On σ-subnormality criteria in finite σ-soluble groups

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Abstract

Let $\sigma = \{\sigma_i : i \in I\}$ be a partition of the set $\mathbb{P}$ of all prime numbers. A subgroup X of a finite group G is called *σ-subnormal* in G if there is a chain of subgroups

$$X = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = G$$

where for every $j = 1, \ldots, n$ the subgroup X_{j-1} normal in X_j or $X_j/Core_{X_j}(X_{j-1})$ is a σ_i-group for some $i \in I$.

In the special case that σ is the partition of $\mathbb{P}$ into sets containing exactly one prime each, the σ-subnormality reduces to the familiar case of subnormality.

In this paper some σ-subnormality criteria for subgroups of σ-soluble groups, or groups in which every chief factor is a σ_i-group, for some $\sigma_i \in \sigma$, are showed.

Mathematics Subject Classification (2010): 20D10, 20D20

Keywords: finite group, σ-solubility, σ-nilpotency, σ-subnormal subgroup, factorised group.

1 Introduction and statements of results.

All groups considered in this paper are finite.

The results of this article are based on a paper of Skiba [15]. There he generalised the concepts of solubility, nilpotency and subnormality introducing σ-solubility, σ-nilpotency, and σ-subnormality in which σ is a partition of the set $\mathbb{P}$, the set of all primes. Hence $\mathbb{P} = \bigcup_{i \in I} \sigma_i$, with $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

We note that in the special case that σ is the partition of $\mathbb{P}$ containing exactly one prime each, the definitions below reduce to the familiar case of soluble groups, nilpotent groups and subnormal subgroups.

From now on let σ denote a partition of $\mathbb{P}$. Given a natural number n, we denote by $\sigma(n)$ the set of all elements of σ including the primes dividing n. Two natural numbers m and n are called *σ-coprime* if $\sigma(m) \cap \sigma(n) = \emptyset$. We say that n is *σ-primary* if $|\sigma(n)| = 1$, that is, if its prime factors all belong to the same member of σ.

A group G is called *σ-primary* if $|G|$ is a σ-primary number.

Definition 1. *A group G is said to be σ-soluble if every chief factor of G is σ-primary. G is said to be σ-*nilpotent *if it is a direct product of σ-primary groups.*

Note that if π is a set of primes and $\sigma = \{\pi, \pi'\}$, then a group G is σ-soluble if and only if G is π-separable. In this case, G is σ-nilpotent if and only if G is π-decomposable. If $\pi = \{p_1, \cdots, p_n\}$, and $\sigma = \{\{p_1\}, \cdots, \{p_n\}, \pi'\}$, then G is σ-soluble if and only if G is π-soluble, and G is σ-nilpotent if and only if G has a normal Hall π'-subgroup and a normal Sylow p_i-subgroup, for all $i = 1, \ldots, n$.

Many normal and arithmetical properties of soluble groups still hold for σ-soluble groups (see [15]). In particular, every σ-soluble group has a conjugacy class of Hall σ_i-subgroups and a conjugacy class of Hall σ_i'-subgroups, for every $\sigma_i \in \sigma$.

The role of the class $\mathcal{N}_\sigma$ of all σ-nilpotent groups in σ-soluble groups is analogous to that of nilpotent groups in soluble groups. In particular, $\mathcal{N}_\sigma$ is a subgroup-closed saturated Fitting formation ([15, Corollary 2.4 and Lemma 2.5]) that is closely related to the subgroup embedding property of σ-subnormality.

Definition 2. *Given a partition σ of the set of prime numbers, a subgroup X of a group G is called σ-subnormal in G if there exists a chain of subgroups*

$$X = X_0 \leq X_1 \leq \cdots \leq X_n = G,$$

with X_{i-1} normal in X_i or $X_i/Core_{X_i}(X_{i-1})$ σ-primary for every $1 \leq i \leq n$.

To know that a non-σ-nilpotent group possesses a non-trivial proper σ-subnormal subgroup is equivalent to know that the group is not simple.

Therefore criteria for the σ-subnormality of a subgroup may have some importance in the study of the normal structure of a group. The close relationship between σ-subnormal subgroups and direct decompositions of a group strongly supports that claim. The significance of the σ-subnormal subgroups in σ-soluble groups is apparent since they are precisely the $\mathcal{N}_\sigma$-subnormal subgroups, and so they are a sublattice of the subgroup lattice of G. They are also important to analyse the structural impact of some permutability properties (see [15]).

In this paper, which is a natural continuation of [3], extensions of some well-known subnormality criteria are presented. For instance, according to a result of Wielandt (see [10, Theorem 7.3.3]), a subgroup X of a group G is subnormal in G if and only if X is subnormal in $\langle X, X^g \rangle$ for all $g \in G$.

In [11, Question 19.84] (see also [18]), Skiba asked whether it is enough to know that X is σ-subnormal in $\langle X, X^g \rangle$ for all $g \in G$ to deduce that X is σ-subnormal in G. It is certainly true in the soluble universe by virtue of [2, Proposition 6.1.10 and Theorem 6.2.17] (see [3, Lemma 2]). Our first main result shows that the answer is also affirmative for σ-soluble groups.

Theorem A. *Suppose that G is a σ-soluble group and X is a subgroup of G that is σ-subnormal in $\langle X, X^g \rangle$ for all $g \in G$. Then X is σ-subnormal in G.*

Theorem A is not true for arbitrary groups. Therefore Question 19.84 in [11] is answered.

Example 1. Let $\pi = \{2, 3\}$ and $\sigma = \{\pi, \pi'\}$. The simple group $G = \mathrm{PSL}_2(7)$ of order $168 = 2^3 \cdot 3 \cdot 7$ has a unique conjugacy class of elements of order 2. Let x be an element of this class. Given $g \in G$, the group $\langle x, x^g \rangle$ is isomorphic to C_2, to $C_2 \times C_2$, to Σ_3 or to D_8. Therefore $X = \langle x \rangle$ is σ-subnormal in $\langle X, X^g \rangle$ for all $g \in G$ but X is not σ-subnormal in G.

Another important subnormality criterion asserts that if $G = AB$ is a group which is the product of the subgroups A and B and X is a subgroup of G contained in $A \cap B$ that is subnormal in A and B, then X is subnormal in G. This result was proved by Maier in [12] for soluble groups and then for arbitrary groups by Wielandt [19]. Applying Theorem A, we show that Maier-Wielandt's result also holds for σ-subnormal subgroups not only in the soluble universe, but also in the σ-soluble one.

Theorem B. *Let the σ-soluble group G be the product of two subgroups A and B. If X is a subgroup of $A \cap B$ which is σ-subnormal in both A and B, then X is σ-subnormal in G.*

Theorem B does not hold in general as the following example shows (see [8]).

Example 2. Let $\pi = \{2, 5\}$ and $\sigma = \{\pi, \pi'\}$. The alternating group of degree five A_5 is the product of the subgroups A and B, where A is the alternating group of degree 4 and B is a dihedral group of order 10. Then $A \cap B$ is σ-subnormal in both A and B, but $A \cap B$ is not σ-subnormal in A_5.

On the other hand, Wielandt [19] conjectured that if X is a subgroup of G such that X is subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$, then X is subnormal in G.

Wielandt's conjecture was proved to be true in the soluble universe by Maier and Sidki [13] for subgroups X of prime power order and then for every subgroup X of a soluble group by Casolo in [4].

In [3, Theorem A], we show that the following σ-version of the aforementioned result holds.

Theorem 1. *Assume that G is a soluble group factorised as a product of the subgroups A and B. Let X be a subgroup of G such that X is σ-subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$. Then X is σ-subnormal in G.*

A natural question to ask is now whether Theorem 1 holds for σ-soluble groups. Unfortunately we have been unable to answer this question; however, our third main result could be regarded as a significant step to solve it.

Theorem C. *Assume that G is a σ-soluble group factorised as a product of the subgroups A and B. Let X be a subgroup of G such that X is σ-subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$. Then X is σ-subnormal in G if one of the following conditions is true:*

(i) $|G : A|$ *and* $|G : B|$ *are σ-primary.*

(ii) $|G : A|$ *is σ-primary and* $|G : A|$ *and* $|G : B|$ *are σ-coprime.*

The proof of Theorem C strongly depends on the following extension of [6, Theorem 3].

Theorem D. *Let G be a σ-soluble group, and A and X two subgroups of G such that X is σ-subnormal in $\langle X, X^a \rangle$ for all $a \in A$. If $|G : A|$ is σ-primary, then X is σ-subnormal in $\langle X, A \rangle$.*

We shall adhere to the notation and terminology of [2] and [5].

2 Preliminaries

Our first lemma collects some basic properties of σ-subnormal subgroups which are very useful in induction arguments.

Lemma 1 ([15]). *Let H, K and N be subgroups of a group G. Suppose that H is σ-subnormal in G and N is normal in G. Then the following statements hold:*

1. *$H \cap K$ is σ-subnormal in K.*
2. *If K is a σ-subnormal subgroup of H, then K is σ-subnormal in G.*
3. *If K is σ-subnormal in G, then $H \cap K$ is σ-subnormal in G.*
4. *HN/N is σ-subnormal in G/N.*
5. *If $N \subseteq K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.*
6. *If $L \leq K$ and K is σ-nilpotent, then L is σ-subnormal in K.*
7. *If $|G : H|$ is a σ_i-number, then $O^{\sigma_i}(H) = O^{\sigma_i}(G)$.*
8. *If N is a σ_i-subgroup of G, then $N \leq N_G(O^{\sigma_i}(H))$.*

A standard induction argument using Lemma 1 allows us to prove the following result.

Lemma 2. *Let X be a subgroup of a σ-soluble group G. Then X is σ-subnormal in G if and only if X is $\mathcal{N}_\sigma$-subnormal in G, that is, there exists a chain of subgroups*

$$X = X_0 \leq X_1 \leq \cdots \leq X_n = G,$$

such that X_{i-1} is a maximal subgroup of X_i and $X_i/Core_{X_i}(X_{i-1}) \in \mathcal{N}_\sigma$, for $1 \leq i \leq n$.

The fact that σ-subnormal subgroups are $\mathcal{N}_\sigma$-subnormal in the σ-soluble universe allows us to prove some relevant properties of these subgroups which are crucial in the proof of our main results.

Lemma 3. *Let X be a subgroup of a group G.*

1. *([2, Lemma 6.1.9 and Proposition 6.1.10]) If X is σ-subnormal in G, then the $\mathcal{N}_\sigma$-residual $X^{\mathcal{N}_\sigma}$ of X is subnormal in G.*

2. *([2, Lemma 6.1.9]) If X is subnormal in G, then X is σ-subnormal in G.*

3. *([2, Lemmas 6.3.11 and 6.3.12 and Example 6.3.13])) $\mathcal{N}_\sigma$ is a lattice formation, that is, the set of all σ-subnormal subgroups of a σ- soluble group G forms a sublattice of the subgroup lattice of G.*

4. *([2, Theorem 6.3.3]) If X is a σ-subnormal σ-nilpotent subgroup of a σ-soluble group G, then X is contained in $F_\sigma(G)$, the $\mathcal{N}_\sigma$-radical of G. In particular, if X is σ_i-group, then $X \leq \mathrm{O}_{\sigma_i}(G)$.*

5. *([2, Theorem 6.5.46]) If $G = \langle A, B\rangle$ is a a σ-soluble group generated by two σ-subnormal subgroups A and B, then $G^{\mathcal{N}_\sigma} = \langle A^{\mathcal{N}_\sigma}, B^{\mathcal{N}_\sigma}\rangle$.*

Note that by Lemmas 1 (2) and 3 (2), subnormal subgroups of σ-subnormal subgroups of a group G are σ-subnormal in G. This fact will be applied in the sequel without further reference.

Our third lemma shows that the residual associated with the class of all σ_i-groups (also called σ_i-residual) respects the σ-subnormal generation of σ-soluble groups.

Lemma 4. *Let $\sigma_i \in \sigma$. If A and B are σ-subnormal subgroups of a σ-soluble group $G = \langle A, B\rangle$, then $O^{\sigma_i}(G) = \langle O^{\sigma_i}(A), O^{\sigma_i}(B)\rangle$.*

Proof. Assume the result is false and let G be a counterexample of least order. Denote $H = \langle O^{\sigma_i}(A), O^{\sigma_i}(B)\rangle$ and $X = O^{\sigma_i}(G)$. Clearly $1 \neq X$. Let N be a minimal normal subgroup of G such that $N \leq X$. Since G is σ-soluble, it follows that N is σ_j-group for some $\sigma_j \in \sigma$. The minimality of G yields $X = HN$ and $Core_G(H) = 1$.

On the other hand, by Lemma 3 (5), we have that $G^{\mathcal{N}_\sigma} = \langle A^{\mathcal{N}_\sigma}, B^{\mathcal{N}_\sigma}\rangle \leq \langle O^{\sigma_i}(A), O^{\sigma_i}(B)\rangle = H$. Since $G^{\mathcal{N}_\sigma}$ is normal in G and $Core_G(H) = 1$, it follows that G is σ-nilpotent.

Then $G = X \times Y$ with $Y = O_{\sigma_i}(G)$. If $Y \neq 1$, then by the minimal choice of G, we have that $G = X \times Y = H \times Y$, and therefore $X = H$. Thus $Y = 1$ and so $G = O^{\sigma_i}(G)$, $A = O^{\sigma_i}(A)$ and $B = O^{\sigma_i}(B)$. This contradiction proves the lemma.

□

Lemma 5. *Let H^* denote either the $\mathcal{N}_\sigma$-residual or the σ_i-residual of a subgroup H of a σ-soluble group G, for $\sigma_i \in \sigma$. Let A be a subgroup of G. If H is a σ-subnormal subgroup of $\langle H, H^a\rangle$ for all $a \in A$, then H normalises $(H^*)^A$.*

Proof. Let $a \in A$. Since H is a σ-subnormal subgroup of $\langle H, H^{a^{-1}}\rangle$, it follows that H^a is σ-subnormal in $\langle H^a, H\rangle = \langle H, H^a\rangle$. By Lemmas 3 (5) and 4, we have $\langle H, H^a\rangle^* = \langle H^*, (H^a)^*\rangle = \langle H^*, (H^*)^a\rangle$, thus

$$[H, (H^*)^a] \leq [H, \langle H, H^a\rangle^*] \leq \langle H, H^a\rangle^* \leq (H^*)^A.$$

□

Lemma 6. *Let G be a σ-soluble group, X a σ_i-subgroup of G and H a Hall σ_i-subgroup of G. If X is σ-subnormal in $\langle X, X^h\rangle$ for all $h \in H$, then $X \leq H$.*

Proof. Suppose that the result is false. Let G be a counterexample of the smallest possible order. Clearly the hypotheses of the lemma hold in $G/O_{\sigma_i}(G)$. Therefore, if $O_{\sigma_i}(G) \neq 1$, we have that $XO_{\sigma_i}(G)/O_{\sigma_i}(G) \leq H/O_{\sigma_i}(G)$ by minimality of G. Hence $X \leq H$, contrary to supposition. Thus $O_{\sigma_i}(G) = 1$. Let N be a minimal normal subgroup of G. Then N is a σ_j-group for some $j \neq i$. Since $X \leq HN$ by the minimal choice of G, there exists $n \in N$ with $X^n \leq H$. Let $x \in X$ and $h = x^{-n} \in H$. Then $[x, h] = [x, n][x^{-1}, n] \in N$ and $[x, h] = x^{-1}x^h \in \langle x, x^h\rangle$. Hence $[x, h] \in N \cap \langle x, x^h\rangle$. Then X is σ-subnormal in $\langle X, X^h\rangle$ by hypothesis. Since X is a σ_i-subgroup, we have that $X \leq O_{\sigma_i}(\langle X, X^h\rangle)$ by Lemma 3 (4). Therefore, $\langle X, X^h\rangle = O_{\sigma_i}(\langle X, X^h\rangle)X^h$ is a σ_i-subgroup of HN. Thus $[x, h] \in N \cap \langle X, X^h\rangle = 1$ and $[x, h] = 1$. In particular, $[x, n] = [x^{-1}, n]$ is a σ_i-element. Since N is a $(\sigma_i)'$-group and $[x, n] \in N$, it follows that $[x, n] = 1$ and $X^n = X \leq H$.

□

Lemma 7. *Let H be a subgroup of a σ-soluble group G such that $O^{\sigma_i}(H) = H$ for some $\sigma_i \in \sigma$. Assume K is a normal σ_i-subgroup of G and $k \in K$ such that H is a σ-subnormal subgroup of $\langle H, H^k\rangle$. Then k normalises H.*

Proof. Denote $L = \langle H, H^k\rangle$. Let Z denote the normal closure of H in L. By Lemma 4, $O^{\sigma_i}(Z) = Z$. Since $O^{\sigma_i}(L/Z) = L/Z$, it follows that $L = O^{\sigma_i}(L)Z$. By [2, Proposition 6.5.5], it follows that $O^{\sigma_i}(L) = O^{\sigma_i}(L)O^{\sigma_i}(Z) = O^{\sigma_i}(L)Z = L$.

On the other hand, $L = L \cap HK = H(L \cap K)$. By Lemma 4, $L = O^{\sigma_i}(L) = O^{\sigma_i}(H)O^{\sigma_i}(L \cap K) = H$. Thus $L = H$ and $H^k = H$. □

3 Proofs of the main theorems

Proof of Theorem A. Suppose the result is not true and let G be a counterexample with $|G|+|X|$ minimal. Then $G^{\mathcal{N}_\sigma} \neq 1$. Let N be a minimal normal subgroup of G contained in $G^{\mathcal{N}_\sigma}$. Then N is a σ_i-group for some $\sigma_i \in \sigma$. Note that XN/N is σ-subnormal in G/N by the minimality of the pair (G, X). If XN were a proper subgroup of G, then X would be σ-subnormal in XN. By Lemma 1, X would be σ-subnormal in G, contrary to our assumption. Hence $G = XN$. Assume that X is a σ_i-group. Then G is a σ_i-group, and X is σ-subnormal in G. This contradiction implies that X is not a σ_i-group, and so $O^{\sigma_i}(X) \neq 1$.

Assume that $O^{\sigma_i}(X) < X$. By minimality of (G, X), it follows that $O^{\sigma_i}(X)$ is σ-subnormal in G. By Lemma 1 (8), N normalises $O^{\sigma_i}(O^{\sigma_i}(X)) = O^{\sigma_i}(X)$. Hence $O^{\sigma_i}(X)$ is a normal subgroup of G. The minimal choice of G implies that $X/O^{\sigma_i}(X)$ is σ-subnormal in $G/O^{\sigma_i}(X)$ and then X is σ-subnormal in G by Lemma 1 (5). This is not possible. Thus $X = O^{\sigma_i}(X)$.

If $n \in N$ then X is σ-subnormal in $U_n = \langle X, X^n \rangle = (U_n \cap N)X$ by hypothesis. By Lemma 1 (7), we have that

$$O^{\sigma_i}(U_n) = O^{\sigma_i}((U_n \cap N)X) = O^{\sigma_i}(X) = X.$$

In particular, X is normal in U_n. Consequently, X is normal in $V = \langle X^n : n \in N \rangle$. Since V is normal in G, we have X is subnormal in G, and we have reached the desired contradiction. $\square$

Proof of Theorem B. Assume the result is false and let G be a counterexample such that $|G : A|+|X|$ is minimal. Suppose that M is a maximal subgroup of G containing A. Then $M = A(M \cap B)$ and X is σ-subnormal in both A and $M \cap B$ by Lemma 1 (1). By minimality of G, X is σ-subnormal in M. On the other hand, $G = MB$. If $|G : M| < |G : A|$, we have X is σ-subnormal in G, which is a contradiction. Therefore $A = M$ is a maximal subgroup of G. Let $K = \mathrm{Core}_G(A)$. If $K \neq 1$, then XK/K is σ-subnormal in G/K by the minimal choice of G. By Lemma 1 (5), XK is σ-subnormal in G. Moreover $X \leq XK \leq A$. Thus X is σ-subnormal in XK by Lemma 1 (1). Thus X is σ-subnormal in G. This contradiction yields $K = 1$ and G is a primitive group. By Lemma 3 (1), $X^{\mathcal{N}_\sigma}$ is a subnormal subgroup of A and B. Applying the result of Maier-Wielandt, we have that $X^{\mathcal{N}_\sigma}$ is a subnormal subgroup of G. By [10, Lemma 7.3.16], $X^{\mathcal{N}_\sigma} \leq \mathrm{Core}_G(A) = 1$. Hence X is σ-nilpotent. By Lemma 1 (6), every subgroup of X is σ-subnormal in X. Therefore every proper subgroup of X is σ-subnormal in A and B by Lemma 1 (2). The minimal choice of X implies that every proper subgroup of X is σ-subnormal in G. By Lemma 3 (3), X is cyclic of prime power order. Assume X is a σ_i-group. Since X is σ-subnormal in A, by Lemma 3 (4), X is contained in $O_{\sigma_i}(A)$. Then X^A, the normal closure of X in A, is a σ_i-group. Analogously, X^B is a σ_i-group. According to [1, Lemma 1.3.2], there exist Hall σ_i-subgroups A_{σ_i} of A and B_{σ_i} of B such that $A_{\sigma_i}B_{\sigma_i}$ is a Hall σ_i-subgroup of G. Then $\langle X^A, X^B\rangle$ is a σ_i-group because it is contained in $A_{\sigma_i}B_{\sigma_i}$. Let $g = ab \in G$ with $a \in A$ and $b \in B$. Then

$$\langle X, X^g\rangle = \langle X^{b^{-1}}, X^a\rangle^b \le \langle X^B, X^A\rangle^b.$$

Consequently $\langle X, X^g\rangle$ is a σ_i-group and then X is σ-subnormal in $\langle X, X^g\rangle$ for every $g \in G$ by Lemma 1 (6). Applying Theorem A, X is σ-subnormal in G, a contradiction. □

Proof of Theorem D. Suppose that the result is false. We choose a counterexample G with $|G| + |X|$ minimal and proceed to derive a contradiction. The minimal choice of G and Theorem A show that $G = \langle X, A\rangle$ and X is not contained in A. Suppose that $|G : A|$ is a σ_i-number for some $\sigma_i \in \sigma$. Then A contains a Hall σ_i'-subgroup of G.

If $C = \mathrm{Core}_G(A) \neq 1$, then XC is a σ-subnormal subgroup of G by minimality of G. Moreover, by Theorem A, X is σ-subnormal in XC. Thus X is σ-subnormal in G by Lemma 1 (2). This contradiction shows that $\mathrm{Core}_G(A) = 1$.

Let N be a minimal normal subgroup of G. Then N is a σ_j-group for some $\sigma_j \in \sigma$. If $i \neq j$, then N is contained in every Hall σ_i'-subgroup of G. In particular, N is contained in A, a contradiction. Therefore N is a σ_i-group, $O_{\sigma_i}(G) \neq 1$, and $O_{{\sigma_i}'}(G) = 1$.

Suppose that X is not σ-nilpotent. Then $1 \neq X^{\mathcal{N}_\sigma}$ is a proper subgroup of X which is σ-subnormal in $\langle X, X^a\rangle$ for all $a \in A$. The choice of the pair (G, X) yields that $X^{\mathcal{N}_\sigma}$ is σ-subnormal in $\langle X^{\mathcal{N}_\sigma}, A\rangle$. Hence $X^{\mathcal{N}_\sigma}$ is σ-subnormal in $(X^{\mathcal{N}_\sigma})^A$. By Lemma 5, X normalises $(X^{\mathcal{N}_\sigma})^A$. Therefore $(X^{\mathcal{N}_\sigma})^A$ is a normal subgroup of G and $X^{\mathcal{N}_\sigma}$ is a σ-subnormal subgroup of G. Since X is not a σ_i-group, it follows that $1 \neq O^{\sigma_i}(X)$. Moreover, since $1 \neq X^{\mathcal{N}_\sigma}$ is a σ-soluble group, it follows that $F_\sigma(X^{\mathcal{N}_\sigma}) \neq 1$. Thus $F_\sigma(X^{\mathcal{N}_\sigma}) \neq 1$ is a σ-nilpotent σ-subnormal subgroup of G. By Lemma 3 (4), $F_\sigma(X^{\mathcal{N}_\sigma}) \le F_\sigma(G) = O_{\sigma_i}(G)$ and then $1 \neq O_{\sigma_i}(X^{\mathcal{N}_\sigma}) \le O_{\sigma_i}(G)$. Hence $Z = X \cap O_{\sigma_i}(G) \neq 1$ and Z^A is a σ-subnormal σ_i-subgroup of G. Let $a \in A$.

Then X is σ-subnormal in $\langle X, Z^a\rangle$ and so $O_{\sigma_i}(\langle X, Z^a\rangle)$ normalises $O^{\sigma_i}(X)$ by Lemma 1 (8). Since $Z^a \leq O_{\sigma_i}(\langle X, Z^a\rangle)$, it follows that Z^a normalises $O^{\sigma_i}(X)$. Therefore Z^A normalises $O^{\sigma_i}(X)$.

Applying Lemma 5, it follows that X normalises $(O^{\sigma_i}(X))^A$. Hence $(O^{\sigma_i}(X))^A$ is a normal subgroup of G. Assume that $O^{\sigma_i}(X)$ is a proper subgroup of X. By minimality of the pair (G, X), we have that $O^{\sigma_i}(X)$ is a σ-subnormal subgroup of $\langle O^{\sigma_i}(X), A\rangle$. Therefore $O^{\sigma_i}(X)$ is a σ-subnormal subgroup of $(O^{\sigma_i}(X))^A$, and so $O^{\sigma_i}(X)$ is σ-subnormal in G. By Lemma 1 (8), $O_{\sigma_i}(G)$ normalises $O^{\sigma_i}(O^{\sigma_i}(X)) = O^{\sigma_i}(X)$ and hence $XO_{\sigma_i}(G)$ normalises $O^{\sigma_i}(X)$. Then $X/O^{\sigma_i}(X)$ is σ-subnormal in $XO_{\sigma_i}(G)/O^{\sigma_i}(X)$. Thus X is σ-subnormal in $XO_{\sigma_i}(G)$ which is σ-subnormal in G by minimality of G and Lemma 1 (5). Lemma 1 (2) yields that X is σ-subnormal in G, contrary to assumption. Hence $O^{\sigma_i}(X) = X$ and so Z^A normalises X. In addition, $[Z^A, X] \leq [\mathrm{N}_G(X) \cap O_{\sigma_i}(G), X] \leq X \cap O_{\sigma_i}(G) = Z \leq Z^A$. Hence Z^A is normalised by X and so it is a normal subgroup of G. Again the minimality of G and Lemma 1 (5) imply that XZ^A is σ-subnormal in G. Since X is normal in XZ^A, we have that X is σ-subnormal in G. This contradiction shows that X is σ-nilpotent.

Suppose that $O^{\sigma_i}(X) \neq 1$. Since X is σ-nilpotent, it follows that either X is a σ_i'-group or $O^{\sigma_i}(X)$ is a proper subgroup of X. Assume that X is a σ_i'-group. Then, by Lemma 6, X is contained in A. Hence $G = A$ and X is σ-subnormal in G by Theorem A, which is not possible. Suppose that $O^{\sigma_i}(X)$ is a proper subgroup of X. By minimality of (G, X), $O^{\sigma_i}(X)$ is σ-subnormal in $\langle O^{\sigma_i}(X), A\rangle$, and, by Lemma 5, X normalises $(O^{\sigma_i}(X))^A$. Therefore $O^{\sigma_i}(X)$ is a σ-subnormal subgroup of $O^{\sigma_i}(X)^A$ which is a normal subgroup of G. Consequently $O^{\sigma_i}(X)$ is a σ-subnormal σ-nilpotent subgroup of G. By Lemma 3 (4), $O^{\sigma_i}(X)$ is contained in $F_\sigma(G) = O_{\sigma_i}(G)$. Hence X is a σ_i-group, contrary to supposition.

Consequently, $O^{\sigma_i}(X) = 1$ and X is a σ_i-group. Since every minimal normal subgroup N of G is a σ_i-group, and XN is σ-subnormal in G, it follows that X is σ-subnormal in G. This final contradiction proves the theorem.

$\square$

Proof of Theorem C. Suppose that the theorem is false and let G be a counterexample for which $|G|+|X|+|G:A|+|G:B|$ is minimal. Note that every proper σ-subnormal subgroup Z of X satifies the hypotheses of the theorem. Therefore Z is a σ-subnormal subgroup of G by the choice of (G, X).

We proceed in a number of steps.

Step 1. If X is not contained in A, then $G = \langle A, X\rangle$ and $|G : A|$ is not σ-primary.

Let $A_0 = \langle A, X\rangle$. We have that $A_0 = A_0 \cap AB = A(A_0 \cap B)$ and $G = A_0B$. If $A_0 \neq G$, then A_0 is not a counterexample to the theorem. Then X is σ-subnormal in A_0, and the 4-tuple (G, X, A_0, B) satisfies the hypotheses of the theorem. The minimal choice of (G, X, A, B) implies that X is σ-subnormal in G. Consequently, $G = \langle A, X\rangle$. If $|G : A|$ were σ-primary, then we would have X is σ-subnormal in G by Theorem D. This is not the case. Thus $|G : A|$ is not σ-primary.

Step 2. Assume that X is contained in A and $|G : A|$ is σ-primary. If X is not contained in B, then $|G : A|$ and $|G : B|$ are not σ-coprime.

Assume that X is not contained in B and $|G : A|$ and $|G : B|$ are σ-coprime and derive a contradiction. Let $B_0 = \langle X, B\rangle = B(B_0 \cap A)$. Then B is a proper subgroup of B_0 and $G = AB_0$. Then $(B_0, X, B_0 \cap A, B)$ satisfies the hypotheses of the theorem. Suppose that B_0 is a proper subgroup of G. Then the theorem holds in B_0, and hence X is σ-subnormal in B_0. Applying Theorem A and Theorem B, we conclude that X is σ-subnormal in G. This contradicts the choice of G, however, and we conclude that $G = \langle X, B\rangle$.

By hypothesis, $|G : A|$ is a σ_i-number, for some $\sigma_i \in \sigma$. Since $|G : A|$ and $|G : B|$ are σ-coprime, it follows that $|G : B|$ is a $\sigma_i^{'}$-number. Therefore B contains a Hall σ_i-subgroup of G.

Let N be a minimal normal subgroup of G. Then N is σ-primary. Assume that N is a σ_j-group, where $j \neq i$. Since $|G : A|$ is σ_i-number, then $N \leq A$. By the choice of G, XN is a σ-subnormal subgroup of G. Moreover, $XN \leq A$. Therefore X is σ-subnormal in XN and then in G, a contradiction. Consequently, every minimal normal subgroup of G is a σ_i-group and $F_\sigma(G) = O_{\sigma_i}(G)$. Moreover, $R = O_{\sigma_i}(G)$ is contained in B.

Suppose that X is not σ-nilpotent. Then $O^{\sigma_i}(X) \neq 1$. Suppose that $O^{\sigma_i}(X)$ is a proper subgroup of X. Then it is σ-subnormal in G. By Lemma 1 (8), $O_{\sigma_i}(G)$ normalises $O^{\sigma_i}(O^{\sigma_i}(X)) = O^{\sigma_i}(X)$ and hence $XO_{\sigma_i}(G)$ normalises $O^{\sigma_i}(X)$. Then $X/O^{\sigma_i}(X)$ is σ-subnormal in $XO_{\sigma_i}(G)/O^{\sigma_i}(X)$. Thus X is σ-subnormal in $XO_{\sigma_i}(G)$ which is σ-subnormal in G by minimality of G and Lemma 1 (5). Lemma 1 (2) yields that X is σ-subnormal in G, contrary to supposition. Thus $O^{\sigma_i}(X) = X$.

On the other hand, since X is not σ-nilpotent, $1 \neq X^{\mathcal{N}_\sigma}$ is σ-subnormal in G. Therefore $1 \neq F_\sigma(X^{\mathcal{N}_\sigma})$ is a σ-nilpotent σ-subnormal subgroup of G contained in $F_\sigma(G) = O_{\sigma_i}(G)$ by Lemma 3 (4). In particular, $O_{\sigma_i}(X) \neq$

1. Applying Lemma 5, we conclude that X normalises $(O^{\sigma_i}(X))^B$. Hence $(O^{\sigma_i}(X))^B$ is a normal subgroup of G. Write $Z = X \cap O_{\sigma_i}(G)$. Then $1 \neq Z$ is a σ-subnormal σ_i-subgroup of G. Let $b \in B$. Then X is σ-subnormal in $\langle X, Z^b \rangle$ and so $O_{\sigma_i}(\langle X, Z^b \rangle)$ normalises $O^{\sigma_i}(X) = X$ by Lemma 1 (8). Since $Z^b \leq O_{\sigma_i}(\langle X, Z^b \rangle)$, it follows that Z^b normalises X. Therefore Z^B normalises X. Then $[Z^B, X] \leq X \cap O_{\sigma_i}(G) = Z \leq Z^B$ and Z^B is normal in G. By the choice of G, it follows that XZ^B is a σ-subnormal subgroup of G and then X is σ-subnormal in G, a contradiction.

Thus X is σ-nilpotent. By assumption every proper subgroup of X is σ-subnormal in G. Applying Lemma 3 (3), X is a cyclic p-group for some prime $p \in \sigma_j$, for some $\sigma_j \in \sigma$. Assume that $i = j$. Then XN is a σ-subnormal σ_i-subgroup of G. Consequently, X is σ-subnormal in G, which contradicts our assumption that G is a counterexample. Thus $i \neq j$ and $O^{\sigma_i}(X) = X$. By Lemma 7, $R = O_{\sigma_i}(G)$ normalises X, and so X is normal in XR. Since XR is σ-subnormal in G by minimality of G and Lemma 1 (5), we conclude that X is σ-subnormal in G, which is not the case.

Step 3. We have a contradiction

Assume that either $|G : A|$ and $|G : B|$ are σ-primary or $|G : A|$ is σ-primary and $|G : A|$ and $|G : B|$ are σ-coprime. Then, by Steps 1 and 2, $X \subseteq A \cap B$. Then, by Theorem A, X is σ-subnormal in A and B. Therefore X is σ-subnormal in G by Theorem B.

□

Acknowledgements

The first and third authors are supported by the grant PGC2018-095140-B-I00 from the *Ministerio de Ciencia, Innovación y Universidades* and the *Agencia Estatal de Investigación*, Spain, and FEDER, European Union and Prometeo/2017/057 of Generalitat (Valencian Community, Spain). The second author was supported by the State Program of Science Researchers of the Republic of Belarus (Grant 19-54 «Convergence-2020»).

The authors are very grateful to the referees for extremely careful reading of the paper and for his numerous suggestions that have contributed to a substantial improvement of it.

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