On $\sigma$-subnormality criteria in finite $\sigma$-soluble groups

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Abstract

Let $\sigma = \{\sigma_i : i \in I\}$ be a partition of the set $\mathbb{P}$ of all prime numbers. A subgroup $X$ of a finite group $G$ is called $\sigma$-subnormal in $G$ if there is a chain of subgroups

$$X = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = G$$

where for every $j = 1, \ldots, n$ the subgroup $X_{j-1}$ normal in $X_j$ or $X_j/\text{Core}_{X_j}(X_{j-1})$ is a $\sigma_i$-group for some $i \in I$.

In the special case that $\sigma$ is the partition of $\mathbb{P}$ into sets containing exactly one prime each, the $\sigma$-subnormality reduces to the familiar case of subnormality.

In this paper some $\sigma$-subnormality criteria for subgroups of $\sigma$-soluble groups, or groups in which every chief factor is a $\sigma_i$-group, for some $\sigma_i \in \sigma$, are showed.

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1 Introduction and statements of results.

All groups considered in this paper are finite.

The results of this article are based on a paper of Skiba [15]. There he generalised the concepts of solubility, nilpotency and subnormality introducing $\sigma$-solubility, $\sigma$-nilpotency, and $\sigma$-subnormality in which $\sigma$ is a partition of the set $\mathbb{P}$, the set of all primes. Hence $\mathbb{P} = \bigcup_{i \in I} \sigma_i$, with $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. 

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We note that in the special case that \( \sigma \) is the partition of \( \mathbb{P} \) containing exactly one prime each, the definitions below reduce to the familiar case of soluble groups, nilpotent groups and subnormal subgroups.

From now on let \( \sigma \) denote a partition of \( \mathbb{P} \). Given a natural number \( n \), we denote by \( \sigma(n) \) the set of all elements of \( \sigma \) including the primes dividing \( n \). Two natural numbers \( m \) and \( n \) are called \( \sigma \)-coprime if \( \sigma(m) \cap \sigma(n) = \emptyset \). We say that \( n \) is \( \sigma \)-primary if \( |\sigma(n)| = 1 \), that is, if its prime factors all belong to the same member of \( \sigma \).

A group \( G \) is called \( \sigma \)-primary if \(|G|\) is a \( \sigma \)-primary number.

**Definition 1.** A group \( G \) is said to be \( \sigma \)-soluble if every chief factor of \( G \) is \( \sigma \)-primary. \( G \) is said to be \( \sigma \)-nilpotent if it is a direct product of \( \sigma \)-primary groups.

Note that if \( \pi \) is a set of primes and \( \sigma = \{\pi, \pi'\} \), then a group \( G \) is \( \sigma \)-soluble if and only if \( G \) is \( \pi \)-separable. In this case, \( G \) is \( \sigma \)-nilpotent if and only if \( G \) is \( \pi \)-decomposable. If \( \pi = \{p_1, \ldots, p_n\} \), and \( \sigma = \{\{p_1\}, \ldots, \{p_n\}, \pi'\} \), then \( G \) is \( \sigma \)-soluble if and only if \( G \) is \( \pi \)-soluble, and \( G \) is \( \sigma \)-nilpotent if and only if \( G \) has a normal Hall \( \pi' \)-subgroup and a normal Sylow \( p_i \)-subgroup, for all \( i = 1, \ldots, n \).

Many normal and arithmetical properties of soluble groups still hold for \( \sigma \)-soluble groups (see [15]). In particular, every \( \sigma \)-soluble group has a conjugacy class of Hall \( \sigma_i \)-subgroups and a conjugacy class of Hall \( \sigma'_i \)-subgroups, for every \( \sigma_i \in \sigma \).

The role of the class \( \mathcal{N}_\sigma \) of all \( \sigma \)-nilpotent groups in \( \sigma \)-soluble groups is analogous to that of nilpotent groups in soluble groups. In particular, \( \mathcal{N}_\sigma \) is a subgroup-closed saturated Fitting formation ([15, Corollary 2.4 and Lemma 2.5]) that is closely related to the subgroup embedding property of \( \sigma \)-subnormality.

**Definition 2.** Given a partition \( \sigma \) of the set of prime numbers, a subgroup \( X \) of a group \( G \) is called \( \sigma \)-subnormal in \( G \) if there exists a chain of subgroups

\[
X = X_0 \leq X_1 \leq \cdots \leq X_n = G,
\]

with \( X_{i-1} \) normal in \( X_i \) or \( X_i/Core_{X_i}(X_{i-1}) \) \( \sigma \)-primary for every \( 1 \leq i \leq n \).

To know that a non-\( \sigma \)-nilpotent group possesses a non-trivial proper \( \sigma \)-subnormal subgroup is equivalent to know that the group is not simple.
Therefore criteria for the $\sigma$-subnormality of a subgroup may have some importance in the study of the normal structure of a group. The close relationship between $\sigma$-subnormal subgroups and direct decompositions of a group strongly supports that claim. The significance of the $\sigma$-subnormal subgroups in $\sigma$-soluble groups is apparent since they are precisely the $N_{\sigma}$-subnormal subgroups, and so they are a sublattice of the subgroup lattice of $G$. They are also important to analyse the structural impact of some permutability properties (see [15]).

In this paper, which is a natural continuation of [3], extensions of some well-known subnormality criteria are presented. For instance, according to a result of Wielandt (see [10, Theorem 7.3.3]), a subgroup $X$ of a group $G$ is subnormal in $G$ if and only if $X$ is subnormal in $\langle X, X^g \rangle$ for all $g \in G$.

In [11, Question 19.84] (see also [18]), Skiba asked whether it is enough to know that $X$ is $\sigma$-subnormal in $\langle X, X^g \rangle$ for all $g \in G$ to deduce that $X$ is $\sigma$-subnormal in $G$. It is certainly true in the soluble universe by virtue of [2, Proposition 6.1.10 and Theorem 6.2.17] (see [3, Lemma 2]). Our first main result shows that the answer is also affirmative for $\sigma$-soluble groups.

**Theorem A.** Suppose that $G$ is a $\sigma$-soluble group and $X$ is a subgroup of $G$ that is $\sigma$-subnormal in $\langle X, X^g \rangle$ for all $g \in G$. Then $X$ is $\sigma$-subnormal in $G$.

Theorem A is not true for arbitrary groups. Therefore Question 19.84 in [11] is answered.

**Example 1.** Let $\pi = \{2, 3\}$ and $\sigma = \{\pi, \pi'\}$. The simple group $G = \text{PSL}_2(7)$ of order $168 = 2^3 \cdot 3 \cdot 7$ has a unique conjugacy class of elements of order 2. Let $x$ be an element of this class. Given $g \in G$, the group $\langle x, x^g \rangle$ is isomorphic to $C_2$, to $C_2 \times C_2$, to $S_3$ or to $D_8$. Therefore $X = \langle x \rangle$ is $\sigma$-subnormal in $\langle X, X^g \rangle$ for all $g \in G$ but $X$ is not $\sigma$-subnormal in $G$.

Another important subnormality criterion asserts that if $G = AB$ is a group which is the product of the subgroups $A$ and $B$ and $X$ is a subgroup of $G$ contained in $A \cap B$ that is subnormal in $A$ and $B$, then $X$ is subnormal in $G$. This result was proved by Maier in [12] for soluble groups and then for arbitrary groups by Wielandt [19]. Applying Theorem A, we show that Maier-Wielandt’s result also holds for $\sigma$-subnormal subgroups not only in the soluble universe, but also in the $\sigma$-soluble one.

**Theorem B.** Let the $\sigma$-soluble group $G$ be the product of two subgroups $A$ and $B$. If $X$ is a subgroup of $A \cap B$ which is $\sigma$-subnormal in both $A$ and $B$, then $X$ is $\sigma$-subnormal in $G$. 

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Theorem B does not hold in general as the following example shows (see [8]).

Example 2. Let $\pi = \{2, 5\}$ and $\sigma = \{\pi, \pi'\}$. The alternating group of degree five $A_5$ is the product of the subgroups $A$ and $B$, where $A$ is the alternating group of degree 4 and $B$ is a dihedral group of order 10. Then $A \cap B$ is $\sigma$-subnormal in both $A$ and $B$, but $A \cap B$ is not $\sigma$-subnormal in $A_5$.

On the other hand, Wielandt [19] conjectured that if $X$ is a subgroup of $G$ such that $X$ is subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$, then $X$ is subnormal in $G$.

Wielandt’s conjecture was proved to be true in the soluble universe by Maier and Sidki [13] for subgroups $X$ of prime power order and then for every subgroup $X$ of a soluble group by Casolo in [4].

In [3, Theorem A], we show that the following $\sigma$-version of the aforementioned result holds.

**Theorem 1.** Assume that $G$ is a soluble group factorised as a product of the subgroups $A$ and $B$. Let $X$ be a subgroup of $G$ such that $X$ is $\sigma$-subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$. Then $X$ is $\sigma$-subnormal in $G$.

A natural question to ask is now whether Theorem 1 holds for $\sigma$-soluble groups. Unfortunately we have been unable to answer this question; however, our third main result could be regarded as a significant step to solve it.

**Theorem C.** Assume that $G$ is a $\sigma$-soluble group factorised as a product of the subgroups $A$ and $B$. Let $X$ be a subgroup of $G$ such that $X$ is $\sigma$-subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$. Then $X$ is $\sigma$-subnormal in $G$ if one of the following conditions is true:

(i) $|G : A|$ and $|G : B|$ are $\sigma$-primary.

(ii) $|G : A|$ is $\sigma$-primary and $|G : A|$ and $|G : B|$ are $\sigma$-coprime.

The proof of Theorem C strongly depends on the following extension of [6, Theorem 3].

**Theorem D.** Let $G$ be a $\sigma$-soluble group, and $A$ and $X$ two subgroups of $G$ such that $X$ is $\sigma$-subnormal in $\langle X, X^a \rangle$ for all $a \in A$. If $|G : A|$ is $\sigma$-primary, then $X$ is $\sigma$-subnormal in $\langle X, A \rangle$.

We shall adhere to the notation and terminology of [2] and [5].
2 Preliminaries

Our first lemma collects some basic properties of \( \sigma \)-subnormal subgroups which are very useful in induction arguments.

**Lemma 1** ([15]). Let \( H, K \) and \( N \) be subgroups of a group \( G \). Suppose that \( H \) is \( \sigma \)-subnormal in \( G \) and \( N \) is normal in \( G \). Then the following statements hold:

1. \( H \cap K \) is \( \sigma \)-subnormal in \( K \).
2. If \( K \) is a \( \sigma \)-subnormal subgroup of \( H \), then \( K \) is \( \sigma \)-subnormal in \( G \).
3. If \( K \) is \( \sigma \)-subnormal in \( G \), then \( H \cap K \) is \( \sigma \)-subnormal in \( G \).
4. \( HN/N \) is \( \sigma \)-subnormal in \( G/N \).
5. If \( N \subseteq K \) and \( K/N \) is \( \sigma \)-subnormal in \( G/N \), then \( K \) is \( \sigma \)-subnormal in \( G \).
6. If \( L \leq K \) and \( K \) is \( \sigma \)-nilpotent, then \( L \) is \( \sigma \)-subnormal in \( K \).
7. If \( |G : H| \) is a \( \sigma_i \)-number, then \( O^{\sigma_i}(H) = O^{\sigma_i}(G) \).
8. If \( N \) is a \( \sigma_i \)-subgroup of \( G \), then \( N \leq N_G(O^{\sigma_i}(H)) \).

A standard induction argument using Lemma 1 allows us to prove the following result.

**Lemma 2.** Let \( X \) be a subgroup of a \( \sigma \)-soluble group \( G \). Then \( X \) is \( \sigma \)-subnormal in \( G \) if and only if \( X \) is \( \mathcal{N}_\sigma \)-subnormal in \( G \), that is, there exists a chain of subgroups

\[
X = X_0 \leq X_1 \leq \cdots \leq X_n = G,
\]

such that \( X_{i-1} \) is a maximal subgroup of \( X_i \) and \( X_i/\text{Core}_{X_i}(X_{i-1}) \in \mathcal{N}_\sigma \), for \( 1 \leq i \leq n \).

The fact that \( \sigma \)-subnormal subgroups are \( \mathcal{N}_\sigma \)-subnormal in the \( \sigma \)-soluble universe allows us to prove some relevant properties of these subgroups which are crucial in the proof of our main results.

**Lemma 3.** Let \( X \) be a subgroup of a group \( G \).
1. ([2, Lemma 6.1.9 and Proposition 6.1.10]) If $X$ is $\sigma$-subnormal in $G$, then the $N_\sigma$-residual $X^{N_\sigma}$ of $X$ is subnormal in $G$.

2. ([2, Lemma 6.1.9]) If $X$ is subnormal in $G$, then $X$ is $\sigma$-subnormal in $G$.

3. ([2, Lemmas 6.3.11 and 6.3.12 and Example 6.3.13]) $N_\sigma$ is a lattice formation, that is, the set of all $\sigma$-subnormal subgroups of a $\sigma$-soluble group $G$ forms a sublattice of the subgroup lattice of $G$.

4. ([2, Theorem 6.3.3]) If $X$ is a $\sigma$-subnormal $\sigma$-nilpotent subgroup of a $\sigma$-soluble group $G$, then $X$ is contained in $F_\sigma(G)$, the $N_\sigma$-radical of $G$. In particular, if $X$ is $\sigma_i$-group, then $X \leq O_{\sigma_i}(G)$.

Note that by Lemmas 1 (2) and 3 (2), subnormal subgroups of $\sigma$-subnormal subgroups of a group $G$ are $\sigma$-subnormal in $G$. This fact will be applied in the sequel without further reference.

Our third lemma shows that the residual associated with the class of all $\sigma_i$-groups (also called $\sigma_i$-residual) respects the $\sigma$-subnormal generation of $\sigma$-soluble groups.

**Lemma 4.** Let $\sigma_i \in \sigma$. If $A$ and $B$ are $\sigma$-subnormal subgroups of a $\sigma$-soluble group $G = \langle A, B \rangle$, then $O^{\sigma_i}(G) = \langle O^{\sigma_i}(A), O^{\sigma_i}(B) \rangle$.

**Proof.** Assume the result is false and let $G$ be a counterexample of least order. Denote $H = \langle O^{\sigma_i}(A), O^{\sigma_i}(B) \rangle$ and $X = O^{\sigma_i}(G)$. Clearly $1 \neq X$. Let $N$ be a minimal normal subgroup of $G$ such that $N \leq X$. Since $G$ is $\sigma$-soluble, it follows that $N$ is $\sigma_j$-group for some $\sigma_j \in \sigma$. The minimality of $G$ yields $X = HN$ and $Core_G(H) = 1$.

On the other hand, by Lemma 3 (5), we have that $G^{N_\sigma} = \langle A^{N_\sigma}, B^{N_\sigma} \rangle \leq \langle O^{\sigma_i}(A), O^{\sigma_i}(B) \rangle = H$. Since $G^{N_\sigma}$ is normal in $G$ and $Core_G(H) = 1$, it follows that $G$ is $\sigma$-nilpotent.

Then $G = X \times Y$ with $Y = O_{\sigma_i}(G)$. If $Y \neq 1$, then by the minimal choice of $G$, we have that $G = X \times Y = H \times Y$, and therefore $X = H$. Thus $Y = 1$ and so $G = O^{\sigma_i}(G)$, $A = O^{\sigma_i}(A)$ and $B = O^{\sigma_i}(B)$. This contradiction proves the lemma.

\[\square\]
Lemma 5. Let $H^*$ denote either the $N_2$-residual or the $\sigma_i$-residual of a subgroup $H$ of a $\sigma$-soluble group $G$, for $\sigma_i \in \sigma$. Let $A$ be a subgroup of $G$. If $H$ is a $\sigma$-subnormal subgroup of $\langle H, H^a \rangle$ for all $a \in A$, then $H$ normalises $(H^*)^A$.

Proof. Let $a \in A$. Since $H$ is a $\sigma$-subnormal subgroup of $\langle H, H^{a^{-1}} \rangle$, it follows that $H^a$ is $\sigma$-subnormal in $\langle H^a, H \rangle = \langle H, H^a \rangle$. By Lemmas 3 (5) and 4, we have $\langle H, H^a \rangle^* = \langle H^*, (H^a)^* \rangle = \langle H^*, (H^*)^a \rangle$, thus

$$[H, (H^*)^a] \leq [H, \langle H, H^a \rangle^*] \leq \langle H, H^a \rangle^* \leq (H^*)^A.$$ 

Lemma 6. Let $G$ be a $\sigma$-soluble group, $X$ a $\sigma_i$-subgroup of $G$ and $H$ a Hall $\sigma_i$-subgroup of $G$. If $X$ is $\sigma$-subnormal in $\langle X, X^h \rangle$ for all $h \in H$, then $X \leq H$.

Proof. Suppose that the result is false. Let $G$ be a counterexample of the smallest possible order. Clearly the hypotheses of the lemma hold in $G/O_{\sigma_i}(G)$. Therefore, if $O_{\sigma_i}(G) \neq 1$, we have that $XO_{\sigma_i}(G)/O_{\sigma_i}(G) \leq H/O_{\sigma_i}(G)$ by minimality of $G$. Hence $X \leq H$, contrary to supposition. Thus $O_{\sigma_i}(G) = 1$.

Let $N$ be a minimal normal subgroup of $G$. Then $N$ is a $\sigma_j$-group for some $j \neq i$. Since $X \leq HN$ by the minimal choice of $G$, there exists $n \in N$ with $X^n \leq H$. Let $x \in X$ and $h = x^{-n} \in H$. Then $[x, h] = [x, n][x^{-1}, n] \in N$ and $[x, h] = x^{-1}x^h \in \langle x, x^h \rangle$. Hence $[x, h] \in N \cap \langle x, x^h \rangle$. Then $X$ is $\sigma$-subnormal in $\langle X, X^h \rangle$ by hypothesis. Since $X$ is a $\sigma_j$-subgroup, we have that $X \leq O_{\sigma_i}(\langle X, X^h \rangle)$ by Lemma 3 (4). Therefore, $\langle X, X^h \rangle = O_{\sigma_i}(\langle X, X^h \rangle)X^h$ is a $\sigma_i$-subgroup of $HN$. Thus $[x, h] \in N \cap \langle X, X^h \rangle = 1$ and $[x, h] = 1$. In particular, $[x, n] = [x^{-1}, n]$ is a $\sigma_i$-element. Since $N$ is a $(\sigma_i)^*$-group and $[x, n] \in N$, it follows that $[x, n] = 1$ and $X^n = X \leq H$.

Lemma 7. Let $H$ be a subgroup of a $\sigma$-soluble group $G$ such that $O^{\sigma_i}(H) = H$ for some $\sigma_i \in \sigma$. Assume $K$ is a normal $\sigma_i$-subgroup of $G$ and $k \in K$ such that $H$ is a $\sigma$-subnormal subgroup of $\langle H, H^k \rangle$. Then $k$ normalises $H$.

Proof. Denote $L = \langle H, H^k \rangle$. Let $Z$ denote the normal closure of $H$ in $L$. By Lemma 4, $O^{\sigma_i}(Z) = Z$. Since $O^{\sigma_i}(L/Z) = L/Z$, it follows that $L = O^{\sigma_i}(L)Z$. By [2, Proposition 6.5.5], it follows that $O^{\sigma_i}(L) = O^{\sigma_i}(L)O^{\sigma_i}(Z) = O^{\sigma_i}(L)Z = L$.

On the other hand, $L = L \cap HK = H(L \cap K)$. By Lemma 4, $L = O^{\sigma_i}(L) = O^{\sigma_i}(H)O^{\sigma_i}(L \cap K) = H$. Thus $L = H$ and $H^k = H$. 

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3 Proofs of the main theorems

Proof of Theorem A. Suppose the result is not true and let $G$ be a counterexample with $|G|+|X|$ minimal. Then $G^{N_G} \neq 1$. Let $N$ be a minimal normal subgroup of $G$ contained in $G^{N_G}$. Then $N$ is a $\sigma_i$-group for some $\sigma_i \in \sigma$. Note that $XN/N$ is $\sigma$-subnormal in $G/N$ by the minimality of the pair $(G, X)$. If $XN$ were a proper subgroup of $G$, then $X$ would be $\sigma$-subnormal in $XN$.

By Lemma 1, $X$ would be $\sigma$-subnormal in $G$, contrary to our assumption. Hence $G = XN$. Assume that $X$ is a $\sigma_i$-group. Then $G$ is a $\sigma_i$-group, and $X$ is $\sigma$-subnormal in $G$. This contradiction implies that $X$ is not a $\sigma_i$-group, and so $O^{\sigma_i}(X) \neq 1$.

Assume that $O^{\sigma_i}(X) < X$. By minimality of $(G, X)$, it follows that $O^{\sigma_i}(X)$ is $\sigma$-subnormal in $G$. By Lemma 1 (8), $N$ normalises $O^{\sigma_i}(O^{\sigma_i}(X)) = O^{\sigma_i}(X)$. Hence $O^{\sigma_i}(X)$ is a normal subgroup of $G$. The minimal choice of $G$ implies that $X/O^{\sigma_i}(X)$ is $\sigma$-subnormal in $G/O^{\sigma_i}(X)$ and then $X$ is $\sigma$-subnormal in $G$ by Lemma 1 (5). This is not possible. Thus $X = O^{\sigma_i}(X)$.

If $n \in N$ then $X$ is $\sigma$-subnormal in $U_n = \langle X, X^n \rangle = (U_n \cap N)X$ by hypothesis. By Lemma 1 (7), we have that

$$O^{\sigma_i}(U_n) = O^{\sigma_i}((U_n \cap N)X) = O^{\sigma_i}(X) = X.$$

In particular, $X$ is normal in $U_n$. Consequently, $X$ is normal in $V = \langle X^n : n \in N \rangle$. Since $V$ is normal in $G$, we have $X$ is subnormal in $G$, and we have reached the desired contradiction. \hfill \Box

Proof of Theorem B. Assume the result is false and let $G$ be a counterexample such that $|G : A|+|X|$ is minimal. Suppose that $M$ is a maximal subgroup of $G$ containing $A$. Then $M = A(M \cap B)$ and $X$ is $\sigma$-subnormal in both $A$ and $M \cap B$ by Lemma 1 (1). By minimality of $G$, $X$ is $\sigma$-subnormal in $M$. On the other hand, $G = MB$. If $|G : M| < |G : A|$, we have $X$ is $\sigma$-subnormal in $G$, which is a contradiction. Therefore $A = M$ is a maximal subgroup of $G$.

Let $K = \text{Core}_G(A)$. If $K \neq 1$, then $XK/K$ is $\sigma$-subnormal in $G/K$ by the minimal choice of $G$. By Lemma 1 (5), $XK$ is $\sigma$-subnormal in $G$. Moreover $X \leq XK \leq A$. Thus $X$ is $\sigma$-subnormal in $XK$ by Lemma 1 (1). Thus $X$ is $\sigma$-subnormal in $G$. This contradiction yields $K = 1$ and $G$ is a primitive group. By Lemma 3 (1), $X^{N_G}$ is a subnormal subgroup of $A$ and $B$. Applying the result of Maier-Wielandt, we have that $X^{N_G}$ is a subnormal subgroup of $G$. By [10, Lemma 7.3.16], $X^{N_G} \leq \text{Core}_G(A) = 1$. Hence $X$ is $\sigma$-nilpotent. By Lemma 1 (6), every subgroup of $X$ is $\sigma$-subnormal in $X$. Therefore every
proper subgroup of $X$ is $\sigma$-subnormal in $A$ and $B$ by Lemma 1 (2). The minimal choice of $X$ implies that every proper subgroup of $X$ is $\sigma$-subnormal in $G$. By Lemma 3 (3), $X$ is cyclic of prime power order. Assume $X$ is a $\sigma_i$-group. Since $X$ is $\sigma$-subnormal in $A$, by Lemma 3 (4), $X$ is contained in $O_{\sigma_i}(A)$. Then $X^A$, the normal closure of $X$ in $A$, is a $\sigma_i$-group. Analogously, $X^B$ is a $\sigma_i$-group. According to [1, Lemma 1.3.2], there exist Hall $\sigma_i$-subgroups $A_{\sigma_i}$ of $A$ and $B_{\sigma_i}$ of $B$ such that $A_{\sigma_i}B_{\sigma_i}$ is a Hall $\sigma_i$-subgroup of $G$. Then $\langle X^A, X^B \rangle$ is a $\sigma_i$-group because it is contained in $A_{\sigma_i}B_{\sigma_i}$. Let $g = ab \in G$ with $a \in A$ and $b \in B$. Then

$$\langle X, X^g \rangle = \langle X^{b^{-1}}, X^a \rangle^b \leq \langle X^B, X^A \rangle^b.$$  

Consequently $\langle X, X^g \rangle$ is a $\sigma_i$-group and then $X$ is $\sigma$-subnormal in $\langle X, X^g \rangle$ for every $g \in G$ by Lemma 1 (6). Applying Theorem A, $X$ is $\sigma$-subnormal in $G$, a contradiction. 

Proof of Theorem D. Suppose that the result is false. We choose a counterexample $G$ with $|G| + |X|$ minimal and proceed to derive a contradiction. The minimal choice of $G$ and Theorem A show that $G = \langle X, A \rangle$ and $X$ is not contained in $A$. Suppose that $|G : A|$ is a $\sigma_i$-number for some $\sigma_i \in \sigma$. Then $A$ contains a Hall $\sigma'_i$-subgroup of $G$.

If $C = \text{Core}_G(A) \neq 1$, then $XC$ is a $\sigma$-subnormal subgroup of $G$ by minimality of $G$. Moreover, by Theorem A, $X$ is $\sigma$-subnormal in $XC$. Thus $X$ is $\sigma$-subnormal in $G$ by Lemma 1 (2). This contradiction shows that $\text{Core}_G(A) = 1$.

Let $N$ be a minimal normal subgroup of $G$. Then $N$ is a $\sigma_j$-group for some $\sigma_j \in \sigma$. If $i \neq j$, then $N$ is contained in every Hall $\sigma'_i$-subgroup of $G$. In particular, $N$ is contained in $A$, a contradiction. Therefore $N$ is a $\sigma_i$-group, $O_{\sigma_i}(G) \neq 1$, and $O_{\sigma_i}(G) = 1$.

Suppose that $X$ is not $\sigma$-nilpotent. Then $1 \neq X^{N_{\sigma}}$ is a proper subgroup of $X$ which is $\sigma$-subnormal in $\langle X, X^a \rangle$ for all $a \in A$. The choice of the pair $(G, X)$ yields that $X^{N_{\sigma}}$ is $\sigma$-subnormal in $\langle X^{N_{\sigma}}, A \rangle$. Hence $X^{N_{\sigma}}$ is $\sigma$-subnormal in $(X^{N_{\sigma}})^A$. By Lemma 5, $X$ normalises $(X^{N_{\sigma}})^A$. Therefore $(X^{N_{\sigma}})^A$ is a normal subgroup of $G$ and $X^{N_{\sigma}}$ is a $\sigma$-subnormal subgroup of $G$. Since $X$ is not a $\sigma_i$-group, it follows that $1 \neq O^\sigma_i(X)$. Moreover, since $1 \neq X^{N_{\sigma}}$ is a $\sigma$-soluble group, it follows that $F_\sigma(X^{N_{\sigma}}) \neq 1$. Thus $F_\sigma(X^{N_{\sigma}}) \neq 1$ is a $\sigma$-nilpotent $\sigma$-subnormal subgroup of $G$. By Lemma 3 (4), $F_\sigma(X^{N_{\sigma}}) \leq F_\sigma(G) = O_{\sigma_i}(G)$ and then $1 \neq O_{\sigma_i}(X^{N_{\sigma}}) \leq O_{\sigma_i}(G)$. Hence $Z = X \cap O_{\sigma_i}(G) \neq 1$ and $Z^A$ is a $\sigma$-subnormal $\sigma_i$-subgroup of $G$. Let $a \in A$. 


Therefore $\sigma$ proper

Then $X$ is $\sigma$-subnormal in $\langle X, Z^a \rangle$ and so $O_{\sigma_i}(\langle X, Z^a \rangle)$ normalises $O^{\sigma_i}(X)$ by Lemma 1 (8). Since $Z^a \leq O_{\sigma_i}(\langle X, Z^a \rangle)$, it follows that $Z^a$ normalises $O^{\sigma_i}(X)$. Therefore $Z^A$ normalises $O^{\sigma_i}(X)$.

Applying Lemma 5, it follows that $X$ normalises $(O^{\sigma_i}(X))^A$. Hence $(O^{\sigma_i}(X))^A$ is a normal subgroup of $G$. Assume that $O^{\sigma_i}(X)$ is a proper subgroup of $X$. By minimality of the pair $(G, X)$, we have that $O^{\sigma_i}(X)$ is a $\sigma$-subnormal subgroup of $(O^{\sigma_i}(X), A)$. Therefore $O^{\sigma_i}(X)$ is a $\sigma$-subnormal subgroup of $(O^{\sigma_i}(X))^A$, and so $O^{\sigma_i}(X)$ is $\sigma$-subnormal in $G$. By Lemma 1 (8), $O_{\sigma_i}(G)$ normalises $O^{\sigma_i}(O^{\sigma_i}(X)) = O^{\sigma_i}(X)$ and hence $XO_{\sigma_i}(G)$ normalises $O^{\sigma_i}(X)$. Then $X/O^{\sigma_i}(X)$ is $\sigma$-subnormal in $XO_{\sigma_i}(G)/O^{\sigma_i}(X)$. Thus $X$ is $\sigma$-subnormal in $XO_{\sigma_i}(G)$ which is $\sigma$-subnormal in $G$ by minimality of $G$ and Lemma 1 (5). Lemma 1 (2) yields that $X$ is $\sigma$-subnormal in $G$, contrary to assumption. Hence $O^{\sigma_i}(X) = X$ and so $Z^A$ normalises $X$. In addition, $[Z^A, X] \leq [N_G(X) \cap O_{\sigma_i}(G), X] \leq X \cap O_{\sigma_i}(G) = Z \leq Z^A$. Hence $Z^A$ is normalised by $X$ and so it is a normal subgroup of $G$. Again the minimality of $G$ and Lemma 1 (5) imply that $XZ^A$ is $\sigma$-subnormal in $G$. Since $X$ is normal in $XZ^A$, we have that $X$ is $\sigma$-subnormal in $G$. This contradiction shows that $X$ is $\sigma$-nilpotent.

Suppose that $O^{\sigma_i}(X) \neq 1$. Since $X$ is $\sigma$-nilpotent, it follows that either $X$ is a $\sigma'_i$-group or $O^{\sigma_i}(X)$ is a proper subgroup of $X$. Assume that $X$ is a $\sigma'_i$-group. Then, by Lemma 6, $X$ is contained in $A$. Hence $G = A$ and $X$ is $\sigma$-subnormal in $G$ by Theorem A, which is not possible. Suppose that $O^{\sigma_i}(X)$ is a proper subgroup of $X$. By minimality of $(G, X)$, $O^{\sigma_i}(X)$ is $\sigma$-subnormal in $(O^{\sigma_i}(X), A)$, and, by Lemma 5, $X$ normalises $(O^{\sigma_i}(X))^A$. Therefore $O^{\sigma_i}(X)$ is a $\sigma$-subnormal subgroup of $O^{\sigma_i}(X)^A$ which is a normal subgroup of $G$. Consequently $O^{\sigma_i}(X)$ is a $\sigma$-subnormal $\sigma$-nilpotent subgroup of $G$. By Lemma 3 (4), $O^{\sigma_i}(X)$ is contained in $F_\sigma(G) = O_{\sigma_i}(G)$. Hence $X$ is a $\sigma_i$-group, contrary to supposition.

Consequently, $O^{\sigma_i}(X) = 1$ and $X$ is a $\sigma_i$-group. Since every minimal normal subgroup $N$ of $G$ is a $\sigma_i$-group, and $XN$ is $\sigma$-subnormal in $G$, it follows that $X$ is $\sigma$-subnormal in $G$. This final contradiction proves the theorem.

\[ \square \]

**Proof of Theorem C.** Suppose that the theorem is false and let $G$ be a counterexample for which $|G| + |X| + |G : A| + |G : B|$ is minimal. Note that every proper $\sigma$-subnormal subgroup $Z$ of $X$ satisifies the hypotheses of the theorem. Therefore $Z$ is a $\sigma$-subnormal subgroup of $G$ by the choice of $(G, X)$.
We proceed in a number of steps.

Step 1. If \( X \) is not contained in \( A \), then \( G = \langle A, X \rangle \) and \( |G : A| \) is not \( \sigma \)-primary.

Let \( A_0 = \langle A, X \rangle \). We have that \( A_0 = A_0 \cap AB = A(A_0 \cap B) \) and \( G = A_0B \). If \( A_0 \neq G \), then \( A_0 \) is not a counterexample to the theorem. Then \( X \) is \( \sigma \)-subnormal in \( A_0 \), and the 4-tuple \((G, X, A_0, B)\) satisfies the hypotheses of the theorem. The minimal choice of \((G, X, A, B)\) implies that \( X \) is \( \sigma \)-subnormal in \( G \). Consequently, \( G = \langle A, X \rangle \). If \( |G : A| \) were \( \sigma \)-primary, then we would have \( X \) is \( \sigma \)-subnormal in \( G \) by Theorem D. This is not the case. Thus \( |G : A| \) is not \( \sigma \)-primary.

Step 2. Assume that \( X \) is contained in \( A \) and \( |G : A| \) is \( \sigma \)-primary. If \( X \) is not contained in \( B \), then \( |G : A| \) and \( |G : B| \) are not \( \sigma \)-coprime.

Assume that \( X \) is not contained in \( B \) and \( |G : A| \) and \( |G : B| \) are \( \sigma \)-coprime and derive a contradiction. Let \( B_0 = \langle X, B \rangle = B(B_0 \cap A) \). Then \( B \) is a proper subgroup of \( B_0 \) and \( G = AB_0 \). Then \((B_0, X, B_0 \cap A, B)\) satisfies the hypotheses of the theorem. Suppose that \( B_0 \) is a proper subgroup of \( G \). Then the theorem holds in \( B_0 \), and hence \( X \) is \( \sigma \)-subnormal in \( B_0 \). Applying Theorem A and Theorem B, we conclude that \( X \) is \( \sigma \)-subnormal in \( G \). This contradicts the choice of \( G \), however, and we conclude that \( G = \langle X, B \rangle \).

By hypothesis, \( |G : A| \) is a \( \sigma_i \)-number, for some \( \sigma_i \in \sigma \). Since \( |G : A| \) and \( |G : B| \) are \( \sigma \)-coprime, it follows that \( |G : B| \) is a \( \sigma_i \)-number. Therefore \( B \) contains a Hall \( \sigma_i \)-subgroup of \( G \).

Let \( N \) be a minimal normal subgroup of \( G \). Then \( N \) is \( \sigma \)-primary. Assume that \( N \) is a \( \sigma_j \)-group, where \( j \neq i \). Since \( |G : A| \) is \( \sigma_i \)-number, then \( N \leq A \). By the choice of \( G \), \( XN \) is a \( \sigma \)-subnormal subgroup of \( G \). Moreover, \( XN \leq A \). Therefore \( X \) is \( \sigma \)-subnormal in \( XN \) and then in \( G \), a contradiction. Consequently, every minimal normal subgroup of \( G \) is a \( \sigma_i \)-group and \( F_{\sigma}(G) = O_{\sigma_i}(G) \). Moreover, \( R = O_{\sigma_i}(G) \) is contained in \( B \).

Suppose that \( X \) is not \( \sigma \)-nilpotent. Then \( O^{\sigma_i}(X) \neq 1 \). Suppose that \( O^{\sigma_i}(X) \) is a proper subgroup of \( X \). Then it is \( \sigma \)-subnormal in \( G \). By Lemma 1 (8), \( O_{\sigma_i}(G) \) normalises \( O^{\sigma_i}(O^{\sigma_i}(X)) = O^{\sigma_i}(X) \) and hence \( XO_{\sigma_i}(G) \) normalises \( O^{\sigma_i}(X) \). Then \( X/O^{\sigma_i}(X) \) is \( \sigma \)-subnormal in \( XO_{\sigma_i}(G)/O^{\sigma_i}(X) \). Thus \( X \) is \( \sigma \)-subnormal in \( XO_{\sigma_i}(G) \) which is \( \sigma \)-subnormal in \( G \) by minimality of \( G \) and Lemma 1 (5). Lemma 1 (2) yields that \( X \) is \( \sigma \)-subnormal in \( G \), contrary to supposition. Thus \( O^{\sigma_i}(X) = X \).

On the other hand, since \( X \) is not \( \sigma \)-nilpotent, \( 1 \neq X^{N_\sigma} \) is \( \sigma \)-subnormal in \( G \). Therefore \( 1 \neq F_\sigma(X^{N_\sigma}) \) is \( \sigma \)-nilpotent \( \sigma \)-subnormal subgroup of \( G \) contained in \( F_\sigma(G) = O_{\sigma_i}(G) \) by Lemma 3 (4). In particular, \( O_{\sigma_i}(X) \neq \)
1. Applying Lemma 5, we conclude that \( X \) normalises \( (O_{\sigma_i}(X))^B \). Hence \( (O_{\sigma_i}(X))^B \) is a normal subgroup of \( G \). Write \( Z = X \cap O_{\sigma_i}(G) \). Then \( 1 \neq Z \) is a \( \sigma \)-subnormal \( \sigma_i \)-subgroup of \( G \). Let \( b \in B \). Then \( X \) is \( \sigma \)-subnormal in \( \langle X, Z^b \rangle \) and so \( O_{\sigma_i}(\langle X, Z^b \rangle) \) normalises \( O_{\sigma_i}(X) = X \) by Lemma 1 (8). Since \( Z^b \leq O_{\sigma_i}(\langle X, Z^b \rangle) \), it follows that \( Z^b \) normalises \( X \). Therefore \( Z^B \) normalises \( X \). Then \( [Z^B, X] \leq X \cap O_{\sigma_i}(G) = Z \leq Z^B \) and \( Z^B \) is normal in \( G \). By the choice of \( G \), it follows that \( XZ^B \) is a \( \sigma \)-subnormal subgroup of \( G \) and then \( X \) is \( \sigma \)-subnormal in \( G \), a contradiction.

Thus \( X \) is \( \sigma \)-nilpotent. By assumption every proper subgroup of \( X \) is \( \sigma \)-subnormal in \( G \). Applying Lemma 3 (3), \( X \) is a cyclic \( p \)-group for some prime \( p \in \sigma_j \), for some \( \sigma_j \in \sigma \). Assume that \( i = j \). Then \( XN \) is a \( \sigma \)-subnormal \( \sigma_i \)-subgroup of \( G \). Consequently, \( X \) is \( \sigma \)-subnormal in \( G \), which contradicts our assumption that \( G \) is a counterexample. Thus \( i \neq j \) and \( O_{\sigma_i}(X) = X \).

By Lemma 7, \( R = O_{\sigma_i}(G) \) normalises \( X \), and so \( X \) is normal in \( XR \). Since \( XR \) is \( \sigma \)-subnormal in \( G \) by minimality of \( G \) and Lemma 1 (5), we conclude that \( X \) is \( \sigma \)-subnormal in \( G \), which is not the case.

**Step 3. We have a contradiction**

Assume that either \( |G : A| \) and \( |G : B| \) are \( \sigma \)-primary or \( |G : A| \) is \( \sigma \)-primary and \( |G : A| \) and \( |G : B| \) are \( \sigma \)-coprime. Then, by Steps 1 and 2, \( X \subseteq A \cap B \). Then, by Theorem A, \( X \) is \( \sigma \)-subnormal in \( A \) and \( B \). Therefore \( X \) is \( \sigma \)-subnormal in \( G \) by Theorem B. 

\( \square \)

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