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A remark on totally smooth renormings

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Abstract E. Oja, T. Viil, and D. Werner showed, in *Totally smooth renormings*, Archiv der Mathematik, **112**, 3, (2019), 269–281, that a weakly compactly generated Banach space $(X, \|\cdot\|)$ with the property that every linear functional on X has a unique Hahn–Banach extension to the bidual X^{**} (the so-called Phelps' property U in X^{**}, also known as the Hahn–Banach smoothness property) can be renormed to have the stronger property that for *every* subspace Y of X, every linear functional on Y has a unique Hahn–Banach extension to X^{**} (the so-called total smoothness property of the space). We mention here that this result holds in full generality —without any restriction on the space— and in a stronger form, thanks to a result of M. Raja, *On dual locally uniformly rotund norms*, Israel Journal of Mathematics **129** (2002), 77–91.

Keywords Renormings · Total smoothness · Hahn–Banach smoothness · Local strict convexity

Mathematics Subject Classification (2010) MSC 46B03 · 46B20 · 46B26 · 46B22

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1 A —even more than— totally smooth renorming

The norm $\|\cdot\|$ of a Banach space $(X,\|\cdot\|)$ is said to be **strictly convex** (or **rotund**), if for $x,y\in S_X$ such that $\|x+y\|=2$ we have $x=y$. The norm is said to be **locally uniformly convex** (or **locally uniformly rotund**) (**LUR**, for short), if $x\in S_X$, $x_n\in S_X$ for $n\in\mathbb{N}$, and $\|x+x_n\|\to 2$ implies $\|x_n-x\|\to 0$. The norm $\|\cdot\|$ in a dual Banach space is said to have **property** w^***-LUR** if $x_n^*\to x_0^*$ in the w^*-topology as soon as x_0^*, $x_n^*\in S_{X^*}$ for $n\in\mathbb{N}$, and $\|x_n^*+x_0^*\|\to 2$.

Attention has been paid to the problem of uniqueness of norm-preserving extensions (also called **Hahn–Banach extensions**) of any continuous functional defined on a closed subspace Y of a Banach space X to the whole of X (a property investigated by R. R. Phelps in [10]). The norm $\|\cdot\|$ of a Banach space $(X,\|\cdot\|)$ has the so-called **Hahn–Banach smooth property (HBS**, for short) if every $x^*\in X^*$ has a unique norm-preserving extension to X^{**}. A result that can be traced back to G. Godefroy [5], and that appears also in [7, Lemma III.2.14], says that this property is equivalent to the coincidence of the topologies w and w^* on the unit sphere S_{X^*} of the dual space X^* (we can call this property **WW*Kadets**, for short). A stronger property, called **total smoothness (TS** for short) is that for *any* closed subspace Y of X, any $y^*\in Y^*$ has a unique Hahn–Banach extension to X^{**}. This is equivalent, by a result of A. E. Taylor and S. R. Foguel [14], [4], to the HBS property plus the rotundness of the dual norm $\|\cdot\|^*$. In [13] it was proved that a *separable* space whose norm has the HBS property has a TS renorming, and in [9] this result was extended to the class of *weakly compactly generated* spaces (i.e., spaces having a weakly compact linearly dense subset).

Here we just point out that the renorming result holds, even in a stronger form, *without any restriction on the space*. This observation is based on the following M. Raja's result: *If X^* is a dual Banach space, then the two following conditions are equivalent:* (i) *X^* admits an equivalent dual LUR norm, and* (ii) *X^* admits an equivalent norm with the WW*Kadets property* [11, Theorem 3.1].

We believe that putting together those results as in Theorem 1 below may help to clarify the connections between the different properties mentioned above.

{thm-main} **Theorem 1** *Let $(X,\|\cdot\|)$ be a Banach space. Then, the following statements are equivalent:*
(i) *X has an equivalent norm with property HBS.*
(ii) *X has an equivalent norm whose dual norm has property WW*Kadets.*
(iii) *X has an equivalent norm whose dual norm is LUR.*
(iv) *X has an equivalent norm with property TS.*

Proof (i)$\Leftrightarrow$(ii) is the aforementioned result of Godefroy [5].

(ii)$\Leftrightarrow$(iii) is the quoted result of Raja above [11]. Although not explicitly stated in this reference, it is simple to observe that the topology induced by a dual LUR norm $\|\cdot\|^*$ coincides with the w^*-topology on the unit sphere S_{X^*}. Indeed, by the LUR property, given $x_0^* \in S_{X^*}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $x^* \in S_{X^*}$ satisfies $\|x_0^* + x^*\| > 2(1-\delta)$, then $\|x_0^* - x^*\| < \varepsilon$. Let $\{x_i^*\}$ be a net in S_{X^*} that w^*-converges to x_0^*. Find, by Riesz's Lemma, $x_0 \in S_X$ such that $\langle x_0, x_0^* \rangle > 1-\delta$. There exists i_0 such that $\langle x_0, x_i^* \rangle > 1-\delta$ for $i \geq i_0$. Thus, $\|x_0^* + x_i^*\| \geq \langle x_0, x_0^* + x_i^* \rangle > 2(1-\delta)$ for $i \geq i_0$, hence $\|x_0^* - x_i^*\| < \varepsilon$ for $i \geq i_0$, and the conclusion follows.

(iii)$\Rightarrow$(iv) follows from the Taylor–Foguel result [14],[4], quoted above, and the observation in the proof of the equivalence (iii)$\Leftrightarrow$(ii) here.

(iv)$\Rightarrow$(i) is obvious. ∎

Remark 1 1. Observe that, in particular, the TS norm defined in (iii) above on every Banach space with a HBS norm is Fréchet differentiable.

2. Banach spaces that satisfy one (and then all) of the conditions (i) to (iv) in Theorem 1 have been characterized in other different ways. Let us mention here that, for example, Theorem 1.4 in [3] provides a few of them, in terms of (a) the existence of a dual norm in X^* such that (S_{X^*}, w^*) is a Moore space, or (b) the existence of an equivalent dual norm such that (S_{X^*}, w^*) is symmetrized by a symmetric ρ such that every point $x^* \in S_{X^*}$ has w^*-neighborhoods of arbitrary small ρ-diameter, or (c) the existence of a dual equivalent norm such that (S_{X^*}, w^*) is metrizable, or even (d) that (B_{X^*}, w^*) is a descriptive compact space (for details, see the op. cit. and the reference list there).

Remark 2 Let us mention here (only with a hint for the proofs) that, for a Banach space $(X, \|\cdot\|)$ whose norm $\|\cdot\|$ has property HBS,

1. The norm, restricted to any closed subspace of X, has property HBS too, a consequence of the w^*-lower semicontinuity of the dual norm.
2. X is Asplund, as it follows from (i) in Theorem 1 and a separable reduction argument.
3. X is **nicely smooth** (i.e., there is no proper 1-norming subspace in X^*) and that, in fact, every James boundary is strong (see, e.g., [1, Paragraph 3.11.8.3]).
4. If $(X, \|\cdot\|) = (C(K), \|\cdot\|_\infty)$, where K is a compact topological space, then K is finite. This follows from the fact that the set of extreme points of $B_{C(K)^*}$ is $\{\pm\delta_k : \ k \in K\}$, that all extreme points are distributed between two closed hyperplanes, the Krein–Milman theorem, and the consequent reflexivity of the space $C(K)$. This observation depends strongly on the fact that the norm on $C(K)$ is the supremum norm. A space $C(K)$, for K an infinite compact space, may admit an equivalent norm $\|\cdot\|$ whose dual is LUR (and so $\|\cdot\|$ has property HBS): Just take K an infinite countable compact space; it is metrizable and scattered (see, e.g., [1, Lemma 14.21]), hence $C(K)$ is Asplund (see, e.g., [1, Theorem 14.25]). Thus, $C(K)^*$ is

separable, and the conclusion follows from a classical result of Kadets (see, e.g., [2, Section 2]).

5. There exists a LUR renorming of X. This follows from the aforementioned Raja's result and a result of R. Haydon in [8]. Note that it is an open problem (see, e.g., [12, Problem 1] and [6, Problem 102]) whether a space X has a LUR renorming as soon as it has a norm whose dual norm has property w^*-LUR.

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