_{n-1} = eb_n$, $\overline{\beta}_{g+1} = e_{g-1}\overline{\beta}_g$ and $\widehat{\overline{\beta}}_{g+1} = \widehat{e}_{g-1}\overline{\beta}_g$. As a consequence

$$\Lambda_{n-1}^2 = e^2\left[2a_nb_n + \delta b_n^2 - \frac{e_{g-1}\widehat{\overline{\beta}}_g}{e}\right] \geq e^2\left[2a_nb_n + \delta b_n^2 - \frac{1}{e} - e_{g-1}\overline{\beta}_g\right]$$

$$= e^2\left[\Lambda_n^2 - \frac{1}{e}\right] > 0,$$

where the first inequality is deduced from the inequality (3.3) and the last one holds since $\Lambda_n^2 > e_{g-1} > 1/e$.

To conclude the proof in this case, it remains to study what happens when $g = 1$. We consider the same subcases as above: The values a_n and b_n are both divisible by $\overline{\beta}_0$, then the fact $\Lambda_{n-1}^2 > 0$ can be proved as before. The value a_n equals $\overline{\beta}_1$, then

$$\Lambda_{n-1}^2 = 2\widehat{\overline{\beta}}_0\widehat{\overline{\beta}}_1 + \delta\widehat{\overline{\beta}}_0^2 - \widehat{\overline{\beta}}_2 = \widehat{\overline{\beta}}_0(2\widehat{\overline{\beta}}_1 + \delta\widehat{\overline{\beta}}_0 - \widehat{\overline{\beta}}_1) = \widehat{\overline{\beta}}_0(\widehat{\overline{\beta}}_1 + \delta\widehat{\overline{\beta}}_0) > 0.$$

Otherwise, then $\Lambda_{n-1}^2 = \widehat{\overline{\beta}}_1(\widehat{\overline{\beta}}_0 + \delta\widehat{\overline{\beta}}_1) > 0$.

Case 2(b): $\widehat{g} = g - 1$. When $g > 1$, it holds

$$\widehat{\overline{\beta}}_{\hat{g}+1} = \frac{\overline{\beta}_{g+1} + 2}{4}$$

and thus

$$\Lambda_{n-1}^2 = \frac{1}{4}\left(2a_n b_n + \delta b_n^2 - \overline{\beta}_{g+1} - 2\right) = \frac{1}{4}\Lambda_n^2 - \frac{1}{2} \geq 0,$$

because $\Lambda_n^2 \geq 2$.

Finally we must assume that $g = 1$ and, as above, when the values a_n and b_n are divisible by $e_0 = 2$, $\Lambda_{n-1}^2 \geq 0$. When $a_n = \overline{\beta}_1$, $\Lambda_{n-1}^2 = \widehat{\overline{\beta}}_1 + \delta \geq 0$, and otherwise,

$$\Lambda_{n-1}^2 = 2\widehat{\overline{\beta}}_1 + \delta\widehat{\overline{\beta}}_1^2 - \widehat{\overline{\beta}}_1 \geq 0,$$

which concludes the proof. □

Next we introduce the concepts of non-positivity and negativity at infinity for special divisorial valuations. Afterwards a similar concept will be given for non-special valuations. We consider a Hirzebruch surface $\mathbb{F}_\delta$, a closed point p in $\mathbb{F}_\delta$ and denote by R the local ring $\mathcal{O}_{\mathbb{F}_\delta,p}$.

Definition 3.5. Let ν be a special divisorial valuation of the fraction field of R centered at R. The valuation ν is called *non-positive* (respectively, *negative*) *at infinity* whenever $\nu(h) \leq 0$ (respectively, $\nu(h) < 0$) for all $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_0))$ (respectively, $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_0))$, $h \notin k$).

We devote the remaining of this section to state two results, Theorems 3.6 and 3.9, which give several equivalent conditions to the fact that a special divisorial valuation of a Hirzebruch surface is non-positive or negative at infinity. We will use the divisor Λ_n and the values a_n and b_n, defined in Proposition 3.3.

Theorem 3.6. *Let ν be a special divisorial valuation of the fraction field of R centered at R. Set Z the surface that ν defines. Consider the divisor Λ_n given in* (3.1) *and the inverse of the volume of ν, $[\mathrm{vol}(\nu)]^{-1}$. Then the following conditions are equivalent:*

- (a) *The valuation ν is non-positive at infinity.*
- (b) *The divisor Λ_n is nef.*
- (c) *The inequality $2a_n b_n + b_n^2\delta \geq [\mathrm{vol}(\nu)]^{-1}$ holds.*
- (d) *The cone of curves $NE(Z)$ is generated by the classes of the strict transforms on Z of the fiber passing through p, the special section and the irreducible exceptional divisors associated with the map π given by ν.*

Proof. Our first step is to prove the equivalence between items (a) and (b), and we start by proving that Item (b) implies Item (a). We assume firstly here that $\delta > 0$ and $p = p_1$ is a special point. Without loss of generality, suppose that the special point p has coordinates $(1 : 0; 1, 0)$. The point p belongs to the fiber F_1 whose equation is $X_1 = 0$, and the special section M_0 is defined by the equation $Y_1 = 0$. Set U_{00} the affine open set of $\mathbb{F}_\delta$ given by $X_0 \neq 0$ and $Y_0 \neq 0$, whose associated affine coordinates are $\{u, v\} = \left\{\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0}\right\}$. Consider also the affine open set of $\mathbb{F}_\delta$, U_{11}, defined by $X_1 \neq 0$ and $Y_1 \neq 0$, with coordinates $\{x, y\} = \left\{\frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1}\right\}$. It holds that $p \in U_{00}$ and F_1 and M_0 have local equations $u = 0$ and $v = 0$, respectively. Denote by $\mathcal{P}$ the set of non-constant functions in $\mathcal{O}_{\mathbb{F}_\delta}(U_{11})$ (up to multiplication by a nonzero element of k) such that neither x nor y divide them. In terms of the coordinates $\{u, v\}$, $f \in \mathcal{P}$ can be expressed as

$$f(x, y) = f(1/u, 1/u^\delta v) = \frac{h_f(u, v)}{u^{\deg_1(h_f)+\delta\deg_2(h_f)} v^{\deg_2(h_f)}}, \tag{3.4}$$

where $h_f(u,v) \in \mathcal{O}_{\mathbb{F}_\delta}(U_{00})$. The bi-homogeneous polynomial

$$X_0^{\deg_1(h_f)} Y_0^{\deg_2(h_f)} \cdot h_f\left(\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0}\right)$$

defines a curve C_f on the surface $\mathbb{F}_\delta$ of degree $(\deg_1(h_f), \deg_2(h_f))$ and, if F' and M' are the fiber and the section on $\mathbb{F}_\delta$ with equations $X_0 = 0$ and $Y_0 = 0$, it holds that the map $f \to C_f$ defines a one-to-one correspondence between $\mathcal{P}$ and the set of the curves on $\mathbb{F}_\delta$ containing no curve in $\{F_1, F', M_0, M'\}$ as a component. Now, the condition Λ_n nef and (3.4) show that

$$\begin{aligned} 0 \le \Lambda_n \cdot C_f &= \Lambda_n \cdot \left[\deg_1(h_f)F^* + \deg_2(h_f)M^* - \sum_{i=1}^n \mathrm{mult}_{p_i}(h_f)E_i^*\right] \\ &= -\left[-(\deg_1(h_f) + \deg_2(h_f)\delta)\nu(u) - \deg_2(h_f)\nu(v) + \nu(h_f)\right] = -\nu(f). \end{aligned}$$

So, to finish the proof of Item (a) in this case (p is a special point), it only remains to assume that either x or y or both are factors of f. Then the proof follows from the existence of non-negative integers α, β with $\alpha + \beta \neq 0$ and $f_1 \in \mathcal{P}$ such that

$$\nu(f) = \nu(x^\alpha y^\beta f_1) = -(\alpha + \beta\delta)\nu(u) - \beta\nu(v) + \nu(f_1) \le 0.$$

If $\delta = 0$ the proof is analogous, and the non-positivity of ν for the case when p is a general point can be proved in a similar way after assuming that p has coordinates $(0:1;0,1)$ and considering local coordinates $\{u,v\} = \left\{\frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1}\right\}$ in the affine open set U_{11} and $\{x,y\} = \left\{\frac{X_1}{X_0}, \frac{Y_0}{X_0^\delta Y_1}\right\}$ in U_{01}.

Now we are going to prove that Item (a) implies Item (b). Assume by contradiction that the divisor Λ_n is not nef and, therefore, that there exists an effective divisor C such that $\Lambda_n \cdot C < 0$. This implies that, with the above notation, if p is a special point –or $p \in \mathbb{F}_0$–, (respectively, p is a general point), then there exists $f \in \mathcal{O}_{\mathbb{F}_\delta}(U_{11})$ (respectively, $f \in \mathcal{O}_{\mathbb{F}_\delta}(U_{01})$) such that $-\nu(f) = \Lambda_n \cdot C < 0$, a contradiction.

The fact that Item (b) implies Item (c) follows easily from previous computations given in the proof of Lemma 3.4.

Let us prove that Item (d) can be deduced from Item (c). Fix any ample divisor H on the surface Z and consider the set

$$A(Z) := \{[D] \in \mathrm{Pic}_{\mathbb{Q}}(Z) \mid [D]^2 \ge 0 \text{ and } [H] \cdot [D] \ge 0\}.$$

Recall that the above defined cone $S_1(Z)$ is generated by the classes $[\tilde{F}_1]$, $[\tilde{M}_0]$ and $[E_i]$, $1 \le i \le n$, and we are going to prove that

$$\overline{NE}(Z) = S_1(Z) + S_1^\vee(Z) = NE(Z) \tag{3.5}$$

and

$$S_1^\vee(Z) \subseteq A(Z) \subseteq S_1(Z) \tag{3.6}$$

hold, which shows Item (d). Our Hypothesis (c) means that $\Lambda_n^2 \ge 0$ and by Lemma 3.4, one has that $\Lambda_i^2 \ge 0$, $1 \le i \le n-1$. Proposition 3.3 proves the first inclusion in (3.6) and the last one follows from the first one and the equality $A(Z)^\vee = A(Z)$ (that holds taking into account the Hodge index theorem [22, Theorem 1.9]). It remains to prove the chain of equalities (3.5). For a start, notice that $A(Z) \subseteq \overline{NE}(Z)$ by [26, Lemma 1.20]. Thus $S_1^\vee(Z) \subseteq \overline{NE}(Z)$. Now, if $[C]$ is the class of an irreducible curve on Z and it is not one of the given generators of $S_1(Z)$, then $[C] \in S_1^\vee(Z)$ because otherwise $[C] \cdot [D] < 0$ for $[D] \in S_1(Z)$ and C and D would have a common component. Therefore we have proved the chain (3.5) with inclusions $\supseteq$ instead of equalities. Taking topological closures we deduce that (3.5) holds.

Finally Item (d) implies Item (b) by Proposition 3.3, which concludes the proof. $\square$

Remark 3.7. The Hirzebruch surface $\mathbb{F}_1$ can be regarded as the projective space $\mathbb{P}^2$ with a point blown-up. If one considers the line at infinity L in $\mathbb{P}^2$ and one regards $\mathbb{F}_1$ as the blow-up of $\mathbb{P}^2$ at a point of L, then it is not difficult to deduce that $M_0 = E_1$ and F is a general divisor in $|\tilde{L}|$. As a consequence, Theorem 3.6 allows us to provide equivalent conditions to the non-positivity of a valuation of $\mathbb{P}^2$ (see Section 2.3). In fact, our Theorem 3.6 recovers Theorem 1 in [18] considering $\delta = 1$ and p_1 a special point, and the above proof is an adaptation and extension to our more general situation of that of [18, Theorem 1].

Remark 3.8. Assume that Z is a surface as in Theorem 3.6 defined by a non-positive at infinity special valuation. Then, on the one hand, Lemma 3.4 and Theorem 3.6 prove that the divisorial valuation ν_i defined by any exceptional divisor E_i given by (2.1) is also special and non-positive at infinity. On the other hand, every divisor Λ_i, $1 \leq i \leq n$, is effective. Indeed, under these conditions, the expression of Λ_i, in the basis of strict transforms $\{\tilde{F}_1, \tilde{M}_0\} \cup \{E_j\}_{j=1}^n$, is

$$\Lambda_i = (\Lambda_i \cdot M^*)\tilde{F}_1 + (\Lambda_i \cdot F^*)\tilde{M}_0 + \sum_{j=1}^{n} (\Lambda_i \cdot \Lambda_j) E_j,$$

which is effective because Λ_i is nef.

We conclude this section by stating a characterization result for negative at infinity special valuations of Hirzebruch surfaces. Argumenting as in Remark 3.7, one can see that our result also proves [18, Theorem 2].

Theorem 3.9. *Let ν (respectively, Z, Λ_n) a divisorial valuation (respectively, a surface, a divisor on Z) as in Theorem 3.6. Then, the following conditions are equivalent:*

(a) *The valuation ν is negative at infinity.*
(b) *It holds that either $2a_nb_n + b_n^2\delta > [\mathrm{vol}(\nu)]^{-1}$, or $2a_nb_n + b_n^2\delta = [\mathrm{vol}(\nu)]^{-1}$ and the Iitaka dimension of the divisor Λ_n vanishes.*
(c) *The inequality $\Lambda_n \cdot \tilde{C} > 0$ holds for the strict transform on Z, $\tilde{C}$, of any curve C on $\mathbb{F}_\delta$, $C \neq F_1, M_0$.*

Proof. For a start, we recall that the Iitaka dimension [23] of a divisor D on Z is the maximum of the projective dimensions of the closures of the images of the rational maps defined by the complete linear systems $|nD|$, when n runs over those positive integers m such that $H^0(Z, \mathcal{O}_Z(mD)) \neq 0$.

We assume that p_1 is a special point. The other cases can be proved similarly. Assume also, without loss of generality, that p_1 has coordinates $(1:0;1,0)$ and consider the same notations as in the proof of Theorem 3.6.

We start by proving by contradiction that Statement (b) can be deduced from Statement (a). Hence, assume that (a) holds but (b) is false (what means, taking into account Theorem 3.6, that $\Lambda_n^2 = 0$ and $\dim |m\Lambda_n| > 0$ for m large enough). Therefore, there exists $f \in \mathcal{P}$ such that the class of $m\Lambda_n - \tilde{C}_f$ is effective for m large enough. This implies that

$$0 \leq \Lambda_n \cdot (m\Lambda_n - \tilde{C}_f) = m\Lambda_n^2 - \Lambda_n \cdot \tilde{C}_f = -\Lambda_n \cdot \tilde{C}_f.$$

Hence $0 = \Lambda_n \cdot \tilde{C}_f = -\nu(f)$ because Λ_n is nef (by Theorem 3.6), and this fact contradicts (a).

To prove that Statement (b) implies Statement (c), we reason again by contradiction and consider C an integral curve on $\mathbb{F}_\delta$ different from F_1 and M_0, and such that $\Lambda_n \cdot \tilde{C} \leq 0$. In fact $\Lambda_n \cdot \tilde{C} = 0$ because, by Theorem 3.6, Λ_n is nef. Let $\mathcal{F}$ be the face of the cone of curves of Z spanned by the classes $[\tilde{F}_1], [\tilde{M}_0]$, $[E_1], \ldots, [E_{n-1}]$, that is, $\mathcal{F} = [\Lambda_n]^\perp \cap NE(Z)$. It is clear that $[\tilde{C}] \in \mathcal{F}$ and, since the extremal rays of $NE(Z)$

are generated by classes of irreducible curves with negative self-intersection, $\tilde{C}^2 = 0$. $\tilde{C}$ is nef, so $[\tilde{C}]^{\perp} \cap NE(Z)$ is a face of $NE(Z)$ which contains $[\tilde{C}]$ and, then, it must coincide with $[\Lambda_n]^{\perp} \cap NE(Z)$. Indeed, this is a consequence of the fact that, in suitable coordinates, $A(Z)$ is the projective cone over an Euclidean ball B (by the Hodge index theorem [22, Theorem 1.9]) and B is strictly convex. Then, $\tilde{C}$ is linearly equivalent to a multiple of Λ_n and, by Remark 3.8, we get a contradiction.

To finish, the fact that Statement (c) implies Statement (a) can be proved as in Theorem 3.6 when proving that Item (b) implies Item (a).

□

Remark 3.10. The concepts of non-positivity (and negativity) at infinity of valuations of $\mathbb{P}^2$ and $\mathbb{F}_\delta$ are different. For instance, by [18, Theorem 1] there is no non-positive at infinity divisorial valuation of $\mathbb{P}^2$ with maximal contact values 3, 11 and 122. However, Theorem 3.6 proves the existence of non-positive at infinity valuations of $\mathbb{F}_\delta$ with those maximal contact values, when $\delta \geq 2$.

4. The sign at infinity of non-special valuations

This section gives results that characterize the non-positivity and negativity at infinity of non-special divisorial valuations (see Definition 4.7 therein) of a Hirzebruch surface $\mathbb{F}_\delta$, $\delta > 0$.

Notice that, when considering non-special valuations, there exists a unique irreducible section that is linearly equivalent to M and whose strict transform on Z has negative self-intersection. We will denote this section by M_1. Notice that its class gives an extremal ray of the cone $NE(Z)$.

For reaching our objectives, we need to describe the dual cone of the strongly convex cone of $\mathrm{Pic}_{\mathbb{Q}}(Z)$, $S_2(Z)$, generated by the classes $[\tilde{F}_1], [\tilde{M}_0], [\tilde{M}_1]$ and $[E_i]$, $1 \leq i \leq n$. Our first result is a lemma which we will use in the forthcoming Proposition 4.2 (that gives generators for the mentioned dual cone).

Lemma 4.1. *The class of the strict transform of M_1 on Z, $[\tilde{M}_1]$, can be written as*

$$[\tilde{M}_1] = \delta[\tilde{F}_1] + [\tilde{M}_0] + (\delta-1)[E_1] + (\delta-2)[E_2] + \dots$$
$$+[E_{\delta-1}] + d_{\delta+1}[E_{\delta+1}] + \dots + d_n[E_n],$$

where $d_i \in \mathbb{Z}$ and $d_i \leq -1$ for all $i = \delta+1, \delta+2, \dots, n$.

Proof. It is clear that we can write $[\tilde{M}_1]$ as

$$[\tilde{M}_1] = d_{01}[\tilde{F}_1] + d_{02}[\tilde{M}_0] + d_1[E_1] + \dots + d_n[E_n],$$

for some values $d_{01}, d_{02}, d_i \in \mathbb{Z}$, $1 \leq i \leq n$. Now, using the equalities

$$[\tilde{F}_1] = [F^*] - [E_1^*],\ [\tilde{M}_0] = \delta[F^*] + [M^*] \text{ and } [E_i] = [E_i^*] - \sum_{p_j \to p_i} [E_j^*],$$

we can compare the above expression with the equality $[\tilde{M}_1] = [M^*] - \sum_{j=1}^{i_{M_1}} [E_j^*]$, where i_{M_1} is the index of the last point of the configuration of infinitely near points given by ν, $\mathcal{C}_\nu$, through which $\tilde{M}_1$ goes. This gives rise to a system of linear equations in the variables $d_{01}, d_{02}, \{d_i\}_{i=1}^n$, whose first equations are

$$d_{01} - \delta d_{02} = 0,\ d_{02} = 1,\ d_1 - d_{01} = -1,\ d_2 - d_1 = -1, \dots,\ d_{\delta-1} - d_{\delta-2} = -1,$$
$$d_\delta - d_{\delta-1} = -1.$$

These equations determine the values of d_{01}, d_{02}, d_i for $i \in \{1, 2, \dots, \delta\}$, that coincide with those given in the statement. The fact that $d_i \leq -1$ for $i \geq \delta+1$ follows from considering the remaining equations and recalling that non-free points can only appear when $j > \delta + 1$. □

Proposition 4.2. *Let Z be the surface given by a non-special divisorial valuation and let $S_2(Z)$ be the cone of $Pic_{\mathbb{Q}}(Z)$ defined before Lemma 4.1. Then the dual cone $S_2^{\vee}(Z)$ of $S_2(Z)$ is generated by the following classes of divisors: $[F^*], [M^*]$, $\{[\Theta_i]\}_{i=1}^{\delta}$, $\{[\Delta_i]\}_{i=\delta+1}^{n}$, $\{[\Gamma_i]\}_{i=\delta+1}^{n}$ and $\{[\Upsilon_{ik}]\}_{i=\delta+1,k=1,\ldots,\delta-1}^{n}$, where*

$$\Theta_i := b_i M^* - \sum_{j=1}^{i} \mathrm{mult}_{p_j}(\varphi_i) E_j^*,$$

$$\Delta_i := (-\delta b_i + c_i)F^* + b_i M^* - \sum_{j=1}^{i} \mathrm{mult}_{p_j}(\varphi_i) E_j^*,$$

$$\Gamma_i := c_i M^* - \sum_{j=1}^{i} \left(\delta\, \mathrm{mult}_{p_j}(\varphi_i)\right) E_j^*, \text{ and}$$

$$\Upsilon_{ik} := (c_i - kb_i)M^* - \sum_{j=1}^{k}(c_i - kb_i)E_j^* - \sum_{j=k+1}^{i} \left((\delta - k)\mathrm{mult}_{p_j}(\varphi_i)\right) E_j^*,$$

and where

$$b_i := (\varphi_{F_1}, \varphi_i)_p, 1 \leq i \leq n, \quad \text{and} \quad c_i := (\varphi_{M_1}, \varphi_i)_p, \delta + 1 \leq i \leq n.$$

Proof. The cone $S_2(Z)$ is the intersection of the half-spaces

$$H_\tau := \{x \in \mathrm{Pic}_{\mathbb{Q}}(Z) \mid u_\tau \cdot x \geq 0\},$$

where τ varies in the set of faces of $S_2(Z)$ of codimension one and $u_\tau \in S_2^{\vee}(Z)$ is such that $\tau = S_2(Z) \cap u_\tau^{\perp}$; then the vectors u_τ generate $S_2^{\vee}(Z)$ (see Section 1.2 of [17]). This shows that it suffices to consider every $(n+1)$-dimensional linear subspace H generated by elements of $S_2(Z)$ and check whether $H^{\perp}$ is generated by an element of $S_2^{\vee}(Z)$. We will see that these generators will be those in the statement.

Denote by $\langle S \rangle$ the linear subspace generated by a set $S \subseteq \mathrm{Pic}_{\mathbb{Q}}(Z)$. Then,

$$\langle \{[\tilde{F}_1]\} \cup \{[E_i]\}_{i=1}^{n} \rangle^{\perp} = \langle [F^*] \rangle, \quad \langle \{[\tilde{M}_0]\} \cup \{[E_i]\}_{i=1}^{n} \rangle^{\perp} = \langle [M^*] \rangle,$$

and $[F^*], [M^*] \in S_2^{\vee}(Z)$. Moreover $\langle \{[\tilde{M}_1]\} \cup \{[E_i]\}_{i=1}^{n} \rangle^{\perp}$ is not generated by an element in $S_2^{\vee}(Z)$.

We have studied spaces H whose generators contain all the classes $[E_i]$. Now we will treat the cases where a class $[E_i]$, $1 \leq i \leq n$, is not considered. Let us start with the linear space $\langle \{[\tilde{F}_1], [\tilde{M}_0]\} \cup \{[E_j]\}_{1 \leq j \leq n, j \neq i} \rangle$, set

$$[D_i] = d_{i01}[F^*] + d_{i02}[M^*] + d_{i1}[E_1^*] + \cdots + d_{in}[E_n^*] \in \mathrm{Pic}_{\mathbb{Q}}(Z)$$

with arbitrary coefficients and impose the conditions:

$$[D_i] \cdot [\tilde{F}_1] = 0, [D_i] \cdot [\tilde{M}_0] = 0, [D_i] \cdot [E_j] = 0, [D_i] \cdot [\tilde{M}_1] \geq 0 \text{ and } [D_i] \cdot [E_i] \geq 0.$$

Then we obtain the system of equalities and inequalities:

$$d_{i02} + d_{i1} = 0,\ d_{i01} + \delta d_{i02} - \delta d_{i02} = 0,\ -d_{ij} + \sum_{p_s \to p_j} d_{is} = 0,$$

$$d_{i01} + \delta d_{i02} + \sum_{j=1}^{\min\{i, i_{M_1}\}} d_{ij} \geq 0 \text{ and } -d_{ii} + \sum_{p_s \to p_i} d_{is} \geq 0,$$

where i_{M_1} is the index defined as in the proof of Lemma 4.1. Solving it, we obtain $d_{ii} = -1; d_{ij} = \sum_{p_s \to p_j} d_{is}$, $1 \leq j \leq i-1$; $d_{ij} = 0$, $i+1 \leq j \leq n$; $d_{i01} = 0$; and

$d_{i02} = -d_{i1}$. This proves that $d_{ij} = -\mathrm{mult}_{p_j}(\varphi_i)$ holds and also

$$\delta \ \mathrm{mult}_{p_1}(\varphi_i) - \sum_{j=1}^{\min\{i,i_{M_1}\}} \mathrm{mult}_{p_j}(\varphi_i) \geq 0$$

by our first inequality, which shows that the classes $\{[\Theta_i]\}_{1\leq i\leq\delta}$ in the statement give generators of the dual cone $S_2^{\vee}(Z)$.

Reasoning as above for the subspace $\langle\{[\tilde{F}_1], [\tilde{M}_1]\} \cup \{[E_j]\}_{1\leq j\leq n, j\neq i}\rangle$ and with the same notation, we get the system of equalities and inequalities:

$$d_{i02} + d_{i1} = 0,\ d_{i01} + \delta d_{i02} + \sum_{j=1}^{\min\{i,i_{M_1}\}} d_{ij} = 0,\ -d_{ij} + \sum_{p_s\to p_j} d_{is} = 0,$$
$$d_{i01} + \delta d_{i02} - \delta d_{i02} \geq 0 \text{ and } -d_{ii} + \sum_{p_s\to p_i} d_{is} \geq 0.$$

Here, the equality $d_{ij} = -\mathrm{mult}_{p_j}(\varphi_i)$ is again true and the first inequality means that

$$\sum_{j=1}^{\min\{i,i_{M_1}\}} \mathrm{mult}_{p_j}(\varphi_i) - \delta \ \mathrm{mult}_{p_1}(\varphi_i) \geq 0$$

must hold. As a consequence, we have proved that the classes $\{[\Delta_i]\}_{\delta+1\leq i\leq n}$ in the statement give extremal rays of $S_2^{\vee}(Z)$.

Repeating the procedure with $\langle\{[\tilde{M}_0], [\tilde{M}_1]\} \cup \{[E_j]\}_{1\leq j\leq n, j\neq i}\rangle$, the obtained system is

$$d_{i01} + \delta d_{i02} - \delta d_{i02} = 0,\ d_{i01} + \delta d_{i02} + \sum_{j=1}^{\min\{i,i_{M_1}\}} d_{ij} = 0,\ -d_{ij} + \sum_{p_s\to p_j} d_{is} = 0,$$
$$d_{i02} + d_{i1} \geq 0 \text{ and } -d_{ii} + \sum_{p_s\to p_i} d_{is} \geq 0.$$

This proves, on the one hand, that $d_{ii} = -1$; $d_{ij} = \sum_{p_s\to p_j} d_{is}$, $1 \leq j \leq i-1$; $d_{ij} = 0$, $i+1 \leq j \leq n$; $d_{i01} = 0$; and $d_{i02} = (1/\delta) \sum_{j=1}^{\min\{i,i_{M_1}\}} -d_{ij}$. On the other hand, reasoning as above, $d_{ij} = \delta \ \mathrm{mult}_{p_j}(\varphi_i)$ and then

$$\sum_{j=1}^{\min\{i,i_{M_1}\}} \mathrm{mult}_{p_j}(\varphi_i) - \delta \ \mathrm{mult}_{p_1}(\varphi_i) \geq 0,$$

which shows that the set of classes $\{[\Gamma_i]\}_{\delta+1\leq i\leq n}$ gives generators of $S_2^{\vee}(Z)$.

It only remains to consider those subspaces $\langle\{[\tilde{F}_1], [\tilde{M}_0], [\tilde{M}_1]\} \cup \{[E_j]\}_{j\in\{1,2,\ldots,n\}\setminus\{k,i\}}\rangle$ attached to pairs of indices k, i, $1 \leq k < i \leq n$. Lemma 4.1 proves the $(n+1)$-dimensionality of these subspaces. Our computations depend on two indices i and k. So, we will write

$$[D_{ik}] = d_{ik01}[F^*] + d_{ik02}[M^*] + d_{ik1}[E_1^*] + d_{ik2}[E_2^*] + \cdots + d_{ikn}[E_n^*].$$

We must impose the following conditions:

$$[D_{ik}] \cdot [\tilde{F}_1] = 0, [D_{ik}] \cdot [\tilde{M}_0] = 0, [D_{ik}] \cdot [\tilde{M}_1] = 0, [D_{ik}] \cdot [E_j] = 0, [D_{ik}] \cdot [E_k] \geq 0$$
$$\text{and } [D_{ik}] \cdot [E_i] \geq 0,$$

which give the equivalent system

$$d_{ik02}+d_{ik1}=0,\ d_{ik01}=0,\ d_{ik01}+\delta d_{ik02}+\sum_{j=1}^{\min\{i,i_{M_1}\}} d_{ikj}=0,$$
$$-d_{ikj}+\sum_{p_s\to p_j} d_{iks}=0, -d_{ikk}+\sum_{p_s\to p_k} d_{iks}\geq 0 \text{ and } -d_{iki}+\sum_{p_s\to p_i} d_{iks}\geq 0.$$

To solve it we can assume that the inequalities are strict because, otherwise, we would obtain that $[D_{ik}]$ either vanishes or it gives the class $[\Theta_\delta]$. Indeed, if both inequalities are equalities, then $[D_{ik}]=0$. Otherwise, taking into account that the first $\delta+1$ points in $\mathcal{C}_\nu$ are free, by considering the third equality and $\delta+1\leq i_{M_1}$, it holds that one of the indices i or k equals δ. This shows that we obtain $[\Theta_\delta]$ as a solution.

The solutions of the system satisfy that $d_{iki}=-1; -d_{ikk}>-\sum_{p_s\to p_k} d_{iks}$; $d_{ikj}=\sum_{p_s\to p_j} d_{iks}$, $1\leq j\neq k\leq i-1$; $d_{ikj}=0$, $i+1\leq j\leq n$; $d_{ik01}=0; d_{ik02}=-d_{ik1}$; and it must hold that

$$-\delta d_{ik1}=-\sum_{j=1}^{\min\{i,i_{M_1}\}} d_{ikj}, \tag{4.1}$$

by the third equation. Note that, for $k+1\leq j\leq i$, $d_{ikj}=-\mathrm{mult}_{p_j}(\varphi_i)$ up to a positive factor, and also that $-d_{ikk}>-\sum_{p_s\to p_k} d_{iks}\geq 0$.

The indices i and k must satisfy that $1\leq k\leq\delta-1$ and $\delta+1\leq i\leq n$. Indeed, with respect to k and reasoning by contradiction, suppose that $k\geq\delta$. By hypothesis, $k<i$, $\delta+1\leq i_{M_1}$, and $d_{ikj}=d_{ik\delta}$ for $1\leq j\leq\delta-1$, because the first $\delta+1$ points in $\mathcal{C}_\nu$ are free, then

$$-\sum_{j=1}^{\min\{i,i_{M_1}\}} d_{ikj}=-\delta d_{ik1}-\sum_{j=\delta+1}^{\min\{i,i_{M_1}\}} d_{ikj},$$

where $-\sum_{j=\delta+1}^{\min\{i,i_{M_1}\}} d_{ikj}>0$, which does not hold by Equality (4.1). Notice that this equality is true by our imposed equalities. With respect to the index i, again reasoning by contradiction, suppose that $i\leq\delta$. As $1\leq k\leq\delta-1$, Equation (4.1) is equivalent to

$$-(\delta-k)d_{ikk}=-\sum_{j=k+1}^{\min\{i,i_{M_1}\}} d_{ikj}, \tag{4.2}$$

because $d_{ikj}=d_{ikk}$ for $1\leq j\leq k$. This implies that $-(\delta-k)d_{ikk}=-(i-k)d_{ikk+1}$, which is a contradiction since $-d_{ikk}>-d_{ikk+1}$.

Notice that (4.1) also gives us the value of d_{ikk}, which can be obtained from the following chain of equalities:

$$d_{ikk}=\delta d_{ik1}-\sum_{j=1,\ j\neq k}^{\min\{i,i_{M_1}\}} d_{ikj}=\delta d_{ikk}-(k-1)d_{ikk}-\sum_{j=k+1}^{\min\{i,i_{M_1}\}} d_{ikj}.$$

Thus, if we take $d_{ikj}=-(\delta-k)\mathrm{mult}_{p_j}(\varphi_i)$, $k+1\leq j\leq i$, one gets that

$$d_{ik1}=\ldots=d_{ikk}=\frac{-(\delta-k)\sum_{j=k+1}^{\min\{i,i_{M_1}\}}\mathrm{mult}_{p_j}(\varphi_i)}{(\delta-k)}=-\sum_{j=k+1}^{\min\{i,i_{M_1}\}}\mathrm{mult}_{p_j}(\varphi_i),$$

and the coefficient of $[M^*]$ is $d_{ik02}=-d_{ik1}$.

As a result, we have that $[D_{ik}] = [\Upsilon_{ik}]$, where

$$\Upsilon_{ik} := \left(\sum_{j=k+1}^{\min\{i,i_{M_1}\}} \mathrm{mult}_{p_j}(\varphi_i)\right) M^* - \sum_{j=1}^{k} \left(\sum_{s=k+1}^{\min\{i,i_{M_1}\}} \mathrm{mult}_{p_s}(\varphi_i)\right) E_j^*$$
$$- \sum_{j=k+1}^{i} \left((\delta - k)\mathrm{mult}_{p_j}(\varphi_i)\right) E_j^*,$$

where $\delta + 1 \leq i \leq n$ and $1 \leq k \leq \delta - 1$. This finishes the proof. □

Remark 4.3. From the above proof, it can be deduced that, when considering the surface $\mathbb{F}_1$ and a non-special valuation ν, no class $[\Upsilon_{ik}]$ appears as a generator of $S_2^\vee(Z)$.

We are interested in determining conditions under which the generators of the cone $NE(Z)$ of the surfaces Z given by non-special valuations are known. The divisors introduced in Proposition 4.2 will be important for this purpose. The next lemma states some of their properties.

Lemma 4.4. *Let Z (respectively, ν) be a rational surface (respectively, valuation) as in Proposition 4.2. Consider the set of divisors there defined. Then $\Delta_{\delta+1}^2 > 0$, $\Gamma_{\delta+1}^2 > 0$ and $\Upsilon_{\delta+1k}^2 > 0$ for all $k \in \{1, 2, \ldots, \delta - 1\}$. In addition, for any index $i \in \{\delta + 2, \delta + 3, \ldots, n\}$ such that $\Delta_i^2 \geq 0$ (respectively, $\Gamma_i^2 \geq 0$, $\Upsilon_{ik}^2 \geq 0$), the following properties are satisfied:*

(a) *If p_i is a satellite point of the configuration $\mathcal{C}_\nu$ that ν defines, it holds $\Delta_i^2 > 0$ (respectively, $\Gamma_i^2 > 0$, $\Upsilon_{ik}^2 > 0$).*
(b) *$\Delta_{i-1}^2 \geq 0$ (respectively, $\Gamma_{i-1}^2 \geq 0, \Upsilon_{i-1k}^2 \geq 0$) and, moreover, if $\Delta_{i-1}^2 = 0$ (respectively, $\Gamma_{i-1}^2 = 0, \Upsilon_{i-1k}^2 = 0$) then p_i is a satellite point and p_{i-1} is free.*

Proof. To prove our first assertion, it suffices to notice that the following three equalities hold:

$$\begin{aligned}
\Delta_{\delta+1}^2 &= 2 + \delta - (\delta + 1) = 1 > 0,\\
\Gamma_{\delta+1}^2 &= \delta(\delta + 1)^2 - \delta^2(\delta + 1) = \delta(\delta + 1)(\delta + 1 - \delta) = \delta(\delta + 1) > 0,\\
\Upsilon_{\delta+1k}^2 &= \delta(\delta + 1 - k)^2 - k(\delta + 1 - k)^2 - (\delta + 1 - k)(\delta - k)^2\\
&= (\delta - k)(\delta + 1 - k)^2 - (\delta - k)^2(\delta + 1 - k)\\
&= (\delta - k)(\delta + 1 - k)[\delta - k + 1 - (\delta - k)]\\
&= (\delta - k)(\delta + 1 - k) > 0.
\end{aligned}$$

Items (a) and (b) can be proved reasoning as in the proof of Lemma 3.4. Indeed, recalling that $g+2$ is the cardinality of the set of maximal contact values of ν, the case $g = 1$ follows as in that proof, and, when $g > 1$, with notations as in that lemma and in Proposition 4.2, it suffices to consider the following equalities and to reason again as we did in the mentioned Lemma 3.4.

$$\begin{aligned}
\Delta_n^2 &= 2(-\delta b_n + c_n)b_n + b_n^2\delta - \overline{\beta}_{g+1} = 2b_nc_n - \delta b_n^2 - \overline{\beta}_{g+1},\\
&= e_{g-1}\left[\frac{2b_nc_n - \delta b_n^2}{e_{g-1}} - \overline{\beta}_g\right],\\
\Gamma_n^2 &= c_n^2\delta - \delta^2\overline{\beta}_{g+1}, \quad \text{and}\\
\Upsilon_{nk}^2 &= (c_n - kb_n)^2\delta - k(c_n - kb_n)^2 - (\delta-k)^2\sum_{j=k+1}^{n}\mathrm{mult}_{p_j}^2(\varphi_n))\\
&= (\delta-k)[(c_n-kb_n)^2 - (\delta-k)(\overline{\beta}_{g+1} - kb_n^2)]\\
&= (\delta-k)[c_n^2 - 2kc_nb_n + \delta kb_n^2 - (\delta-k)\overline{\beta}_{g+1}]\\
&= (\delta-k)[c_n^2 - k(2c_nb_n - \delta b_n^2) - (\delta-k)\overline{\beta}_{g+1}].
\end{aligned}$$

□

Remark 4.5. Lemma 4.4 allows us to get numerical conditions which imply the non-negativity of the self-intersection of the divisors, appearing in Proposition 4.2, whose classes generate the above defined dual cone $S_2^\vee(Z)$. Let us show which are those numerical conditions.

It is clear that the divisors Θ_i, $1 \leq i \leq \delta$, satisfy $\Theta_i^2 = \delta - i \geq 0$ because each p_j, $1 \leq j \leq i$, is a free point.

Now, $2b_nc_n - \delta b_n^2 \geq [\mathrm{vol}(\nu)]^{-1}$ implies that, for all $i \in \{1, 2 \ldots, n\}$, $\Delta_i^2 \geq 0$, which is equivalent to the fact that $2b_ic_i - \delta b_i^2 \geq [\mathrm{vol}(\nu_i)]^{-1}$, where ν_i is the divisorial valuation defined as in Remark 3.8. In a similar way, it holds that if $c_n^2 \geq \delta[\mathrm{vol}(\nu)]^{-1}$, then $\Gamma_i^2 \geq 0$ or, equivalently, $c_i^2 \geq \delta[\mathrm{vol}(\nu_i)]^{-1}$ for all $i \in \{1, 2, \ldots, n\}$.

Finally, for each integer k, $1 \leq k \leq \delta - 1$, if one assumes

$$c_n^2 - k(2c_nb_n - \delta b_n^2) \geq (\delta - k)[\mathrm{vol}(\nu)]^{-1},$$

one can deduce that, for all $i \in \{1, 2, \ldots, n\}$, $\Upsilon_{ik}^2 \geq 0$ or, equivalently, $c_i^2 - k(2c_ib_i - \delta b_i^2) \geq (\delta - k)[\mathrm{vol}(\nu_i)]^{-1}$.

Before giving our main result in this section, we need to state a last lemma.

Lemma 4.6. *Let ν be a non-special divisorial valuation of a Hirzebruch surface and Z the surface that it defines. Consider the divisors Δ_i, Γ_i and Υ_{ik}, $\delta + 1 \leq i \leq n$; $1 \leq k \leq \delta - 1$, given in Proposition 4.2. Then, for each index i, $\Delta_i^2 \geq 0$ implies $\Gamma_i^2 \geq 0$ and $\Upsilon_{ik}^2 \geq 0$ for all $k \in \{1, 2, \ldots, \delta - 1\}$.*

Proof. Our proof follows from the following two properties:

Property 1: If the self-intersections of the divisors Δ_i and $\Upsilon_{i\delta-1}$ are non-negative, then the same property holds for the divisors Γ_i and Υ_{ik}, $1 \leq k \leq \delta - 1$.

Property 2: If the self-intersection of the divisor Δ_i is non-negative, so is the self-intersection of $\Upsilon_{i\delta-1}$.

For proving Property 1, our hypothesis are, by Remark 4.5,

$$[\mathrm{vol}(\nu_i)]^{-1} \leq 2c_ib_i - \delta b_i^2 \quad \text{and} \tag{4.3}$$

$$(\delta - 1)(2c_ib_i - \delta b_i^2 - [\mathrm{vol}(\nu_i)]^{-1}) \leq c_i^2 - \delta[\mathrm{vol}(\nu_i)]^{-1}. \tag{4.4}$$

The inequality in (4.4) and the following one

$$[\mathrm{vol}(\nu_i)]^{-1} \leq c_i^2 - (\delta - 1)(2c_ib_i - \delta b_i^2)$$

are equivalent. From the last inequality and the one in (4.3), we get that $c_i^2 \geq \delta[\mathrm{vol}(\nu_i)]^{-1}$ and then $\Gamma_i^2 \geq 0$. Finally, $\Upsilon_{ik}^2 \geq 0$, $1 \leq k \leq \delta - 1$, if and only if the inequality

$$k(2c_ib_i - \delta b_i^2 - \overline{\beta}^{\,i}_{g+1}) \leq c_i^2 - \delta\overline{\beta}^{\,i}_{g+1}$$

holds, fact that follows straightforwardly from the inequalities (4.3) and (4.4).

To conclude we prove Property 2. It suffices to check that the following inequalities

$$[\mathrm{vol}(\nu_i)]^{-1} \leq 2c_ib_i - \delta b_i^2 < c_i^2 - (\delta-1)(2c_ib_i - \delta b_i^2) \tag{4.5}$$

are true. In fact, the first inequality comes from our hypothesis $\Delta_i^2 \geq 0$ and the inequality given by the first and the last sides in (4.5) allows us to show $\Upsilon^2_{i\delta-1} \geq 0$. To prove the second inequality in (4.5), set $c_i = x$ and $b_i = b$ for simplicity. We are considering non-special valuations, which means that $x > \delta b$. In our new notation we want to prove that

$$2bx - \delta b^2 < x^2 - (\delta-1)(2bx - \delta b^2).$$

This inequality is equivalent to

$$0 < x^2 - (2b\delta)x + \delta^2 b^2,$$

and it holds for all $x \neq \delta b$ since the point $(\delta b, 0)$ is the vertex of the parabola given by the right-hand side of the inequality. □

Theorem 3.6 considered special valuations of Hirzebruch surfaces. There we gave equivalent conditions to the non-positivity at infinity of valuations of that type. Our next result gives the corresponding conditions for non-special valuations. In fact, it gives an easy to check numerical and local condition, and two global properties concerning the surfaces that these valuations define. Before stating our result, we introduce the concepts of non-positive, and negative, at infinity, non-special valuation.

Definition 4.7. Let ν be a non-special divisorial valuation of a Hirzebruch surface $\mathbb{F}_\delta$ and keep the above notation. The valuation ν is called to be *non-positive* (respectively, *negative*) at infinity if $\nu(h) \leq 0$ (respectively, $\nu(h) < 0$) for all $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_1))$ (respectively, $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_1))$ such that $h \notin k$).

Theorem 4.8. *Let ν be a non-special divisorial valuation of the fraction field of $R = \mathcal{O}_{\mathbb{F}_\delta,p}$ centered at R and $\mathcal{C}_\nu = \{p_i\}_{i=1}^n$ the configuration of infinitely near points defined by ν. Let Z be the surface that ν defines and consider the divisor Δ_n on Z defined in Proposition 4.2. Then, the following conditions are equivalent:*

(a) *The valuation ν is non-positive at infinity.*
(b) *The divisor Δ_n is nef.*
(c) *It holds the following inequality $2c_nb_n - \delta b_n^2 \geq [\mathrm{vol}(\nu)]^{-1}$.*
(d) *The cone of curves of Z is generated by $[\tilde{F}_1], [\tilde{M}_0], [\tilde{M}_1], [E_1], [E_2], \ldots, [E_n]$.*

Proof. Our proof uses a close reasoning to that of Theorem 3.6. Keeping the notation as in that theorem, we are going to give a sketch of the proof emphasizing only the main differences.

To prove that Item (a) can be deduced from Item (b), we can suppose that p is a general point of $\mathbb{F}_\delta$ with coordinates $(0:1;0,1)$. Consider local coordinates $\{x,y\} = \left\{\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0}\right\}$ in the affine open set U_{00} and $\{u,v\} = \left\{\frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1}\right\}$ in U_{11}. Notice that, with our notation, F_1 and M_1 are defined by the equations $X_0 = 0$ and $Y_0 = 0$, $p \in U_{11}$, and F_1 and M_1 have local equations $u = 0$ and $v = 0$, respectively.

If now $\mathcal{S}$ denotes the set of non-constant polynomials in $\mathcal{O}_{\mathbb{F}_\delta}(U_{00})$ (up to multiplication by a nonzero element of k) such that neither x nor y divide them, $f \in \mathcal{S}$ satisfies

$$f(x,y) = f(1/u, u^\delta/v) = \frac{h_f(u,v)}{u^{\deg_1(h_f)}v^{\deg_2(h_f)}}, \tag{4.6}$$

where $h_f(u,v) \in \mathcal{O}_{\mathbb{F}_\delta}(U_{11})$.

The bi-homogeneous polynomial $X_1^{\deg_1(h_f)+\delta\deg_2(h_f)}Y_1^{\deg_2(h_f)}h_f(\frac{X_0}{X_1},\frac{Y_0}{X_1^\delta Y_1})$ defines a curve C_f on $\mathbb{F}_\delta$ of degree $(\deg_1(h_f),\deg_2(h_f))$ and $f\mapsto C_f$ is a one-to-one correspondence between $\mathcal{S}$ and the set of curves on $\mathbb{F}_\delta$ containing no curve F_1,F',M_0,M_1 as a component, where F' and M_0 are defined by the equations $X_0=0$ and $Y_0=0$. Then $\Delta_n\cdot C_f=-\nu(f)$ and by Item (b), $-\nu(f)\geq 0$. The case when $f\in\mathcal{O}_{\mathbb{F}_\delta}(U_{00})$ and x or y or both are factors of f follows as in Theorem 3.6 and Item (a) is proved.

A proof of the fact that Item (a) implies Item (b), Item (b) implies Item (c) and Item (d) implies Item (b) can be done as in Theorem 3.6.

To see that Item (c) implies Item (d), it suffices to notice that, by Lemmas 4.4 and 4.6,

$$S_2^\vee(Z)\subseteq\{[D]\in\mathrm{Pic}_{\mathbb{Q}}(Z)\mid [D]^2\geq 0\text{ and }[H]\cdot[D]\geq 0\}=:A(Z),$$

where $S_2^\vee(Z)$ is the dual cone defined in Proposition 4.2 and H an ample divisor on Z. Finally, the fact

$$S_2^\vee(Z)\subseteq A(Z)\subseteq(S_2^\vee(Z))^\vee=S_2(Z)$$

and a reasoning as in Theorem 3.6 completes our proof. □

An immediate consequence of the above result is the following one.

Corollary 4.9. *Let ν be a non-positive at infinity non-special divisorial valuation of $\mathbb{F}_\delta$. Consider the divisorial valuations ν_i defined by the divisors E_i associated to the simple sequence of point blowing-ups that ν defines. Then, the valuations ν_i, $\delta+1\leq i\leq n-1$, are non-positive at infinity.*

Remark 4.10. Let Z be a surface as in Theorem 4.8 defined by a non-positive at infinity non-special valuation. Then, all the divisors Θ_i, $i=1,2,\ldots,\delta$; Δ_i,Γ_i and Υ_{ik}, $i=\delta+1,\delta+2,\ldots,n$ and $k=1,2,\ldots,\delta-1$ are effective. Indeed, under this assumption, all the divisors $\Theta_i,\Delta_i,\Gamma_i$ and Υ_{ik} can be expressed as

$$\Theta_i=(\Theta_i\cdot F^*)\tilde{M}_1+\sum_{j=1}^n(\Theta_i\cdot\Delta_j)E_j,$$

$$\Delta_i=(\Delta_i\cdot M_0^*)\tilde{F}_1+(\Delta_i\cdot F^*)\tilde{M}_1+\sum_{j=1}^n(\Delta_i\cdot\Delta_j)E_j,$$

$$\Gamma_i=(\Gamma_i\cdot F^*)\tilde{M}_1+\sum_{j=1}^n(\Gamma_i\cdot\Delta_j)E_j,$$

$$\Upsilon_{ik}=(\Upsilon_{ik}\cdot F^*)\tilde{M}_1+\sum_{j=1}^n(\Upsilon_{ik}\cdot\Delta_j)E_j,$$

which are effective divisors since $\Theta_i,\Delta_i,\Gamma_i$ and Υ_{ik} are nef divisors.

Example 4.11. Let ν be a non-special divisorial valuation of the Hirzebruch surface $\mathbb{F}_2$ whose sequence of maximal contact values is $\{15,51,262,786\}$. Set $\mathcal{C}_\nu=\{p_i\}_{i=1}^{12}$ the configuration of infinitely near points of ν. Let F_1 be the fiber of $\mathbb{F}_2$ that goes through p_1, M_0 the special section and M_1 the section that is linearly equivalent to M and passes through p_1,p_2 and p_3. Then $b_{12}=15, c_{12}=45$ and $[\mathrm{vol}(\nu)]^{-1}=786$, and so, Item (c) in Theorem 4.8 is satisfied. Therefore, the cone of curves of the surface Z defined by ν is generated by $\{[\tilde{F}_1],[\tilde{M}_0],[\tilde{M}_1]\}\cup\{E_i\}_{i=1}^{12}$ and the divisors Δ_i, $1\leq i\leq 12$, defined in Proposition 4.2 are nef.

We finish this paper with a result that gives two equivalent properties to the fact of that a non-special valuation is negative at infinite. It can be proved as we did in Theorem 3.9.

Theorem 4.12. *Keeping the same assumptions and notations as in Theorem 4.8. Then the following conditions are equivalent:*

(a) *The valuation ν is negative at infinity.*
(b) *It holds that either $2c_nb_n - b_n^2\delta > [\mathrm{vol}(\nu)]^{-1}$, or $2c_nb_n - b_n^2\delta = [\mathrm{vol}(\nu)]^{-1}$ and the Iitaka dimension of the divisor Δ_n vanishes.*
(c) *The inequality $\Delta_n \cdot \tilde{C} > 0$ is satisfied for the strict transform on Z, $\tilde{C}$, of any curve C on $\mathbb{F}_\delta$, $C \neq F_1, M_1$.*

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Instituto Universitario de Matemáticas y Aplicaciones de Castellón, y Departamento de Matemáticas, Jaume I University, Spain.
E-mail address: galindo@mat.uji.es
E-mail address: cavila@uji.es

Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia (Spain).
E-mail address: framonde@mat.upv.es