INTEGRAL REPRESENTATION OF PRODUCT FACTORABLE BILINEAR OPERATORS AND SUMMABILITY OF BILINEAR MAPS ON $\mathcal{C}(K)$-SPACES

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Abstract. We present a constructive technique to represent classes of bilinear operators that allow a factorization through a bilinear product, providing a general version of the well-known characterization of integral bilinear forms as elements of the dual of an injective tensor product. We show that this general method fits with several known situations coming from different contexts—-harmonic analysis, C^*-algebras, $\mathcal{C}(K)$-spaces, operator theory, polynomials—, providing a unified approach to the integral representation of a broad class of bilinear operators. Some examples and applications are also shown, regarding for example operator spaces and summability properties of bilinear maps.

1. Introduction and preliminaries

Consider Banach spaces E and F. It is known that every bounded linear functional on the injective tensor product $E\hat{\otimes}_\varepsilon F$ is the linearization of a continuous bilinear form on $E \times F$ that has an integral representation. The present paper is an investigation of the bounded quadratic forms defined on $E \times E$ for which there is a factorization through some canonical product $\circledast : E \times E \to F$, which plays the role of the injective tensor product for the above mentioned integral bilinear maps. As we will see, the problem can be reduced sometimes to analyze the case of products defined on $C(K)$-spaces and integral representation of linear operators in such spaces, just by using the isomorphic representation of any Banach space F as a subspace of such a function space. A similar characterization was investigated in [25], but using as reference the projective tensor product instead of the injective one, obtaining also some duality formulas as the ones that will be shown here.

We are interested in studying this type of results in a broad context, seeking a unified approach to many of the developments and results that have been obtained in different fields. The main reference, that provides the starting point of our analysis, is the space of Grothendieck's integral bilinear

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forms, that gives an isometric representation of the dual of the injective tensor product (see for example [14, Ch.4]). Zero product preserving bilinear operators and convolution-orthogonal polynomials, which are defined using convolution in Lebesgue spaces of locally compact groups, define other class of examples that fits with our procedure (see [2, 1, 3, 17] and the references therein). Some classical constructions with spaces of operators can also be adapted to our setting, using for example the so called trace duality.

Thus, this work is mainly synthetic, although most of the results we show are new. As we will see, our construction can also be applied in the context of orthogonal polynomials. For instance, Sundaresan proved in [29] that a real valued n-homogeneous polynomial P defined on the Lebesgue space $L^p[0,1]$ ($L^\infty[0,1]$), with $n \leqslant p$, is orthogonally additive —that is, $P(f+g) = P(f) + P(g)$ for disjoint functions f and g— if and only if there is a function $h \in L^{p/p-n}[0,1]$ ($L^\infty[0,1]$) such that $P(f) = \int_{[0,1]} h f^n \, dx$ for all $f \in L^p[0,1]$ ($f \in L^\infty[0,1]$). This result was extended by Benyamini et al for the case of polynomials on order continuous Banach function spaces in [10]. A similar result was also obtained for the case of vector valued polynomials defined on $\mathcal{C}(K)$-spaces by Pérez García and Villanueva, who showed that a Banach valued n-homogeneous orthogonally additive polynomial on $\mathcal{C}(K)$ can be represented by an integral as $P(f) = \int_K f^n d\nu$, $f \in \mathcal{C}(K)$, where ν is a finite additive vector measure on K, which is countably additive under certain restrictions [20]. We will find these results —for the case of 2-homogeneous polynomials— as consequences of the application of our construction. Summability properties of bilinear maps acting in products of $\mathcal{C}(K)$-spaces will be also studied.

This paper is organised as follows. After explaining some definitions and notations, in Section 2 we analyze a quotient of the injective tensor product defined by a continuous product operation. We show that, for any product, it is always associated by duality with a space of bilinear forms that allow some kind of integral representation (Theorem 2.5). Section 3 is devoted to explain the main examples of quotient tensor products. It is shown that some of the well known classes of forms —such as the classical Grothendieck's integral forms, the orthogonally additive 2-homogeneous polynomials and the convolution-orthogonal bilinear forms— can be represented by means of this duality. In Section 4, general symmetric and non-symetric products on spaces of operators are faced, providing a general structure for understanding integral bilinear forms on these spaces; although the representation results that are shown in the previous section deal with tensor product as $E \otimes E$, it is easily seen that they work for the case $E \otimes F$ as well, allowing to analyze non-symmetric products too. Thus, they can be applied to analyze to the natural integral forms associated to some fundamental non-symmetric products, as the one given by the composition of operators. We have tried to present an exhaustive analysis of the classical structures that can be studied with our method. With the aim of providing some applications, we give in Section 5 a generalized version of the Pietsch integral bilinear operators and analyze some of their main properties, and in Section 6 we study summability of some new classes of integral bilinear maps defined on function spaces $\mathcal{C}(K)$. A factorization of these maps is given by using Pisier's Theorem and Pietsch's Domination Theorem.

The notations and terminology used throughout the paper are standard. Nevertheless, before going any further we remind the reader some terminology. We will use capital letters E, F, G, X, Y, Z to denote Banach spaces. B_E and E^* are the unit ball and the topological dual of the Banach space E, respectively. The space of the linear continuous operators from E to F will be denoted by $L(E, F)$. The Banach space of real valued continuous functions defined on the compact set K endowed with the usual supremum norm will be written as $\mathcal{C}(K)$. $\mathcal{M}(K)$ denotes the space of regular Borel measures on K. $L^p(\mu)$ $(p \geqslant 1)$ is the Banach lattice of functions for which the p-th power of the absolute value is μ-integrable, equipped with its standard norm $\|f\| = (\int_\Omega |f|^p d\mu)^{1/p}$. $L^\infty(\mu)$ denote the Banach space of the μ-essentially bounded functions. $L_{p,q}(\mu)$ $(1 \leqslant p, q < \infty)$ will denote the Lorentz space on the measure space (Ω, Σ, μ) equipped with $\|f\|_{p,q} = \Big(\int_0^{\mu(\Omega)} (t^{(1/p)-1} f^*(t))^q dt \Big)^{1/q}$, where $f^*(t) = \inf\{s > 0 : \mu(\{|f| > s\}) \leqslant 1\}$, $t \in [0, \mu(\Omega))$, is the decreasing rearrangement of $|f|$. Note that $L_{p,p}(\mu) = L_p(\mu)$.

Recall that a linear operator between Banach spaces $T : X \to Y$ is called (q,p)-*summing* $(T \in \Pi_{q,p}(X,Y))$ if there is a constant $k > 0$ such that for every $x_1, ..., x_n \in X$ and for all positive integers n,

$$\Big(\sum_{i=1}^n \|T(x_i)\|_Y^q \Big)^{1/q} \leqslant k \sup_{x^* \in B_{X^*}} \Big(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \Big)^{1/p}.$$

As usual, we say that an operator $T : X \to Y$ is p-summing if it is (p,p)-summing.

If E, F and G are Banach spaces, we will use often the word *product* for a bilinear map $\circledast : E \times F \to G$ if it is in some sense canonical in the setting that we are considering; this name is intended to emphasize that $\circledast$ is fixed in the given context and induces relevant properties in other associated bilinear maps. If X is another Banach space, we will call a bilinear map $B : E \times F \to X$ *zero product preserving* if it is zero valued for the couples whose product is zero, that is, $B(x,y) = 0$ whenever $x \circledast y = 0$.

A particular class of products that we will consider is given by the norm preserving products. A product $\circledast$ is norm preserving (n.p. product for short) if $\|x \circledast y\| \leqslant \|x\|\|y\|$ and for every $z \in B_G$,

$$\|z\| = \inf\{\|x\|\|y\| : x \in B_E,\ y \in B_F,\ x \circledast y = z\}.$$

The reader can find information about this kind of products in [18, p.2].

Let E and Y be Banach spaces. A continuous Banach-valued map $P : E \to Y$ is called an n- homogeneous polynomial if there is a continuous n-linear symmetric operator $B : E \times \cdots \times E \to Y$ such that $P(x) = B(x, ..., x)$

for all $x \in E$. An n-homogeneous polynomial P is called *orthogonally additive* if it satisfies $P(x+y) = P(x) + P(y)$ for every disjoint couple $x, y \in E$ –by disjointness we mean that their algebraic (pointwise μ-a.e.) multiplication is zero if E is a Banach algebra (a Banach function space). We will denote the spaces of n-homogeneous polynomials and n-homogeneous orthogonally additive polynomials by $\mathcal{P}(^nE, Y)$ and $\mathcal{P}_0(^nE, Y)$, respectively. We will simply write $\mathcal{P}(^nE)$ and $\mathcal{P}_0(^nE)$ if Y is the real line $\mathbb{R}$.

2. Diagonal bilinear functionals and tensor products: integral representations of bilinear forms

Let E be a Banach space. In this section we will consider a class of (quasi) norms for the tensor product $E \otimes E$ which generalizes the injective tensor norm, and that has as a particular case an associated quotient space that coincides with the symmetric injective tensor product that appears in the duality theory of homogeneous polynomials. In order to do this, we need a continuous product operation $\circledast$ in the space E. The given norm —we will denote it by $\varepsilon_\circledast$— is defined for a quotient space of the injective tensor product. We will prove in this section that it can be identified with a space of bilinear functionals that can be factored through the pointwise product.

Let E and Z be Banach spaces and a $\circledast : E \times E \to Z$ is a bilinear continuous map. Consider the tensor product $E \otimes E$ and the kernel subspace of tensors defined as

$$Ker\circledast_L := \Big\{ t = \sum_{i=1}^n x_i \otimes y_i : \sum_{i=1}^n x_i \circledast y_i = 0 \Big\}.$$

Note that the requirement for the tensor t is independent of its representation, so the subspace is well-defined. Of course, this subspace defines an equivalence relation in the usual way: two tensors t_1, $t_2 \in E \otimes E$ are equivalent —we write $t_1 \sim t_2$— if there is another tensor $t_3 \in Ker\circledast_L$ such that $t_1 = t_2 + t_3$. Thus, we can define the equivalence class of a given tensor t_1 as

$$[t_1] = \{t_2 : t_2 \sim t_1\} = \{t_2 : \, t_1 - t_2 \in Ker\circledast_L\}.$$

In this paper we will deal with *symmetric* products —that is, symmetric bilinear operators—, but this is not needed in some cases, as the standard case of the injective tensor product itself that will be explained later on.

We will write $E \otimes_{/\circledast} E$ for the corresponding quotient space, that is obviously a linear space. Let us define now a "quotient injective" norm for it. Recall that we are always assuming that $\circledast$ is continuous.

Definition 2.1. Let E be a Banach space and let $\circledast : E \times E \to F$ be a continuous product on it. Consider the (algebraic) symmetric tensor product $E \otimes E$. We define the functional $\varepsilon_\circledast$ by

$$\varepsilon_\circledast(t) := \Big\| \sum_{i=1}^n x_i \circledast y_i \Big\|_F, \quad t = \sum_{i=1}^n x_i \otimes y_i \in E \otimes E.$$

Lemma 2.2. *The functional* $\varepsilon_{\circledast}$ *is a norm on* $E \otimes_{/\circledast} E$.

Proof. First, note that $\varepsilon_{\circledast}$ is well-defined. Indeed, it is independent of the representation of the tensor t and of the particular element of the equivalence class of t, since if r is another tensor of the equivalence class, we have that there is a tensor p such that $t = r + p$, where $p = \sum_{i=1}^n x_{i,p} \otimes y_{i,p}$ satisfies that $\| \sum_{i=1}^n x_{i,p} \circledast y_{i,p} \|_F = 0$. It is homogeneous with respect to multiplication by scalars, and also subadditive, since it is defined by means of a norm. So we only need to prove that it separates points. Take a nontrivial tensor $t = \sum_{i=1}^n x_i \otimes y_i$; if $\| \sum_{i=1}^n x_i \circledast y_i \|_F = 0$, then we have that $t \in [0]$, and so it is zero. □

As usual, we will identify the classes of tensors in each space $E \otimes_{/\circledast} E$ with the tensor that define the class itself; so we will write t instead of $[t]$ if there is no risk of confusion. The following lemma provides alternative formulas for $\varepsilon_{\circledast}$ in the case of subspaces of $\mathcal{C}(K)$-spaces.

Lemma 2.3. *Suppose that* E *and* F *are isometric to subspaces of a space* $\mathcal{C}(K)$ *for a certain (Hausdorff) compact set* K*, and the isometries are defined by the operators* $i_E : E \to \mathcal{C}(K)$ *and* $i_F : F \to \mathcal{C}(K)$*, respectively. If* $i_F(x \circledast y) = i_E(x) \cdot i_E(y)$ *is satisfied for* $\circledast : E \times E \to F$ *and for all* $x, y \in E$*, then*

$$\varepsilon_{\circledast}(t) = \sup_{w \in K} \Big| \sum_{i=1}^n \lambda_i \, (i_E(x_i) \cdot i_E(y_i))(w) \Big| = \sup_{\mu \in B_{(\mathcal{C}(K))^*}} \Big| \sum_{i=1}^n \lambda_i \int_K i_E(x_i) \cdot i_E(y_i) \, d\mu \Big|,$$

for $t = \sum_{i=1}^n \lambda_i \, x_i \otimes y_i \in E \otimes E$.

Indeed, note that if F is a subspace of $\mathcal{C}(K)$, the norm is given by the first formula. The second one holds just by duality.

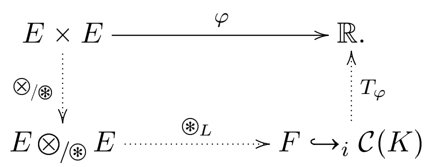
Inspired by the definition of the norm $\varepsilon_{\circledast}$ and the previous result, we define the following class of continuous real valued bilinear forms. Note that, in the situation described in Lemma 2.3 and for $i_E : E \to \mathcal{C}(K)$, $F \subseteq \mathcal{C}(K)$, $i_F = i : F \hookrightarrow \mathcal{C}(K)$ and the product $x \circledast y = i_E(x) \cdot i_E(y)$, the next definition gives the classical integral bilinear forms.

Definition 2.4. Let $\varphi : E \times E \to \mathbb{R}$ be a continuous bilinear form and consider an F-valued product $\circledast$ in E. We say that φ is $\circledast$*-integral* if there is a compact set K such that F is isometrically isomorphic to a subspace of $\mathcal{C}(K)$ —we write i for this identification, and we omit it in case F is already a subspace—, and there is a Borel regular measure η on K such that

$$\varphi(x, y) = \int_K i(x \circledast y)(w) \, d\eta(w), \quad x, y \in E.$$

We will write $\mathcal{IF}_{\circledast}(E \times E)$ for the space of all these bilinear forms, that is clearly linear and becomes a normed space with the usual supremum norm for bilinear functionals.

The following factorization scheme shows the nature of the $\circledast$-integral bilinear forms, and will be useful to prove the general duality theorem that is the main result of this section.



Theorem 2.5. *Let E be a Banach space and $\circledast : E \times E \to F$ be a product on it. Then the following equality holds isomorphically,*

$$(E \otimes_{/\circledast, \varepsilon_\circledast} E)^* = \mathcal{IF}_\circledast(E \times E).$$

In particular, the compact set K in the integral representation of the functionals in this space can be chosen to be B_{F^*}. Moreover, if $\circledast$ is an n.p. product, the measure η in the integral representation of $\varphi \in (E \otimes_{/\circledast, \varepsilon_\circledast} E)^*$ can be chosen to satisfy

$$\|\eta\| = |\eta|(B_{F^*}) = \|\varphi\|_{(E \otimes_{/\circledast, \varepsilon_\circledast} E)^*}.$$

Proof. First, take a bilinear functional $\varphi \in (E \otimes_{/\circledast, \varepsilon_\circledast} E)^*$. Then we have that there is a constant $K > 0$ such that for every tensor $t = \sum_{i=1}^n \lambda_i x_i \otimes y_i$,

$$|\varphi(t)| \leqslant K \Big\| \sum_{i=1}^n \lambda_i x_i \circledast y_i \Big\|_F.$$

This implies that φ is $\circledast-$factorable, so by Lemma 1 in [18] we have that it can be factored as $\varphi = h \circ \circledast$ for a certain $h \in F^*$. Recall that any Banach space is isometric to a subspace of the space $C(K)$ for a suitable compact K; indeed, the identification $z \mapsto i(z) := \langle z, \cdot \rangle \in C(B_{F^*})$, $z \in F$, clearly provides such an isometry. Thus, we can consider the subspace $i(F) \subset \mathcal{C}(B_{F^*})$, and so we get that there is a functional $h' : i(F) \to \mathbb{R}$ such that $h(z) = h'(i(z))$ for all $z \in F$. The Hahn-Banach extension of this functional gives a regular Borel measure $\eta \in (C(B_{F^*}))^*$ such that

$$\varphi(x, y) = \int_{B_{F^*}} i(x \circledast y)\, d\eta, \quad x, y \in F,$$

and the result is obtained.

Conversely, take a $\circledast$-integral bilinear operator φ. Then there is a regular Borel measure η over K such that for every tensor as $t = \sum_{i=1}^n \lambda_i x_i \otimes y_i$,

$$|\varphi(t)| = \Big| \int_K i(\sum_{i=1}^n \lambda_i x_i \circledast y_i)\, d\eta \Big| \leqslant |\eta|(K) \cdot \sup_{w\in K} \Big| i(\sum_{i=1}^n \lambda_i x_i \circledast y_i)(w) \Big|$$

$$= |\eta|(K) \cdot \Big\| i(\sum_{i=1}^n \lambda_i x_i \circledast y_i) \Big\|_{\mathcal{C}(K)}$$

$$= |\eta|(K) \cdot \Big\| \sum_{i=1}^n \lambda_i x_i \circledast y_i \Big\|_F .$$

Therefore, φ is $\varepsilon_\circledast$-continuous, and so the result holds. Looking at the definition of the norms in the involved spaces, it can be easily seen that the identification is in fact an isometry if $\circledast$ is an n.p. product. $\square$

Theorem 2.6. *Let E, F be Banach spaces and let $\circledast$ be an F-valued n.p. product. A bilinear form φ on $E \times E$ is $\circledast$-integral if and only if there exist a finite measure space (Ω, Σ, μ) and an operator $A : F \to L^\infty(\mu)$ such that*

$$\varphi(x, y) = \int_\Omega A(x \circledast y) d\mu$$

for every $x, y \in E$. Moreover, $\|\varphi\|_{(E\otimes_{/\circledast,\varepsilon_\circledast} E)^*} = \inf \|A\|\mu(\Omega)$, where the infimum is taken over all such kind of factorizations of φ.

Proof. Suppose that φ is a $\circledast$-integral bilinear form. Then by Definition 2.4 and Theorem 2.5, there is a regular Borel measure η on B_{F^*} such that $\varphi(x, y) = \int_{B_{F^*}} i(x \circledast y)(z')d\eta(z')$, $z' \in B_{F^*}$, every $x, y \in E$ and $\|\varphi\|_{(E\otimes_{/\circledast,\varepsilon_\circledast} E)^*} = \|\eta\|$. That is, using the canonical identification $f(z') = z'(x \circledast y)$ for all $f \in \mathcal{C}(B_{F^*})$ and $x \circledast y \in F$, we obtain $\varphi(x, y) = \int_{B_{F^*}} \langle x \circledast y, z'\rangle d\eta$, $z' \in B_{F^*}$. By the Radon-Nikodym Theorem, there is a Borel measurable function ϕ on B_{F^*} such that $|\phi(z')| = 1$ for every $z' \in B_{F^*}$ and $d\eta = \phi d|\eta|$. Let $\mu = |\eta|$ and $\Omega = B_{F^*}$ with the Borel σ-algebra Σ.

Let us define the map $A : F \to L^\infty(\mu)$ given by $A(z)(z') = \phi(z') \cdot \langle z, z'\rangle$, $z \in F$, $z' \in B_{F^*}$. Therefore, $\varphi(x, y) = \int_\Omega A(x \circledast y)d\mu$ for all $x, y \in E$. Since, $\|A\| \leqslant 1$, we obtain $\|\varphi\|_{(E\otimes_{/\circledast,\varepsilon_\circledast} E)^*} = \|\eta\| = \mu(\Omega) \geqslant \|A\|\mu(\Omega)$.

Conversely, if φ has such a factorization, it clearly follows that φ is a $\circledast$-integral bilinear form and $\|\varphi\|_{(E\otimes_{/\circledast,\varepsilon_\circledast} E)^*} \leqslant \|A\|\mu(\Omega)$. $\square$

3. Main classes of generalized integral forms and polynomials

Let us present now the main examples of our duality formula for integral bilinear forms and quotient tensor products. Our aim is to show that most of the cases of integral representations of bilinear forms that can be found in the scientific literature can be understood in our setting: classical Grothendieck's integral forms, symmetric integral bilinear forms —in the context of the scalar valued 2-homogeneous polynomials–, disjointness preserving bilinear functionals and polynomials —called orthosymmetric bilinear forms in the context of the Banach lattices—, bilinear forms involving convolution, and some other examples coming from different fields.

3.1. **Integral bilinear forms.** The canonical context in which the formula provided in Theorem 2.5 holds is the well-known duality theorem due to Grothendieck for the injective tensor product, that asserts that the dual space $(E \otimes_\varepsilon E)^*$ can be identified with the space $\mathcal{IF}(E \times E)$ of all the integral bilinear functionals φ which can be written as

$$\varphi(x,y) := \int_{B_{E^*} \times B_{E^*}} \langle x, x' \rangle \langle y, y' \rangle \, d\eta(x', y'), \quad x, y \in E,$$

for a certain regular Borel measure η on $B_{E^*} \times B_{E^*}$ (see for example [14, pp.52-54]). Recall that the injective norm is given by

$$\varepsilon\Big(\sum_{i=1}^n x_i \otimes y_i \Big) = \sup_{(x',y') \in B_{E^*} \times B_{E^*}} \Big| \sum_{i=1}^n \langle x_i, x' \rangle \langle y_i, y' \rangle \Big|.$$

In this case, the (in general non-symmetric) product $\circledast$ can be defined as follows. Consider as F the subspace of $\mathcal{C}(B_{E^*} \times B_{E^*})$ defined by the linear hull of the functions $B_{E^*} \times B_{E^*} \ni (x', y') \mapsto \langle x, x' \rangle \langle y, y' \rangle \in F$. Then, the product is given by the following rule: in $x, y \in E$, we can define a function in F by the formula

$$x \circledast y \, (x', y') := \langle x, x' \rangle \langle y, y' \rangle, \quad x', y' \in B_{E^*}.$$

In this case, if we consider a tensor $t = \sum_{i=1}^n x_i \times y_i$, we have that

$$\varepsilon_\circledast(t) = \Big\| \sum_{i=1}^n \langle x_i, \cdot \rangle \langle y_i, \cdot \rangle \Big\|_{\mathcal{C}(B_{E^*} \times B_{E^*})}$$

$$= \sup_{(x',y') \in B_{E^*} \times B_{E^*}} \Big| \sum_{i=1}^n \langle x_i, x' \rangle \langle y_i, y' \rangle \Big| = \varepsilon(t).$$

Therefore, Theorem 2.5 gives the well-known formula

$$(E \otimes_\varepsilon E)^* = \mathcal{IF}(E \times E).$$

3.2. **Real valued 2-homogeneous polynomials in Banach spaces.** A similar construction can be done for the symmetric version of the injective tensor product. Usually, this is applied in the context of the analysis of 2-homogeneous polynomials by means of the definition of the so called symmetric tensor product $E \otimes_s E$. The elements of such space are symmetric tensors, which are invariant by changing the order of the factors, as for example $x \otimes y + y \otimes x$. It can be proved —by a direct application of the Sylvester's Law of Inertia [30]— that all these tensors allow a diagonal representation, that is, they can be written as sums of single tensors as $\lambda \, x \otimes x$, $\lambda \in \mathbb{R}$, $x \in E$. The linear span of all the symmetric tensors —equivalently, of all the finite sums of single tensors as $\lambda\, x \otimes x$—, define the so-called symmetric tensor product, in which a symmetric injective norm can be defined by

$$\varepsilon_s(t) := \sup_{x' \in B_{E^*}} \Big| \sum_{i=1}^n \langle x_i, x' \rangle \langle y_i, x' \rangle \Big|, \quad t = \sum_{i=1}^n x_i \otimes y_i \in E \otimes_s E$$

(see [19] for more information). Since it is independent of the representation, this formula coincides with the one that is usually given, in which only sums of tensors as $\lambda\, x \otimes x$ appear; it is well-known that it is a norm.

However, it can also be understood as a consequence of our construction. We can define a first linear map S from $E \otimes E$ to $E \otimes_s E$ given by the symmetrization of the tensors, that is $S(x \otimes y) = (x \otimes y + y \otimes x)/2$ for $x, y \in E$, and extended by linearity. The kernel of S and the corresponding space of equivalence classes with respect to the quotient by S can be identified with the symmetric tensor product $E \otimes_s E$. Note that, by Sylvester's Law, there is always an element as $\sum_{i=1}^n \lambda_i x_i \otimes x_i$ in each equivalence class.

Take the subspace F of $\mathcal{C}(B_{E^*})$ that is defined by all the symmetric tensors $t = \sum_{i=1}^n x_i \otimes y_i$ when considered as functions acting in B_{E^*} as $t(x') = \sum_{i=1}^n \langle x_i, x' \rangle \langle y_i, x' \rangle$. Write I for the corresponding map $I : E \otimes_s E \to F$. Thus, we can consider the product $\circledast : E \times E \to F \subseteq \mathcal{C}(B_{E^*})$ as the composition of the (continuous) maps

$$\circledast : E \times E \to^{\otimes} E \otimes_\varepsilon E \to^{S} E \otimes_{s,\,\varepsilon_s} E \to^{I} F \hookrightarrow \mathcal{C}(B_{E^*}).$$

That is, $x \circledast y\, (x') = \langle x, x' \rangle \langle y, x' \rangle$ for every $x, y \in E$.

Note that, in this case, an integral bilinear functional $\varphi \in (E \otimes_{s,\varepsilon_s} E)^*$ can be represented as

$$\varphi(t) = \sum_{i=1}^n \lambda_i \int_{B_{F^*}} \langle x_i, x' \rangle \langle y_i, x' \rangle \, d\eta(x'), \quad t = \sum_{i=1}^n \lambda_i x_i \otimes y_i \in E \otimes_s E,$$

for a certain Borel regular measure η over B_{F^*}. Write $K = B_{E^*}$, and note that $\varepsilon_\circledast = \varepsilon_s$. Thus, the known duality relation $(E \otimes_{s,\varepsilon_s} E)^* = \mathcal{IF}_s(E \times E)$ is a particular case of our representation Theorem 2.5, since $E \otimes_{s,\varepsilon_s} E = E \otimes_{/\circledast, \varepsilon_\circledast} E$, and for every $x, y \in E$,

$$\int_{B_{F^*}} \langle x, x' \rangle \langle y, x' \rangle \, d\eta(x') = \int_K x \circledast y\, (x')\, d\eta(x').$$

3.3. **Orthogonally additive polynomials.** If we add the property of being zero product preserving to the fact of being symmetric, we obtain the so called class of orthogonally additive polynomials. We can find this notion for n-homogeneous polynomials in Banach lattices, Banach algebras —in particular $C(K)$-spaces—, convolution algebras and Fourier algebras (see [3, 13, 20, 29] and the references therein).

1) 2-*homogeneous polynomials on the function space* $\mathcal{C}(K)$.

If we consider the pointwise product $\odot : \mathcal{C}(K) \times \mathcal{C}(K) \to \mathcal{C}(K)$, we can construct the quotient space of the injective tensor product $\mathcal{C}(K)\otimes_{/\odot,\varepsilon_\odot}\mathcal{C}(K)$ with the quotient injective norm $\varepsilon_\odot$ given by

$$\varepsilon_\odot(h) = \Big\| \sum_{i=1}^{n} f_i \odot g_i \Big\|_{\mathcal{C}(K)}, \quad h = \sum_{i=1}^{n} f_i \otimes g_i \in \mathcal{C}(K) \otimes \mathcal{C}(K).$$

This quotient injective tensor product $\mathcal{C}(K) \otimes_{/\odot,\varepsilon_\odot} \mathcal{C}(K)$ can be identified with the function space $\mathcal{C}(K)$. If we consider the canonical linear map $J : \mathcal{C}(K) \otimes_{/\odot,\varepsilon_\odot} \mathcal{C}(K) \to \mathcal{C}(K)$ defined by $J(h)(t) = \sum_{i=1}^n f_i(t)g_i(t)$, where $\sum_{i=1}^n f_i \otimes g_i$ is a representation of h and $t \in K$, we have

$$\begin{aligned} \|J(h)\|_{\mathcal{C}(K)} &= \sup_{t\in K} \Big| \sum_{i=1}^{n} f_i(t)g_i(t) \Big| \\ &= \Big\| \sum_{i=1}^{n} f_i \odot g_i \Big\|_{\mathcal{C}(K)} = \varepsilon_\odot\Big(\sum_{i=1}^{n} f_i \otimes g_i \Big) = \varepsilon_\odot(h). \end{aligned}$$

Let us show that $J(\mathcal{C}(K)\otimes_{/\odot,\varepsilon_\odot}\mathcal{C}(K))$ is dense in $\mathcal{C}(K)$. Consider $h \in \mathcal{C}(K)$. Let us write $\mathbf{1}$ for the unit element of $\mathcal{C}(K)$, that is, $\mathbf{1}(t) = 1$ for all $t \in K$. Let $g = \mathbf{1} \otimes h \in \mathcal{C}(K) \otimes_{/\odot,\varepsilon_\odot} \mathcal{C}(K)$. Then,

$$\|J(g) - h\|_{\mathcal{C}(K)} = \sup_{t\in K} |h(t) - J(g)(t)| = \sup_{t\in K} |h(t) - \mathbf{1}(t)h(t)| = 0.$$

Therefore, $J : \mathcal{C}(K) \otimes_{/\odot,\varepsilon_\odot} \mathcal{C}(K) \to \mathcal{C}(K)$ is an isometric isomorphism. As a result $(\mathcal{C}(K) \otimes_{/\odot,\varepsilon_\odot} \mathcal{C}(K))^*$ and $(\mathcal{C}(K))^*$ are isomorphic. Since the dual space of $\mathcal{C}(K)$ is the space $\mathcal{M}(K)$ of the regular Borel measures on K, we get $\mathcal{IF}_\odot(\mathcal{C}(K) \times \mathcal{C}(K)) = (\mathcal{C}(K) \otimes_{/\odot,\varepsilon_\odot} \mathcal{C}(K))^* = (\mathcal{C}(K))^* = \mathcal{M}(K)$ by using Lemma 2.5.

Corollary 3.1. *The equality $\mathcal{IF}_\odot(\mathcal{C}(K) \times \mathcal{C}(K)) = \mathcal{P}_0(^2\mathcal{C}(K))$ holds isomorphically. In particular, every 2-homogeneous orthogonally additive polynomial on $\mathcal{C}(K)$ has an associated bilinear map that is $\varepsilon_\odot$-continuous.*

Proof. Let φ be a $\odot$-integral bilinear form. Notice that there is a unique orthogonally additive 2-homogeneous polynomial $P_\varphi : \mathcal{C}(K) \to \mathbb{R}$ such that $\varphi(f, f) = P_\varphi(f)$ for all $f \in \mathcal{C}(K)$. Indeed, by the integral representation of φ, it is easily seen that it is symmetric and zero product preserving. So, it defines a 2-homogeneous polynomial. Proposition 2.2. in [20] states that a 2-homogeneous polynomial form on $\mathcal{C}(K)$ is orthogonally additive if and only if its associated bilinear map is zero product preserving. Therefore, by the zero product preservation of φ, it is obtained that P_φ is orthogonally additive.

Now, let us show that the correspondence $\varphi \longleftrightarrow P_\varphi$ is an isomorphism. Let us consider a 2-homogeneous orthogonally additive polynomial P and its associated bilinear form B. By the first theorem in [13], it is seen that P has an integral representation as $P(f) = \int_K f^2 d\nu$, where ν is a regular

Borel measure on K. Therefore its associated bilinear map has the following integral representation;

$$B(f,g) = \frac{P(f+g) - P(f) - P(g)}{2} = \int_K fg d\nu.$$

This shows that B is a $\odot$-integral bilinear form and it is clearly unique. Using the results proved in Section 2, we also obtain that the associated bilinear form B of P is $\varepsilon_\odot$-continuous. □

Remark 3.2. The isomorphism given above can be also seen by using the relation between orthogonally additive polynomials and regular Borel measures. The integral representation $P(f) = \int_K f^2 d\nu$ given in [13] for the orthogonally additive 2-homogeneous polynomial $P \in \mathcal{P}_0(^2\mathcal{C}(K))$ implies a canonical isomorphism between $\mathcal{P}_0(^2\mathcal{C}(K))$ and the space $\mathcal{M}(K)$ of regular Borel measures on K. From this isomorphisn and the isomorhism between $\mathcal{C}(K) \otimes_{/\odot,\varepsilon_\odot} \mathcal{C}(K)$ and $\mathcal{C}(K)$, we get

$$\mathcal{IF}_\odot(\mathcal{C}(K)\times\mathcal{C}(K)) = (\mathcal{C}(K)\otimes_{/\odot,\varepsilon_\odot}\mathcal{C}(K))^* = (\mathcal{C}(K))^* = \mathcal{M}(K) = \mathcal{P}_0(^2\mathcal{C}(K)),$$

and this relation holds isomorphically by Lemma 2.5.

2) *Bilinear operators on Banach function spaces that factor through the pointwise product.*

Let us show that orthogonally additive 2-homogeneous real-valued polynomials that act in Banach function spaces —which coincide with bilinear maps factoring through the pointwise product when the factor spaces are the same—, can be characterized also using Theorem 2.5. For the aim of simplicity we will consider the pointwise product in $L^2[0,1]$, but the reader can notice easily that the same construction works for every pointwise self-product of Banach function spaces (see [18, 22] and references therein). Let μ be Lebesgue measure on $[0,1]$ and consider the μ-a.e. pointwise product $\odot : L^2[0,1] \times L^2[0,1] \to L^1[0,1]$. The quotient tensor product $L^2[0,1] \otimes_{/\odot} L^2[0,1]$ is defined by the equivalence classes

$$[t] = \Big\{ \sum_{i=1}^m f_i' \otimes g_i' \in L^2[0,1] \otimes L^2[0,1] : \sum_{i=1}^n f_i \cdot g_i - \sum_{i=1}^m f_i' \cdot g_i' = 0 \, \mu - a.e. \Big\},$$

for every tensor $t = \sum_{i=1}^n f_i \otimes g_i$. A $\odot$-integral bilinear functional φ is then given by the factorization through the μ-a.e. pointwise product. In fact, for this case we can obtain two different representations for such a functional.

First, the continuity of the map φ with respect to $\varepsilon_\odot$ gives that it factors through $L^1[0,1]$ as $\varphi = T_\varphi \circ \odot$ for a certain linear and continuos operator $T_\varphi : L^1[0,1] \to \mathbb{R}$ (see [18, Corollary 1]). Therefore, there is a function

$h \in L^\infty[0,1]$ such that

$$\begin{aligned}\varphi(f,g) = T_\varphi(f \odot g) &= \int_{[0,1]} \big(f(w)\cdot g(w)\big)\cdot h(w)\,d\mu(w) \\ &= \int_{[0,1]} f(w)\,g(w)\,h(w)\,d\mu(w), \quad f,g \in L^2[0,1].\end{aligned}$$

On the other hand, $L^1[0,1]$ can be embedded isometrically in $\mathcal{C}(B_{L^\infty[0,1]})$ by the canonical inclusion $i : L^1[0,1] \to \mathcal{C}(B_{L^\infty[0,1]})$, and so the product can be considered as taking values in a subspace of this space. This gives an alternate representation of φ as

$$\varphi(f,g) = \int_{B_{L^\infty[0,1]}} \Big(\int_{[0,1]} f(w)\,g(w)\,h(w)\,d\mu(w) \Big)\ d\eta(h), \quad f,g \in L^2[0,1],$$

for a certain regular Borel measure η.

Note that this class of bilinear forms coincides with the orthogonally additive homogeneous polynomials of degree 2 with respect to the pointwise product. Indeed, for a $\odot$-integral bilinear form φ, define the map $p : L^2[0,1] \to \mathbb{R}$ by $p(f) = \varphi(f,f)$. The operator p defines a 2-homogeneous polynomial since φ is symmetric. By the zero product preservation of the functional φ we get that there is a function $h \in L^\infty[0,1]$ such that for disjoint functions $f,g \in L^2[0,1]$,

$$\begin{aligned}p(f+g) &= \varphi(f+g, f+g) \\ &= \int_{[0,1]} f(w)^2\,h(w)\,d\mu(w) + 2\int_{[0,1]} f(w)\,g(w)\,h(w)\,d\mu(w) \\ &+ \int_{[0,1]} g(w)^2\,h(w)\,d\mu(w) = \varphi(f,f) + \varphi(g,g) = p(f) + p(g).\end{aligned}$$

It shows that p is an orthogonally additive polynomial. On the other hand, if p is an orthogonally additive polynomial it can be written as an integral as

$$p(f) := \int_{[0,1]} f^2\,h\,d\mu, \quad f \in L^2[0,1],$$

for a given $h \in L^\infty[0,1]$ (see [29, Theorem 2] or [10, Corollary 2.5]). Its associated bilinear form, that is given by

$$\varphi(f,g) := \frac{p(f+g) - p(f) - p(g)}{2}, \quad f,g \in L^2[0,1],$$

is clearly an $\odot$-integral bilinear form.

3) *Convolution-orthogonal 2-homogeneous polynomials.*

As in the pointwise-product case, it is possible to obtain integral representations for bilinear forms defined on the convolution algebras on a given compact group. Let $\mathbb{T}$ denote the circle group —the real line mod 2π— and let $*_c$ be the convolution operation defined by

$$f *_c g(x) = \int_{\mathbb{T}} f(x-y)g(y)d\mu(y)$$

for scalar-valued measurable functions f, g satisfying the necessary integration requirements.

Let us consider the convolution $*_c$ defined from $L^1(\mathbb{T}) \times L^1(\mathbb{T})$ to $L^1(\mathbb{T})$. It is known that $L^1(\mathbb{T}) *_c L^1(\mathbb{T}) = L^1(\mathbb{T})$ by the Cohen's factorization theorem. We consider the quotient tensor product $L^1(\mathbb{T}) \otimes_{/*_c} L^1(\mathbb{T})$ given by the equivalence classes

$$[t] = \Big\{ \sum_{i=1}^{m} f_i' \otimes g_i' \in L^1(\mathbb{T}) \otimes L^1(\mathbb{T}) : \sum_{i=1}^{n} f_i *_c g_i - \sum_{i=1}^{m} f_i' *_c g_i' = 0 \Big\}, \quad t = \sum_{i=1}^{n} f_i \otimes g_i.$$

So, by Theorem 2.5 the $*_c$-integral bilinear functionals are defined by convolution as

$$\phi(f,g) = \int_{B_{L^\infty(\mathbb{T})}} i(f *_c g) d\eta = \int_{B_{L^\infty(\mathbb{T})}} i\Big(\int_{\mathbb{T}} f(x-y)g(y)d\mu(y) \Big) d\eta$$

with a regular Borel measure η on $B_{L^\infty(\mathbb{T})}$. As in the pointwise-product case, this can be improved if we consider the zero product preservation. In this case, ϕ factors through $L^1(\mathbb{T})$ by convolution as $\phi = T \circ *_c$ (see [17, Theorem 3.4]). Thus, there is a functional in $(L^1(\mathbb{T}))^*$ —that is, a function $h \in L^\infty(\mathbb{T})$— such that

$$\phi(f,g) = \int_{\mathbb{T}} (f *_c g)(x) h(x) d\lambda(x). \tag{3.1}$$

On the other hand, by using the canonical inclusion of $L^1(\mathbb{T})$ in $\mathcal{C}(B_{(L^1(\mathbb{T}))^*})$ we also get

$$\begin{aligned}\phi(f,g) &= \int_{B_{L^\infty(\mathbb{T})}} \int_{\mathbb{T}} (f *_c g)(x) h(x) d\lambda(x) d\eta(h) \\ &= \int_{B_{L^\infty(\mathbb{T})}} \int_{\mathbb{T}} \Big(\int_{\mathbb{T}} f(x-y)g(y)d\mu(y) \Big) h(x) d\lambda(x) d\eta(h).\end{aligned}$$

These bilinear forms coincide with the convolution-orthogonally additive 2-homogeneous polynomials defined on $L^1(\mathbb{T})$. To show this let us use the integral representation given in (3.1). If we define the map $p : L^1(\mathbb{T}) \to \mathbb{R}$ by $p(f) = \phi(f,f)$ for all $f \in L^1(\mathbb{T})$, by commutativity of the convolution it defines a 2-homogeneous polynomial which is convolution-orthogonally additive, since

$$p(f+g) = \phi(f+g, f+g) = \int_{\mathbb{T}} ((f+g) *_c (f+g))(x) h(x) d\lambda(x)$$
$$= \int_{\mathbb{T}} (f *_c f)(x) h(x) d\lambda(x) + 2 \int_{\mathbb{T}} (f *_c g)(x) h(x) d\lambda(x)$$
$$+ \int_{\mathbb{T}} (g *_c g)(x) h(x) d\lambda(x) = \phi(f,f) + \phi(g,g) = p(f) + p(g)$$

for each pair of functions $f, g \in L^1(\mathbb{T})$ whose convolution is zero.

Conversely, consider a 2-homogeneous convolution-orthogonally additive polynomial p defined on $L^1(\mathbb{T})$. Then, there is a unique continuous linear operator $T : L^1(\mathbb{T}) \to \mathbb{R}$ such that $p(f) = T(f *_c f)$ (see [3, Theorem 3.1]). By the Riesz's representation theorem we get an integral representation

$$T(f *_c f) = \int_{\mathbb{T}} (f *_c f) h d\lambda,$$

where $h \in L^\infty(\mathbb{T})$. Consequently, the associated bilinear form

$$\phi(f,g) = T \circ *_c(f,g) = \frac{p(f+g) - p(f) - p(g)}{2} \quad (f, g \in L^1(\mathbb{T}))$$

defines a $*_c$-integral bilinear form.

Remark 3.3. We can also give a class of $*_c$-integral bilinear forms *which does not coincide with the corresponding class of orthogonally additive* 2-*homogeneous polynomials.* Let $U(\mathbb{T})$ denote the Banach spaces $L^2(\mathbb{T})$ or $\mathcal{C}(\mathbb{T})$. The equivalence classes

$$[t] = \Big\{ \sum_{i=1}^m f_i' \otimes g_i' \in U(\mathbb{T}) \otimes U(\mathbb{T}) : \sum_{i=1}^n f_i *_c g_i - \sum_{i=1}^m f_i' *_c g_i' = 0 \Big\}, \quad t = \sum_{i=1}^n f_i \otimes g_i.$$

define the quotient tensor product $U(\mathbb{T}) \otimes_{/*_c} U(\mathbb{T})$. It is known that $L^2(\mathbb{T}) *_c L^2(\mathbb{T}) = W(\mathbb{T})$ and $\mathcal{C}(\mathbb{T}) *_c \mathcal{C}(\mathbb{T}) \subset W(\mathbb{T})$, where $W(\mathbb{T})$ is the so-called Wiener algebra, the Banach space of the functions having absolutely summable Fourier series. It is well-known that $W(\mathbb{T})$ is isometrically isomorphic to $\ell^1(\mathbb{Z})$, and the isomorphism is given by the Fourier transform. It is a subset of both $\mathcal{C}(\mathbb{T})$ and $L^2(\mathbb{T})$ (see [21, §34.40]), and so $U(\mathbb{T}) \otimes_{/*_c} U(\mathbb{T}) \subseteq W(\mathbb{T})$.

Now, consider a $*_c$-integral bilinear form $\phi : U(\mathbb{T}) \times U(\mathbb{T}) \to \mathbb{R}$. As in the other cases, the Wiener algebra $W(\mathbb{T})$ can be embedded isometrically in $\mathcal{C}(B_{(W(\mathbb{T}))^*})$ by the canonical inclusion $i : W(\mathbb{T}) \to \mathcal{C}(B_{W(\mathbb{T})^*})$. Using our representation technique, for every $*_c$-integral bilinear functional ϕ we get a regular Borel measure η on $B_{(W(\mathbb{T}))^*}$ such that

$$\phi(f,g) = \int_{B_{(W(\mathbb{T}))^*}} \langle f *_c g, \varphi \rangle d\eta(\varphi) = \int_{B_{(W(\mathbb{T}))^*}} \Big\langle \int_{\mathbb{T}} f(x-y) g(y) d\mu(y), \varphi \Big\rangle d\eta(\varphi).$$

Such a $*_c$-integral bilinear form defines a convolution-orthogonally additive homogeneous polynomial of degree 2. Indeed, based on the integral representation of ϕ, some computations as the ones given above shows that the formula $p(f) = \phi(f, f)$, $f \in U(\mathbb{T})$, provides an orthogonally additive polynomial $p : U(\mathbb{T}) \to \mathbb{R}$.

However, the converse is not true in general. In fact, for a convolution-orthogonally additive 2-homogeneous polynomial p defined on $L^2(\mathbb{T})$ or $\mathcal{C}(\mathbb{T})$, it is not always possible to find a continuous linear map T such that $p(f) = T(f *_c f)$ (see [3, Section 4] for counterexamples). Therefore, these polynomials does not define in general $*_c$-integral bilinear forms.

4. Symmetric and non-symmetric products on spaces of operators

In this section and following the method we have explained in the previous ones, we will describe the natural spaces of integral forms that can be associated to the composition product on spaces of operators. As we will show, two opposite cases appear, the first one connected with the classical formula of the integral operators and a factorization through a $\mathcal{C}(K)$ space, and the second one with a factorization through an L^1-space.

Let E be a Banach space and let $L(E, E)$ the space of continuous linear operators on it. Let $U(E, E)$ be a —not necessarily closed— subspace of $L(E, E)$, for example a component of an operator ideal. Although the non-symmetric nature of the products that will be given in this section allows the use of different factor spaces, we choose coincidence of both of them for the aim of simplicity —$U(E, E)$ instead of $U(E, F)$, and $U(E, E)$ in both sides of the product—.

4.1. **Main products on spaces of operators.** Let us explain the main examples of products that can be defined for spaces of operators in a natural way and show the characterization of the associated dual spaces, that are identified with some classes of bilinear operators.

1) The *canonical composition product* on spaces of operators can be constructed as follows. It is given by the composition product in $U(E, E)$,

$$\circ : U(E, E) \times U(E, E) \to L(E, E), \quad (T, S) \mapsto T \circ S, \quad S, T \in L(E, E).$$

The natural topology associated to the product $\circ$ for the tensor product is given by

$$\varepsilon_\circ\Big(\sum_{i=1}^n T_i \otimes S_i\Big) = \Big\| \sum_{i=1}^n T_i \circ S_i \Big\|_{L(E,E)},$$

and the linearization of the corresponding $\circ$-integral operators $\varphi : U(E, E) \times U(E, E) \to \mathbb{R}$ have the formula

$$\varphi\Big(\sum_{i=1}^n T_i \otimes S_i\Big) = \int_{\phi \in B_{\left(L(E,E)\right)^*}} \Big(\sum_{i=1}^n \langle T_i \circ S_i, \phi \rangle\Big)\, d\eta(\phi),$$

$\sum_{i=1}^n T_i \otimes S_i \in U(E, E) \otimes U(E, E)$, for a certain Borel regular measure η. As a consequence of Theorem 2.5, we get the isomorphism between the space

$\big(U(E) \otimes_{/\circ,\varepsilon_\circ} U(E)\big)^*$ and the space of all such integral bilinear forms. However, note that a direct representation by means of a fixed $\phi_0 \in (L(E,E))^*$ is also available: a bilinear form belonging to the dual of $U(E) \otimes_{/\circ,\varepsilon_\circ} U(E)$ factors through a subspace of $L(E,E)$, and so it can be extended to the whole space by means of Hahn-Banach Theorem, giving a representation as

$$\varphi(T,S) = \big\langle T \circ S, \phi_0 \big\rangle, \quad S,T \in U(E,E).$$

2) *Trace duality.* Given two linear operators $T,S \in L(E,E)$, we can always define a duality relation among them by means of the trace $tr(T \circ S)$. This is a real number but it is on the basis of the so called trace duality for operator ideals (see for example [15, Ch.6]). In certain particular cases in which a concrete representation of the duality in the space E is available, we can obtain a product having values in a Banach space. Let us explain an easy example that involves Hilbert-Schmidt operators. Let H be the Hilbert space $L^2[0,1]$, write μ for Lebesgue measure and consider $U(E,E)$ to be the space $\mathcal{S}_2(L^2[0,1], L^2[0,1])$ of Hilbert-Schmidt operators, that is a Hilbert space. If $T,S \in \mathcal{S}_2(L^2[0,1], L^2[0,1])$, the inner product is defined using the trace as

$$(T,S) = \sum_{i=1}^{\infty} \int_{[0,1]} T(f_i) S(f_i)\, d\mu$$

where $\{f_i : i \in \mathbb{N}\}$ is an orthonormal basis for $L^2[0,1]$. This formula is independent of the basis, and allows to define a product $I : L^2[0,1] \times L^2[0,1] \to L^1[0,1]$ by fixing the basis and defining

$$I(T,S) = \sum_{i=1}^{\infty} T(f_i)(w) S(f_i)(w) \in L^1[0,1], \quad T,S \in \mathcal{S}_2(L^2[0,1], L^2[0,1]).$$

Thus, the associated integral bilinear forms are given by a formula as

$$\varphi(T,S) := \int_{[0,1]} T(f_i)(w) S(f_i)(w)\, h(w)\, d\mu(w), \quad T,S \in U(E,E),$$

where $h \in L^\infty(\mu)$.

3) The *pointwise evaluation product.* There are several ways of defining classes of integral bilinear forms associated to the pointwise evaluation of the composition of operators.

Take first a non-null vector $x \in E$. A pointwise product $b_x : U(E,E) \times U(E,E) \to E$ can be defined by the formula

$$T\, b_x\, S = (T \circ S)(x) \in E, \quad T,S \in U(E,E).$$

Using our procedure, we can easily get that a b_x-integral bilinear forms allow an expression as

$$\varphi(T,S) = \big\langle T \circ S, x'_\varphi \big\rangle, \quad T,S \in U(E,E)$$

for a certain functional $x'_\varphi \in E^*$.

As usual, in the case of $E = \mathcal{C}(K)$ we obtain an explicit formula involving integrals over K. If $0 \neq f_0 \in \mathcal{C}(K)$, the product $b_{f_0} : U(\mathcal{C}(K), \mathcal{C}(K)) \times U(\mathcal{C}(K), \mathcal{C}(K)) \to \mathcal{C}(K)$ is given by $b_{f_0}(T,S) = T \circ S(f_0) \in \mathcal{C}(K)$. Thus, a b_{f_0}-integral operator φ allows an integral representation formula as

$$\varphi(T,S) = \int_K (T \circ S)(f_0)\, d\eta, \quad T, S \in U(\mathcal{C}(K), \mathcal{C}(K)),$$

for a certain Borel regular measure η on K.

4) The *average products.* There are several ways of defining a product based on the average of pointwise evaluations of operators.

(a) Let (Ω, Σ, μ) be a probability measure space and take a μ-Bochner integrable function $f : \Omega \to \mathbb{E}$. Recall that, if $T : E \to E$ is a linear and continuous operator, we can always define the Bochner integrable function $T(f)$, that is given by $T(\sum_{i=1}^n x_i\, \chi_{A_i}) = \sum_{i=1}^n T(x_i)\, \chi_{A_i}$ for simple functions. Due to Hille's Theorem (see [16, Chapter II, Theorem 2.6]), we have that $T(f)$ is also μ-Bochner integrable, and

$$T\Big(\int_\Omega f(w)\, d\mu(w) \Big) = \int_\Omega T\big(f(w)\big)\, d\mu(w).$$

Fix a function $f_0 \in L^1(\mu, E)$. A first way of defining a Bochner space valued composition product

$$b_{f_0} : U(E,E) \times U(E,E) \to^{\circ} L^1(\mu, E) \to^{I_\mu} E,$$

where I_μ is the integration map on $L^1(\mu, E)$, is

$$b_{f_0}(T,S) := \int_\Omega T \circ S(f_0)(w)\, d\mu(w) \in E, \quad T, S \in U(E).$$

However, note that by the above mentioned theorem of Hille, we have that

$$b_{f_0}(T,S) = \int_\Omega T \circ S(f_0)(w)\, d\mu(w) = T \circ S\Big(\int_\Omega f_0(w)\, d\mu(w) \Big),$$

and so $b_{f_0} = b_{(\int_\Omega f_0 d\mu)}$, and so it is as described above in 3).

(b) However, there is another way of defining an $L^1(\mu)$-valued product. For this aim, fix two Bochner 2-integrable functions $f \in L^2(\mu, E)$ and $g \in L^2(\mu, E^*)$, and consider the product formula

$$\begin{aligned} b_{f,g}(T,S) &= \int_\Omega \langle T \circ S(f(w)), g(w) \rangle\, d\mu(w) \\ &= \int_\Omega \langle S(f(w)), T'(g(w)) \rangle\, d\mu(w), \quad T, S \in U(E,E). \end{aligned}$$

We have that

$$b_{f,g}(T,S) \leqslant \Big(\int_\Omega \|S(f(w))\|^2 d\mu(w) \Big)^{1/2} \Big(\int_\Omega \|T'(g(w))\|^2 d\mu(w) \Big)^{1/2}$$
$$\leqslant \|S\| \, \|T\| \, \|f\|_{L^2(\mu,E)} \, \|g\|_{L^2(\mu,E^*)}.$$

and so the product is well-defined and continuous. The corresponding integral bilinear forms can be represented as

$$\varphi(T,S) := \int_\Omega \langle S(f(w)), T'(g(w)) \rangle \, h_\varphi(w) \, d\mu(w), \quad T,S \in U(E,E),$$

where $h_\varphi \in L^\infty(\mu)$.

(c) A measure average product. It can also coincide with the pointwise evaluation product. Let $A_E \subset B_E$ be a Borel set, and consider a measure space $(A_E, \mathcal{B}(A_E), \eta)$, where $\mathcal{B}(A_E)$ is the class of all the Borel sets of A_E —A_E is endowed with the norm topology—, and η is a Borel measure. Consider the space of Bochner integrable functions $L^1(\eta, E)$, and assume that the E-valued functions $T \circ S$ are strongly $\mathcal{B}(A_E)$-measurable; we can assume for example that the identity map $I : A_E \to A_E$ is η-integrable. Then we have that all of these functions belong to $L^1(\eta, E)$, since

$$\int_{A_E} \big\|T \circ S(x)\big\|_E \, d\eta(x) \leqslant \big(\sup_{A_E} \big\|T \circ S(x)\big\|_E \big) \cdot \eta(A_E) \leqslant \|T\| \, \|S\| \, \eta(A_E).$$

Then the product is given by

$$b(T,S) := \int_{A_E} \big(T \circ S\big)(x) \, d\eta(x) \in E, \quad T,S \in U(E).$$

The simplest example is given by a finite set of vectors $A_E = \{x_i : i = 1,...,n\} \subseteq B_E$, and a probability measure $\eta(C) := \sum_{i=1}^n \alpha_i \delta_{x_i}(C)$, where $\sum_{i=1}^n \alpha_i = 1$ and δ_{x_i} is the Dirac's delta on x_i. All the functions $T \circ S(\cdot)$ are obviously strongly measurable, and so Bochner integrable. Then the product coincides again with a pointwise evaluation product. The same construction can be adapted to the case of norm compact sets of E.

4.2. **Some non-standard products and the associated dual spaces of product integral linear operators.** In the classical cases, the dual of the tensor products involved are well-known, and have provided some of the most relevant results in the theory of spaces of operators. This is the case for example of the dual space of the injective tensor product, which produce the classical space of integral operators, and the case of the adjoint operator ideals—between, for example, summing and integral operators—that are obtained by trace duality. There are, however, some other cases that could be interesting and are associated with non-standard products. We will explain here some of them.

1) *Operators between $\mathcal{C}(K)$-spaces and the pointwise product.* Consider a norm one function $f_0 \in \mathcal{C}(K)$ and the pointwise product

$$b_{f_0} : L(\mathcal{C}(K), \mathcal{C}(K)) \times L(\mathcal{C}(K), \mathcal{C}(K)) \to \mathbb{R}.$$

Then for every functional in

$$\big(L(\mathcal{C}(K), \mathcal{C}(K)) \otimes_{/b_{f_0}, \varepsilon_{b_{f_0}}} L(\mathcal{C}(K), \mathcal{C}(K))\big)^*$$

there is a Borel regular measure $\eta \in (\mathcal{C}(K))^*$ such that

$$\varphi_\eta(T, S) = \int_K T \circ S(f_0)\, d\eta$$

and each such an η defines a b_{f_0}-integral operator. A standard construction using Hahn-Banach Theorem gives that for every norm one function $f_1 \in \mathcal{C}(K)$ there is a norm one operator $S_{0,1}$ such that $S_{0,1}(f_0) = f_1$. Since the same can be done for T, if f_2 is a norming element for η (up to an $\epsilon > 0$) we can find an operator $T_{1,2}$ such that $T_{1,2}(f_1) = f_2$, and so we have that there is a pair of norm one operators such that $T_{1,2} \circ S_{0,1}(f_0) = f_2$, and

$$\langle T_{1,2} \circ S_{0,1}(f_0), \eta\rangle + \epsilon > \|\eta\|.$$

On the other hand, using Lemma 2.3, we get

$$\|\varphi_\eta\| = \sup_{\varepsilon_{b_{f_0}}(\sum T_i \otimes S_i) \leqslant 1} \Big| \sum_{i=1}^n \int_K T_i(S_i(f_0)) d\eta \Big|$$
$$\leqslant \sup_{\varepsilon_{b_{f_0}}(\sum T_i \otimes S_i) \leqslant 1} \Big\| \sum_{i=1}^n \int_K T_i(S_i(f_0)) \Big\|_{\mathcal{C}(K)} \|\eta\| \leqslant \|\eta\|.$$

Therefore, the norm of φ_η satisfies that $\|\varphi_\eta\| = \|\eta\|_{(\mathcal{C}(K))^*}$, and so we have the isometry

$$\big(L(\mathcal{C}(K), \mathcal{C}(K)) \otimes_{/b_{f_0}, \varepsilon_{b_{f_0}}} L(\mathcal{C}(K), \mathcal{C}(K))\big)^* = (\mathcal{C}(K))^* = \mathcal{M}(K),$$

the space of regular Borel measures over K.

2) *Composition-integral maps between spaces of operators.* Let us characterize the space of operators associated to the bilinear forms studied in Section 4.1. In order to do that, we need to introduce a non-symmetric versions of Theorem 2.5. It is easy to check in the proof of this result that the duality formula given there is still valid if the factors do not coincide. That is, we have that, if $\circledast : E \times H \to F$ is a product,

$$(E \otimes_{/\circledast, \varepsilon_\circledast} H)^* = \mathcal{IF}_\circledast(E \times H).$$

Take now as first space $L(F, G)$, and for the second one take the closure $\mathcal{N}_0(E^*, F)$ of the tensor product $E \otimes_\pi F$ endowed with the projective norm π considered as a subspace of nuclear operators (see [14, 3.6]). The product $\circ$ is then well-defined from $L(F, G) \times \mathcal{N}_0(E^*, F) \to L(E^*, G)$. We have that

$$\Big(L(F, G) \otimes_{/\circ, \varepsilon_\circ} \mathcal{N}_0(E^*, F)\Big)^* = \mathcal{IF}_\circ(L(F, G) \times \mathcal{N}_0(E^*, F)).$$

Therefore, we can represent each element of this dual space as $(T,S) \mapsto B_\phi(T,S) = \langle \phi, T \circ S\rangle$, where $\phi \in (L(E^*,G))^*$. Recall that $(E \otimes_\pi F)^* = L(E,F^*)$. Then we have that each bilinear form B_ϕ defines a linear and continuous operator

$$R_\phi : L(F,G) \to \big(\mathcal{N}_0(E^*,F)\big)^* = L(E,F^*)$$

by

$$\langle R_\phi(T), S\rangle := \langle \phi, T \circ S\rangle, \quad T \in L(F,G),\ S \in \mathcal{N}_0(E^*,F).$$

Let us write $\mathcal{I}_\circ\big(L(F,G), L(E,F^*)\big)$ for the space of all the $\circ$-integral operators that are given by these operators just described. Thus, we have shown the following *representation formula:*

$$\mathcal{I}_\circ\big(L(F,G), L(E,F^*)\big) = \Big(L(F,G) \otimes_{/\circ,\varepsilon_\circ} \mathcal{N}_0(E^*,F)\Big)^*.$$

In particular, if $S = \sum_{i=1}^n x_i \otimes y_i$, where $x_i \in E$ and $y_i \in F$, we have that the *formula for the $\circ$-integral linear operators from $L(F,G)$ to $L(E,F^*)$ is*

$$\langle R_\phi(T), S\rangle := \langle \phi, T \circ (\sum_{i=1}^n \langle x_i, \cdot\rangle \otimes y_i)\rangle = \langle \phi, \sum_{i=1}^n \langle x_i, \cdot\rangle T(y_i)\rangle.$$

Moreover, note that $E^* \otimes_\pi G^* \hookrightarrow L(E^*,G)^*$, and so if $\phi = \sum_{j=1}^m x'_i \otimes z'_i$, we have that

$$R_\phi(T)(S) := \Big\langle \sum_{j=1}^m x'_i \otimes z'_i, \sum_{i=1}^n \langle x_i, \cdot\rangle T(y_i)\Big\rangle = \sum_{j=1}^m \sum_{i=1}^n \langle T(y_i), z'_j\rangle \langle x_i, x'_j\rangle.$$

5. Integral representations of operators through Pietsch integral maps and applications

In this section, we will analyze a particular class of factorable summing multilinear maps that are related to the multilinear version of the so called Pietsch integral operators. Our aim is to show how our general representation formulas can be applied. We will assume again that $\circledast$ is an F-valued n.p. product, for a Banach space F.

Definition 5.1. Let X, F, Z be Banach spaces and $1 \leqslant p, q < \infty$. A bilinear operator $B : X \times X \to Z$ is said to be *$\circledast$-factorable (q,p)-summing* for the *surjective* product $\circledast : X \times X \to F$ if there exists a constant $C > 0$ such that for every couple of positive integers A, D and all $A \times D$ matrices (x_{ik}), $(y_{ik}) \in X$, the following equality holds,

$$\Big(\sum_{i=1}^A \Big\| \sum_{k=1}^D B(x_{ik}, y_{ik})\Big\|_Z^q\Big)^{1/q} \leqslant C \sup_{f \in B_{F^*}} \Big(\sum_{i=1}^A \Big| \sum_{k=1}^D \langle x_{ik} \circledast y_{ik}, f\rangle\Big|^p\Big)^{1/p}.$$

This notion is inspired by the definition of *factorable (p,q)-summing multilinear operators* given in [23].

The well-known class of the Pietsch integral operators was extended to the multilinear case and to the framework of the homogeneous polynomials by Alencar, and was also studied by Villanueva (see [4, 31]). We introduce the notion of $\circledast$-Pietsch integral bilinear map in a similar way.

Definition 5.2. We will say that a bilinear operator $B : X \times X \to Z$ is $\circledast$-Pietsch integral for the product $\circledast : X \times X \to F$ if there is a regular (countable additive) Z-valued Borel measure η on B_{F^*} such that

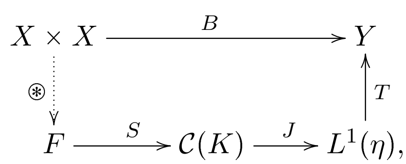
$$B(x, y) = \int_{B_{F^*}} \langle x \circledast y, f \rangle \, d\eta(f), \qquad x, y \in X.$$

The space of $\circledast$-Pietsch integral operators $\mathcal{PI}_{\circledast}(X \times X, Y)$ is a Banach space under the norm $\|B\|_{\mathcal{PI}_{\circledast}} = \inf |\eta|(B_{F^*})$, where the infimum is computed over all the measures satisfying the requirements above.

Remark 5.3. It is clearly seen that a bilinear operator B is $\circledast$-Pietsch integral if and only if it factors through the n.p. product $\circledast$ and a Pietsch integral linear operator. Indeed, consider a $\circledast$-Pietsch integral B and define the map $T : F \to Y$ given by $T(x \circledast y) = B(x, y) = \int_{B_{F^*}} \langle x \circledast y, f \rangle d\eta(f)$ for all $x \circledast y = z \in F$. It is easily seen that T is well defined, linear and independent of the elements x, y appearing in the representation of $x \circledast y$. Moreover, it is continuous since $\|T\| \leqslant \inf |\eta|(B_{F^*})$, and it is a Pietsch integral operator by the representation $T(z) = \int_{B_{F^*}} \langle z, f \rangle \, d\eta(f)$, $x \circledast y = z \in F$, and $B = T \circ \circledast$. The converse is obvious.

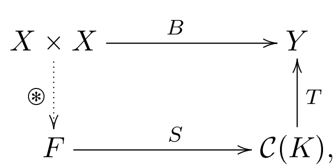
Example 5.4. The bilinear map $B : \mathcal{C}(K) \times \mathcal{C}(K) \to L^1(\mu)$ defined by $B(f, g) = f \cdot g$ is a $\odot$-Pietsch integral operator, where μ is a regular Borel measure on K. Indeed, the bilinear map B factors through a linear operator $T : C(K) \to L^1(\mu)$ defined by $B(f, g) = T(f \cdot g)$, since $B(f \cdot g, h) = (f \cdot g) \cdot h = f \cdot (g \cdot h) = B(f, g \cdot h)$ for all $f, g, h \in \mathcal{C}(K)$ (see [2, pp.133]). It is also clear that the linear map T is the natural injection from $C(K)$ to $L^1(\mu)$, which is a Pietsch integral operator (see [16, Ch. VI, Example 3.10]).

Theorem 5.5. *A continuous bilinear operator $B : X \times X \to Y$ is $\circledast$-Pietsch integral if and only if B has the following factorization,*



where K is a compact Hausdorff space, η is a regular Borel positive measure on K, R and S are bounded linear operators and J is the inclusion map.

In particular, B is a $\circledast$-Pietsch integral operator if and only if the following diagram commutes,



where S is a bounded linear map and T is an absolutely summing operator.

Proof. A bilinear map B is $\circledast$-Pietsch integral if and only if it factors through a Pietsch integral operator. Since a linear map is a Pietsch integral operator if and only if it admits a factorizations through $L^1(\eta)$ and $\mathcal{C}(K)$ (see [16, Theorem VI.3.11]), the diagrams are obtained. $\square$

Theorem 5.6. *Every $\circledast$-Pietsch integral operator is $\circledast$-factorable q-summing.*

Proof. Fix natural numbers A and D, and consider $A \times D$ matrices (x_{ik}), $(y_{ik}) \in X$ and a $\circledast$-Pietsch integral operator $T : X \times X \to Z$ with associated Z-valued Borel measure η. Recall that T is a Pietsch integral operator if and only if its representing Borel vector measure has finite variation (see for example [27, Proposition 5.28]). Then

$$\begin{aligned}
\Big(\sum_{i=1}^{A} \Big\| \sum_{k=1}^{D} T(x_{ik}, y_{ik}) \Big\|_Z^q \Big)^{1/q} &= \left(\sum_{i=1}^{A} \Big\| \int_{B_{F^*}} \sum_{k=1}^{D} \langle x_{ik} \circledast y_{ik}, f \rangle d\eta(f) \Big\|_Z^q \right)^{1/q} \\
&\leqslant \left(\sum_{i=1}^{A} \Big(\int_{B_{F^*}} \Big| \sum_{k=1}^{D} \langle x_{ik} \circledast y_{ik}, f \rangle \Big| d|\eta|(f) \Big)^q \right)^{1/q} \\
&= \sup_{\lambda_i \in B_{\ell^{q'}}} \left(\sum_{i=1}^{A} \lambda_i \int_{B_{F^*}} \Big| \sum_{k=1}^{D} \langle x_{ik} \circledast y_{ik}, f \rangle \Big| \, d|\eta|(f) \right) \\
&\leqslant \int_{B_{F^*}} \sup_{\lambda_i \in B_{\ell^{q'}}} \Big(\sum_{i=1}^{A} \lambda_i \Big| \sum_{k=1}^{D} \langle x_{ik} \circledast y_{ik}, f \rangle \Big| \Big) \, d|\eta|(f) \\
&\leqslant \sup_{f \in B_{F^*}} \sup_{\lambda_i \in B_{\ell^{q'}}} \Big(\sum_{i=1}^{A} \lambda_i \Big| \sum_{k=1}^{D} \langle x_{ik} \circledast y_{ik}, f \rangle \Big| \Big) \; |\eta|(B_{F^*}) \\
&= \sup_{f \in B_{F^*}} \left(\sum_{i=1}^{A} \Big| \sum_{k=1}^{D} \langle x_{ik} \circledast y_{ik}, f \rangle \Big|^q \right)^{1/q} \; |\eta|(B_{F^*}),
\end{aligned}$$

where $|\eta|(B_{F^*}) < \infty$ is the total variation of the measure. Thus, T is $\circledast$-factorable q-summing. $\square$

Let us finish this section by giving some new properties of $\circledast$–Pietsch integral bilinear operators. Let us consider a class of bilinear maps which do not allow integral representations but satisfy integral dominations. By using the definition of q-semi integrals given by Alencar and Matos in [5], we can introduce a class of bilinear maps in a similar manner. If $1 \leqslant q < \infty$, we say that a bilinear operator $B : X \times X \to Z$ is $\circledast$-*q-semi integral* for the product $\circledast : X \times X \to F$ if there is a constant K and a regular probability measure η on B_{F^*} such that

$$\|B(x, y)\| \leqslant K \Big(\int_{B_{F^*}} |\langle x \circledast y, f \rangle|^q d\eta(f) \Big)^{1/q}.$$

Let us show that *every $\circledast$-Pietsch integral bilinear operator $B : X \times X \to Z$ is $\circledast$-q-semi integral.* For instance, a standard domination inequality involving the variation of the vector measure that represents B gives the result. It can also be proved using the associated vector norm inequality. Indeed, since every $\circledast$-Pietsch integral operator on $X \times X$ is $\circledast$-factorable q-summing by Theorem 5.6, if we assume $D = 1$ in Definition 5.1 we get that there is a constant $C > 0$ such that for every finite family $x_1, ..., x_n, y_1, ..., y_n \in X$,

$$\Big(\sum_{i=1}^{n} \Big\| B(x_i, y_i) \Big\|_Z^q \Big)^{1/q} \leqslant C \sup_{f \in F^*} \Big(\sum_{i=1}^{n} \Big| \langle x_i \circledast y_i, f \rangle \Big|^q \Big)^{1/q}.$$

Taking into account that the product $\circledast$ is n.p. product by assumption, this inequality gives that the linear operator $T : F \to Z$ appearing in its linearization is q-summing. Pietsch Domination Theorem gives then that the bilinear map is $\circledast$-q-semi integral.

Corollary 5.7. *Every $\circledast$-integral bilinear functional on $X \times X$ is $\circledast$-Pietsch integral, and therefore, $\circledast$-factorable q-summing map and $\circledast$-q-semi integral.*

6. Applications: summability properties of bilinear maps on $\mathcal{C}(K)$-spaces

In this section we will investigate summability conditions of bilinear maps defined on the Cartesian product of $\mathcal{C}(K)$-spaces. Although $(q, 1)$-summing bilinear operators on these products have recently been analyzed in [23] (see also [24] for the polynomial version), the results presented here and the type of summing inequalities we consider differ from those studied there. However, some related arguments and additional explanations about the general framework of summability of multilinear operators can be found in these papers.

The definition of the *positive (q, p)-summing operator* was given by Blasco in [11] as follows; a linear operator T from a Banach lattice X to a Banach space Y is called positive (q, p)-summing $(1 \leqslant p, q < \infty)$ if there exists a constant $C > 0$ such that for every choice of the finitely many positive elements $x_1, \ldots, x_n \in X$

$$\Big(\sum_{i=1}^{n} \|Tx_i\|_Y^q \Big)^{1/q} \leqslant C \sup_{x^* \in B_{X^*}} \Big(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p \Big)^{1/p}.$$

It is clear that every (q, p)-summing operator is positive (q, p)-summing $(1 \leqslant p, q < \infty)$. In particular, if we consider the Banach lattice X as the function space $\mathcal{C}(K)$ for a compact Hausdorff space K, this allows us to obtain the following result.

Lemma 6.1. *Any linear operator $T : \mathcal{C}(K) \to Y$ is (q, p)-summing if and only if it is positive (q, p)-summing.*

Proof. The *if* assertion is trivial. The *only if* assertion is proved as follows. If we consider a positive (q, p)-summing operator T and arbitrary elements

$f_1, \dots, f_n$ in $\mathcal{C}(K)$, by the positive (q,p)-summability of the operator T, we get

$$\begin{aligned}
\Big(\sum_{i=1}^n \|Tf_i\|_Y^q\Big)^{1/q} &\leqslant \Big(\sum_{i=1}^n \|T(f_i^+)\|_Y^q\Big)^{1/q} + \Big(\sum_{i=1}^n \|T(f_i^-)\|_Y^q\Big)^{1/q} \\
&\leqslant \sup_{x^* \in B_{(\mathcal{C}(K))^*}} \Big(\sum_{i=1}^n |\langle f_i^+, x^*\rangle|^p\Big)^{1/p} + \sup_{x^* \in B_{(\mathcal{C}(K))^*}} \Big(\sum_{i=1}^n |\langle f_i^-, x^*\rangle|^p\Big)^{1/p} \\
&\leqslant \Big\|\Big(\sum_{i=1}^n |f_i^+|^p\Big)^{1/p}\Big\|_{\mathcal{C}(K)} + \Big\|\Big(\sum_{i=1}^n |f_i^-|^p\Big)^{1/p}\Big\|_{\mathcal{C}(K)} \\
&\leqslant 2\Big\|\Big(\sum_{i=1}^n |f_i^+|^p + |f_i^-|^p\Big)^{1/p}\Big\|_{\mathcal{C}(K)} \\
&= 2\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\|_{\mathcal{C}(K)} \\
&= 2 \sup_{x^* \in B_{(\mathcal{C}(K))^*}} \Big(\sum_{i=1}^n |\langle f_i, x^*\rangle|^p\Big)^{1/p}.
\end{aligned}$$

Thus, we obtain (q,p)-summability of the linear map T. □

In order to use disjointness arguments in the framework of the $C(K)$-spaces, we need to adapt the notion of zero product preserving bilinear operator. The problem is that, for technical reasons, we require the disjointness preserving property for characteristic functions of measurable sets, that are not of course continuous functions. So, let us write $Bd(K)$ for the space of (real valued) bounded Borel measurable functions with the supremum norm. Recall that, in the context of the Arens extensions of multilinear operators, every such map can be extended —in several ways— to the bidual spaces, with an extension having values in the bidual of the original range space (see [7, 6, 8]). This allows to extend any bilinear operator $B : \mathcal{C}(K) \times \mathcal{C}(K) \to Y$ to a bilinear (continuous) map $\overline{B} : Bd(K) \times Bd(K) \to Y^{**}$ such that $\|\overline{B}\| = \|B\|$ (see Theorem 9 in [12]) which is very much related to integral representation of multilinear maps by polymeasures (see [12] and Section 5 in [28]). A complete characterization of the properties of such extended multilinear maps (including uniqueness and norm preservation) using the Aron-Berner extension as technical tool [8] can be found in these papers.

In this setting, we will say that a bilinear operator $B : \mathcal{C}(K) \times \mathcal{C}(K) \to Y$ *has zero product preserving extension* if the (an) extension $\overline{B} : Bd(K) \times Bd(K) \to Y^{**}$ satisfies that, if $A \cap B = \varnothing$, then $\overline{B}(\chi_A, \chi_B) = 0$. In other words, the bimeasure that provides the integral representation of B preserves disjointness. Recall that B is weakly compact if $B(B_{C(K)}, B_{C(K)})$ is a relatively weakly compact subset of Y. This will imply, following the results in the classical paper of Bartle, Dunford and Schwartz [9], that the linear factorization operator is also weakly compact and then representable by an integral with respect to a countably additive vector measure, which will be needed to extend the operator to the space $Bd(K)$.

Theorem 6.2. *Let us consider a weakly compact bilinear operator* $B : \mathcal{C}(K) \times \mathcal{C}(K) \to Y$, $1 \leqslant q < \infty$, *and let* $p = 1$ *or* $p = q$. *The following statements are equivalent:*

(1) *B has a zero product preserving extension and there is a constant $K > 0$ satisfying*

$$\Big(\sum_{i=1}^{n} \|B(f_i, f_i)\|^q \Big)^{1/q} \leqslant K \, \Big\| \Big(\sum_{i=1}^{n} (f_i^2)^p \Big)^{1/p} \Big\|_{\mathcal{C}(K)},$$

for every finite sequence $f_1, \dots f_n$ in $\mathcal{C}(K)$.

(2) *The operator B has a zero product preserving extension, is symmetric —that is, $B(f,g) = B(g,f)$ for all $f, g \in \mathcal{C}(K)$—, and there exists a constant $K > 0$ such that for every choice of finitely many functions $f_1, \dots, f_n, g_1, \dots, g_n \in \mathcal{C}(K)$,*

$$\Big(\sum_{i=1}^{n} \|B(f_i, f_i) - B(g_i, g_i)\|^q \Big)^{1/q} \leqslant K \, \Big\| \Big(\sum_{i=1}^{n} |f_i^2 - g_i^2|^p \Big)^{1/p} \Big\|_{\mathcal{C}(K)}.$$

(3) *B admits a zero product preserving extension $\overline{B}$ and there exists a constant $K > 0$ satisfying that for every choice of finitely many functions $f_1, \dots, f_n, g_1, \dots, g_n \in \mathcal{C}(K)$,*

$$\Big(\sum_{i=1}^{n} \|B(f_i, g_i)\|^q \Big)^{1/q} \leqslant K \, \Big\| \Big(\sum_{i=1}^{n} |f_i \cdot g_i|^p \Big)^{1/p} \Big\|_{\mathcal{C}(K)}.$$

(4) *The operator B admits a factorization of the form*

$$\mathcal{C}(K) \times \mathcal{C}(K) \xrightarrow{\ \odot\ } \mathcal{C}(K) \xrightarrow{\ T\ } Y,$$

where T is a (q,p)-summing linear operator and $\odot$ is the pointwise product.

(5) *There is a probability measure λ on K such that the bilinear map B can be factored through the Lorentz space $L_{q,p}(\lambda)$ as*

$$\begin{array}{ccc} \mathcal{C}(K) \times \mathcal{C}(K) & \xrightarrow{\quad B \quad} & Y. \\ \Big\downarrow \odot & & \Big\uparrow \tilde{T} \\ \mathcal{C}(K) & \xrightarrow{\quad [i] \quad} & L_{q,p}(\lambda), \end{array}$$

where $[i]$ is the inclusion/quotient map and $\tilde{T}$ is a continous map.

Proof. (1) $\implies$ (4) We will show that, since B has zero product preserving extension and $\mathcal{C}(K) \odot C(K) = \mathcal{C}(K)$, B can be factored through the $\mathcal{C}(K)$-space by an operator $T : \mathcal{C}(K) \to Y$ and the pointwise product $\odot : \mathcal{C}(K) \times \mathcal{C}(K) \to \mathcal{C}(K)$. This T can be defined by $T(f \cdot g) := B(\mathbf{1}, f \cdot g)$ for all $f, g \in$

$\mathcal{C}(K)$, where $\mathbf{1}$ is the unit element of $\mathcal{C}(K)$. It is clearly seen that T is well-defined, linear and continuous. We need to prove that $B(\mathbf{1}, f\cdot g) = B(f,g)$ for all $f,g \in \mathcal{C}(K)$.

Note first that, by assumption, the bilinear map $\overline{B}$ is defined for couples of characteristic functions of Borel sets A and B, and $\overline{B}(\chi_A,\chi_B) = 0$ if $A\cap B = \varnothing$. Since f,g are continuous –hence uniformly continuous– on the compact Hausdorff space K, we can define a partition $\bigcup_{i=1}^N A_i$ of disjoint sets for K such that for every $\varepsilon > 0$ and every $x,y \in A_i$, $|f(x)-f(y)| < \varepsilon$, $|g(x)-g(y)| < \varepsilon$, $|fg(x)-fg(y)| < \varepsilon$, for all $i = 1,\dots,N$. Let us represent the unit element as $\mathbf{1} = \chi_{\bigcup_{i=1}^N A_i}$, and so $\mathbf{1} = \sum_{i=1}^N \chi_{A_i}$. Choose an x_i from each A_i and define the functions $s_f = \sum_{i=1}^N f(x_i)\chi_{A_i}$, $s_g = \sum_{i=1}^N g(x_i)\chi_{A_i}$ and $s_{fg} = \sum_{i=1}^N f(x_i)g(x_i)\chi_{A_i}$. Thus, the following inequalities involving computations in Y^{**},

$$\begin{aligned}\|B(f,g)-\overline{B}(s_f,s_g)\| &\leqslant \|\overline{B}(f-s_f,g)+\overline{B}(s_f,g-s_g)\|\\ &\leqslant \|B\|(\|f-s_f\|_\infty\|g\|_\infty + \|s_f\|_\infty\|g-s_g\|_\infty)\\ &= \|B\|\varepsilon\,(\|f\|_\infty+\|g\|_\infty)\end{aligned}$$

and,

$$\|\overline{B}(\mathbf{1},s_{fg}) - B(\mathbf{1},f\cdot g)\| = \|\overline{B}(\mathbf{1},s_{fg}-f\cdot g)\| \leqslant \|B\|\varepsilon$$

are obtained. By using that $\overline{B}$ is zero product preserving, we get

$$\begin{aligned}\overline{B}(s_f,s_g)-\overline{B}(\mathbf{1},s_{fg}) &= \overline{B}\Big(\sum_{i=1}^N f(x_i)\chi_{A_i}, \sum_{j=1}^N g(x_j)\chi_{A_j}\Big)\\ &\quad - \overline{B}\Big(\sum_{i=1}^N \chi_{A_i}, \sum_{j=1}^N f(x_j)g(x_j)\chi_{A_j}\Big)\\ &= \sum_{i=1}^N f(x_i)g(x_i)\overline{B}(\chi_{A_i},\chi_{A_i}) - \sum_{i=1}^N f(x_i)g(x_i)\overline{B}(\chi_{A_i},\chi_{A_i}) = 0.\end{aligned}$$

Therefore, we get for all $f,g\in\mathcal{C}(K)$,

$$\begin{aligned}&\|B(f,g)-B(\mathbf{1},f\cdot g)\|\\ \leqslant\ &\|B(f,g)-\overline{B}(s_f,s_g)\| + \|\overline{B}(s_f,s_g)-\overline{B}(\mathbf{1},s_{fg})\| + \|\overline{B}(\mathbf{1},s_{fg})-B(\mathbf{1},f\cdot g)\|,\end{aligned}$$

which is as small as we want by choosing adequate simple functions s_f, s_g and s_{fg}.

To finish the proof, just note that the inequality given in (1) implies the positive (q,p)-summability of the linear operator T, since each non-negative function in $C(K)$ can be written as the square root of a function in $C(K)$. By Lemma 6.1, it is seen that T is (q,p)-summing.

(4) $\implies$ (5) For $p=1$, by Pisier's theorem we get that for the Banach space valued linear operator $T:\mathcal{C}(K)\to Y$, there is a probability measure λ on K such that T can be factored through the Lorentz space $L_{q,1}(\lambda)$ $(q\geqslant 1)$ by the inclusion/quotient map and a continuous linear operator (see [26,

Theorem 2.4.]). For $p = q$, this factorization is obtained by using the well-known Pietsch's domination theorem which states that every p-summing operator defined on $\mathcal{C}(K)$ is factored through $L_p(\lambda)$ space with a regular Borel probability measure λ on K (see [15, Corollary 2.15]). Therefore, the required factorization is obtained.

(5) $\implies$ (3) First note that the pointwise product $\odot$ can be extended to $\odot : Bd(K) \times Bd(K) \to Bd(K)$ in the obvious way. Now we use that B is weakly compact to prove that the factorization operator $T := \tilde{T} \circ [i]$ in (5) can be written as an integral with respect to a countably additive vector measure having values in Y (Theorem 3.2 in [9], see also Theorem 13 in [28] and the comments therein regarding multilinear versions of this result). This integral provides an extension $\overline{T}$ of T to the space $Bd(K)$. Therefore, we obtain the extension $\overline{B} = \overline{T} \circ \odot$, that is clearly zero product preserving.

On the other hand, by using the (q,p)-summability of $\tilde{T} \circ [i]$ we get

$$\begin{aligned}\Big(\sum_{i=1}^{n} \|B(f_i, g_i)\|^q\Big)^{1/q} &= \Big(\sum_{i=1}^{n} \|\tilde{T} \circ [i](f_i \cdot g_i)\|^q\Big)^{1/q} \\ &\leqslant C \sup_{x^* \in B_{(\mathcal{C}(K))^*}} \Big(\sum_{i=1}^{n} |\langle f_i \cdot g_i, x^*\rangle|^p\Big)^{1/p} \\ &\leqslant K \Big\| \Big(\sum_{i=1}^{n} |f_i \cdot g_i|^p\Big)^{1/p} \Big\|_{\mathcal{C}(K)}.\end{aligned}$$

(3) $\implies$ (1) is obvious.

(4) $\implies$ (2) Since B factors through the linear operator $T : \mathcal{C}(K) \to Y$ defined by $B(f,g) = T(f \cdot g)$, it is seen that the map B is symmetric. To show the inequality, consider finitely many elements $\{f_i\}_{i=1}^n$, $\{g_i\}_{i=1}^n \in \mathcal{C}(K)$ and define the finite sets of functions $\{\psi_i\}_{i=1}^n = \{f_i + g_i\}_{i=1}^n$, $\{\phi_i\}_{i=1}^n = \{f_i - g_i\}_{i=1}^n$. By the symmetry of the map B, we get $B(\psi_i, \phi_i) = B(f_i, f_i) - B(g_i, g_i)$ and $\psi_i \cdot \phi_i = f_i^2 - g_i^2$ by the pointwise product of the functions. Using the (q,p)-summability of the linear map T, the inequality is obtained as follows.

$$\begin{aligned}\Big(\sum_{i=1}^{n} \|B(f_i, f_i) - B(g_i, g_i)\|^q\Big)^{1/q} &= \Big(\sum_{i=1}^{n} \|B(\psi_i, \phi_i)\|^q\Big)^{1/q} \\ &= \Big(\sum_{i=1}^{n} \|T(\psi_i \cdot \phi_i)\|^q\Big)^{1/q} \\ &\leqslant K \Big\| \Big(\sum_{i=1}^{n} |\psi_i \cdot \phi_i|^p\Big)^{1/p} \Big\|_{\mathcal{C}(K)} \\ &= K \Big\| \Big(\sum_{i=1}^{n} |f_i^2 - g_i^2|^p\Big)^{1/p} \Big\|_{\mathcal{C}(K)}.\end{aligned}$$

(2) $\implies$ (1) is obvious, and so the proof is finished.

□

Remark 6.3. Note that in the case $q = p$ the weak compactness requirement for B is not needed in Theorem 6.2; since every p-summing operator is weakly compact (what is not true for the case of $(q,1)$-summing operators), we do not need this assumption. A look to the proof shows that once we have the factorization in (4), this implies that B is weakly compact.

On the other hand, the requirement on the existence of a zero product preserving extension can be avoided under further assumptions on B. For example, if Y is a Banach lattice and B is positive, that is, $B(f,g) \geqslant 0$ for all non-negative elements f and g, an argument based on the use of a partition of the unit provides also the factorization in (5) just assuming (1) without any mention to the extension. However, for the general case it seems to be necessary to explicitly require zero product preservation of $\overline{B}$.

Let us finish the paper by writing the representation of the bilinear operator that is implicitly used in the previous arguments. Recall that if K is a compact Hausdorff space and $\mathcal{B}(K)$ is the σ-algebra of its Borel sets, every vector measure $\mu : \mathcal{B}(K) \to Y$ defines an integration operator $I_\mu : \mathcal{C}(K) \to Y$ by $I_\mu(f) = \int_K f d\mu$, $f \in \mathcal{C}(K)$, that can be extended to $Bd(K)$.

Corollary 6.4. *Let us consider a weakly compact bilinear map $B : \mathcal{C}(K) \times \mathcal{C}(K) \to Y$ and for $1 \leqslant q < \infty$, let $p = 1$ or $p = q$. Any of the assertions of Theorem 6.2 holds if and only if B can be written as an integral*

$$B(f,g) = \int_K f \cdot g \, d\mu, \qquad f, g \in \mathcal{C}(K),$$

for a countably additive vector measure such that the corresponding integration map $I_\mu : C(K) \to Y^{**}$ is (q,p)-summing.

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