On factorised finite groups

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Abstract

A subgroup H of a finite group G is called $\mathbb{P}$-subnormal in G if either $H = G$ or it is connected to G by a chain of subgroups of prime indices. In this paper some structural results of finite groups which are factorised as the product of two $\mathbb{P}$-subnormal subgroups are showed.

Mathematics Subject Classification (2010): 20D10, 20D25.

Keywords: finite group, factorised group, w-supersoluble group, $\mathbb{P}$-subnormal subgroup

1 Introduction and statement of results.

Assume that A and B are subgroups of a group G. We say that A and B permute if the product AB is a subgroup of G. If $G = AB$, then we say G is the product of the factors A and B. A natural question to ask is whether properties of $G = AB$ can be deduced from properties of A and B. There is an extensive literature on this question. Many properties and further restrictions on the products have been considered (see the books [2], [4] and the seminal papers [3], [7], [8]). We want to concentrate here on some particular properties and we will consider only finite groups.

Our starting point is a series of interesting papers of Vasil'ev, Vasil'eva, and Tyutyanov, where groups which are $\mathbb{P}$-*subnormal* products were studied ([10], [11], [12], [13]).

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$\mathbb{P}$-subnormality is a subgroup embedding property that naturally emerges in the study of supersolubility, and was introduced by the aforecited authors in [10].

Definition 1. A subgroup H of a group G is $\mathbb{P}$-*subnormal* in G whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for every $i = 1, \ldots, n$.

Note that in the soluble universe the $\mathbb{P}$-subnormality is the $\mathcal{U}$-subnormality associated with the saturated formation $\mathcal{U}$ of all supersoluble groups (see [5, Chapter 6]).

Vasil'ev, Vasil'eva, and Tyutyanov defined the class $w\mathcal{U}$ of *widely supersoluble* groups, w-supersoluble for short, as the class of all groups whose Sylow subgroups are $\mathbb{P}$-subnormal, and proved the following interesting result:

Theorem 2 ([10, Theorem 4.7])**.** *Let $G = AB$ be a group which is the product of two w-supersoluble subgroups A and B. If A and B are $\mathbb{P}$-subnormal in G and $G^{\mathcal{A}}$ is nilpotent, then G is w-supersoluble.*

Here $G^{\mathcal{A}}$ denotes the residual of G with respect to the formation $\mathcal{A}$ of all groups with abelian Sylow subgroups.

The proof of Theorem 2 depends on the properties of w-supersoluble groups as a class of groups showed in [10, Section 2]. It turns out that $w\mathcal{U}$ is a subgroup-closed saturated formation of soluble groups containing $\mathcal{U}$ and it is locally defined by a formation function f such that for every prime p, $f(p)$ is composed of all soluble groups G whose Sylow subgroups are abelian of exponent dividing $p-1$. Moreover, $\mathcal{U}$ is a proper subclass of $w\mathcal{U}$ (see [10, Example 1]).

Theorem 2 was the motive behind the results of [6]. We studied there mutually sn-permutable products of w-supersoluble groups. The idea to consider these products arises naturally from [11, Lemma 4.5]: if $G = AB$ is a product of two subgroups A and B, and B permutes with every subnormal subgroup of A and A is soluble, then B is $\mathbb{P}$-subnormal in G.

As a consequence, every mutually sn-permutable product is a $\mathbb{P}$-subnormal product. The group constructed in [10, Example 1] shows that the converse does not hold in general.

The main goal of this paper is to show some structural results of groups which are products of $\mathbb{P}$-subnormal subgroups. They can be regarded as $\mathbb{P}$-subnormal versions of the results of [6].

Our first main theorem analyses the behaviour of the abelian normal subgroups of groups which are the product of two $\mathbb{P}$-subnormal subgroups with respect to subgroup-closed saturated formations containing all supersoluble groups. It is a significant and useful extension of [1, Theorem A] and [6, Theorem 3].

Theorem A. *Let $\mathcal{F}$ be a subgroup-closed saturated formation containing $\mathcal{U}$. Let the group $G = AB$ be the product of the $\mathbb{P}$-subnormal $\mathcal{F}$-subgroups A and B. If N is an abelian normal subgroup of G, then both AN and BN are $\mathcal{F}$-groups.*

As Example 1 in [6] shows, the class of all w-supersoluble groups is not closed under taking products of $\mathbb{P}$-subnormal subgroups even if one of the factors is nilpotent. However, the w-supersolubility is guaranteed if the product $G = AB$ is mutually sn-permutable, A is w-supersoluble, B is nilpotent and B permutes with the Sylow subgroups of A ([6, Theorem 4]). Our second main result shows that this result also holds under much weaker hypotheses.

Theorem B. *Let $G = AB$ be the product of the $\mathbb{P}$-subnormal subgroups A and B, where A is w-supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is w-supersoluble.*

Theorem B does not hold for subgroup-closed saturated formations containing all supersoluble groups.

Example 3. Consider the subgroup-closed saturated formation $\mathcal{F}$ of all metanilpotent groups. It is known that $\mathcal{F}$ contains $\mathcal{U}$. Let G be the symmetric group of degree 4. Then $G = AB$, where A is the alternating group of degree 4, and B a Sylow 2-subgroup of G. It is clear that A and B are $\mathbb{P}$-subnormal in G. Moreover, A belongs to $\mathcal{F}$, B is nilpotent and permutes with every Sylow subgroup of A. However $G \notin \mathcal{F}$.

Similar arguments to those used in the proof of Theorem B allow us to prove the following

Theorem C. *Let $G = AB$ be the product of the $\mathbb{P}$-subnormal subgroups A and B. Suppose that A is supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then G is supersoluble.*

Our last theorem is a $\mathbb{P}$-subnormal version of [6, Theorem 5].

Theorem D. *Let $G = AB$ be the product of the $\mathbb{P}$-subnormal w-supersoluble subgroups A and B. If $(|A/A^{\mathcal{A}}|, |B/B^{\mathcal{A}}|) = 1$, then G is w-supersoluble.*

2 Proofs

Proof of Theorem A. Let F and U be the canonical local definitions of $\mathcal{F} = LF(F)$ and $\mathcal{U} = LF(U)$ respectively. Then, by [9, Proposition IV.3.11], $U(q) \subseteq F(q)$ for all primes q. Moreover, by [9, Example IV.3.4(f) and Proposition IV.3.8], $U(q) = \mathcal{S}_p\mathcal{A}_{q-1}$, where $\mathcal{A}_{q-1}$ is the class of all abelian groups of exponent dividing $q-1$.

Assume the result is not true and let G be a counterexample with $|G|+|N|$ as small as possible. Clearly we may suppose that $A < G$, $N \neq 1$ and $AN \notin \mathcal{F}$.

Suppose that $|N| = p$, p a prime number. Then $AN/\operatorname{C}_{AN}(N)$ is cyclic of order dividing $p-1$ and so $AN/\operatorname{C}_{AN}(N) \in F(p)$. Since $A \in \mathcal{F}$, we can apply [9, Remark IV.3.5(c)] to conclude that $AN \in \mathcal{F}$, which is a contradiction. Therefore, N is not of prime order.

Assume that N_1 is a non-trivial normal subgroup of G such that $N_1 \leq N$ and $|N : N_1| = q$, q a prime number. Then the assumption about the pair (G, N) gives that $AN_1 \in \mathcal{F}$. Let

$$1 = K_0 \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_{m-1} = N_1 \trianglelefteq K_m = N$$

be part of a chief series of AN below N. Let $1 \leq i \leq m-1$. Since N is abelian, it follows that K_i/K_{i-1} is a chief factor of AN_1. Since $AN_1 \in \mathcal{F} = LF(F)$, we have that $AN/\operatorname{C}_{AN}(K_i/K_{i-1}) \cong AN_1/\operatorname{C}_{AN_1}(K_i/K_{i-1}) \in F(p)$, where p is the prime dividing $|K_i/K_{i-1}|$, $1 \leq i \leq m-1$. Moreover, $AN/\operatorname{C}_{AN}(N/N_1) \in F(q)$. Applying the Jordan-Hölder Theorem, we have that $AN/C_{AN}(H/K) \in F(p)$, for all chief factor H/K of AN below N, and every prime p dividing the order of H/K. We conclude that $AN \in LF(F) = \mathcal{F}$. This contradicts our choice of G. Consequently, $|N : N_1|$ is not a prime for every non-trivial normal subgroup N_1 of G contained in N.

Since A is $\mathbb{P}$-subnormal in G and $A < G$, there exists a chain of subgroups

$$A = A_0 \leq A_1 \leq \cdots \leq A_{n-1} \leq A_n = G,$$

such that $|A_{i+1} : A_i|$ is a prime, for every $i = 0, 1, \cdots, n-1$.

Assume that $N \nleq A_{n-1}$. Then $G = A_{n-1}N$, and $N_1 = A_{n-1} \cap N$ is normal in G. If $N_1 = 1$, then $|G : A_{n-1}| = |NA_{n-1} : A_{n-1}| = |N : N_1| = |N|$ is a prime number. If $N_1 \neq 1$, then $|N : N_1| = |A_{n-1}N : A_{n-1}| = |G : A_{n-1}|$ is a prime. In both cases, we have a contradiction. Therefore $N \leq A_{n-1}$.

On the other hand, $A_{n-1} = G \cap A_{n-1} = AB \cap A_{n-1} = A(B \cap A_{n-1})$. It is clear that A is $\mathbb{P}$-subnormal in A_{n-1} and, by [11, Lemma 4.1], $B \cap A_{n-1}$ is $\mathbb{P}$-subnormal in A_{n-1}. Moreover $B \cap A_{n-1} \in \mathcal{F}$ because $\mathcal{F}$ is subgroup-closed. Hence A_{n-1} satisfies the hypotheses of the theorem. The minimal choice of G yields $AN \in \mathcal{F}$. This final contradiction proves the theorem. □

Proof of Theorem B. Assume the result is not true and let G be a counterexample with $|G|$ as small as possible. We derive a contradiction through the following steps:

Step 1: $N = \mathrm{Soc}(G)$ is a minimal normal subgroup of G; N is an elementary abelian p-group, for some prime p, $N = \mathrm{C}_G(N) = \mathrm{F}(G) = O_p(G)$ and N is a Sylow p-subgroup of G.

Let L be a minimal normal subgroup of G. Then, by [11, Lemma 3.1], we have that $G/L = (AL/L)(BL/L)$ is the product of the $\mathbb{P}$-subnormal subgroups AL/L and BL/L. Moreover, AL/L is w-supersoluble, BL/L is nilpotent and BL/L permutes with each Sylow subgroup of AL/L. The minimal choice of G implies that G/L is a $w\mathcal{U}$-group. Since $w\mathcal{U}$ is a saturated formation, it follows that G is a primitive group. Hence $N = \mathrm{Soc}(G)$ is a minimal normal subgroup of G and $G/N \in w\mathcal{U}$.

Let p be the largest prime dividing the order of G. By [11, Theorem 4.4], a Sylow p-subgroup P of G is normal in G. Therefore N is a p-group and $P = N = \mathrm{F}(G) = \mathrm{C}_G(N) = O_p(G)$.

Step 2: N is not contained in B and B is a p'-group.

Assume that N is contained in B. Since B is nilpotent, N is a Sylow p-subgroup of G and $N = C_G(N)$, we have that $B = N$. Hence $G = AN$ and G is w-supersoluble by Theorem A, contrary to assumption. Therefore N is not contained in B and BN is a proper subgroup of G by Theorem A. Then every Hall p'-subgroup $A_{p'}$ of A is not trivial. Since B permutes with every Sylow subgroup of A and N is not contained in B, it follows that $A_{p'}B$ is a proper subgroup of $G = NA_{p'}B$. Hence $N \cap A_{p'}B = N \cap B$ is a normal subgroup of G. Since N is a minimal normal subgroup of G, it follows that $N \cap B = 1$. Consequently, B is a p'-group.

Step 3: The Sylow r-subgroups of G are abelian of exponent dividing $p-1$ for all primes $r \neq p$.

Let $r \neq p$ be a prime dividing the order of G. By Theorem A, $X = AN$ is w-supersoluble. Since $O_{p'}(X) = 1$, it follows that $O_{p',p}(X) = N$ and X/N has abelian Sylow subgroups of exponent dividing $p-1$. Therefore the Sylow r-subgroups of A are abelian of exponent dividing $p-1$, and the same is true for B.

Let $r \neq p$ be a prime dividing the order of G. By [4, Theorem 1.1.19], there exist Sylow r-subgroups A_r and B_r of A and B respectively such that $G_r = A_rB_r$ is a Sylow r-subgroup of G. By hypothesis, $B{A_r}^a$ is a subgroup of G for every $a \in A$. Let $g \in G$. Then $g = ab$, where $a \in A$ and $b \in B$. Thus $(B{A_r}^a)^b = B{A_r}^{ab} = B{A_r}^g$ is a subgroup of G for every $g \in G$. Applying

[2, Lemma 2.5.1], we conclude that $[A_r, B]$ is a subnormal p'-subgroup of G. Consequently $[A_r, B] \leq O_{p'}(G) = 1$. Since A_r and B_r are abelian of exponent $p-1$, we have that G_r is abelian of exponent dividing $p-1$.

Step 4: The final contradiction.

Since G/N is w-supersoluble and the Sylow r-subgroups of G are abelian of exponent dividing $p-1$, we can apply [9, Remark IV.3.5(c)] to conclude that G is w-supersoluble, the final contradiction.

□

Proof of Theorem D. We prove the theorem by induction on the order of G. Arguing as in Step 1 of Theorem B, we may assume that G is a primitive soluble group. Then $G = NM$, where N is the unique minimal normal subgroup of G, $C_G(N) = N$ and G/N is w-supersoluble. Let p be the largest prime dividing the order of G. By [11, Theorem 4.4], a Sylow p-subgroup P of G is normal in G. Therefore $P = N$. By Theorem A, AN and BN are w-supersoluble. Since $O_{p',p}(AN) = O_{p',p}(BN) = 1$, we have that $A/A \cap N \simeq AN/N$ and $B/B \cap N \simeq BN/N$ have abelian Sylow subgroups of exponent dividing $p-1$. Therefore $A^{\mathcal{A}} \leq N$ and $B^{\mathcal{A}} \leq N$. Hence $(|AN/N|, |BN/N|) = (|A/A \cap N|, |B/B \cap N|) = 1$. Consequently $G/N = (AN/N)(BN/N)$ has abelian Sylow subgroups of exponent dividing $p-1$. Applying [9, Remark IV.3.5(c)], we conclude that G is w-supersoluble, as desired.

□

Acknowledgements

The first and third authors are supported by the grant PGC2018-095140-B-I00 from the *Ministerio de Ciencia, Innovación y Universidades* and the *Agencia Estatal de Investigación*, Spain, and FEDER, European Union and by Prometeo/2017/057 of Generalitat (Valencian Community, Spain).

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