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UNIVERSITAT POLITÈCNICA DE VALÈNCIA

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Contents

Analysing nonlinear oscillators subject to Gaussian inputs via the random perturbation technique

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1 Introduction

The aim of this paper is to perform a stochastic analysis of nonlinear oscillators subject to stationary Gaussian forcing sources using the random perturbation technique along with the Maximum Entropy Principle. By combining these two stochastic techniques, we construct reliable approximations of the probability density function of the solution, which is a stochastic process.

Specifically, we will deal with a random nonlinear oscillator subject to small perturbations upon both the position, $X(t)$, and velocity, $\dot{X}(t)$, of the form

$$\ddot{X}(t) + 2\zeta\omega_0\dot{X}(t) + \epsilon X^2(t)\dot{X}(t) + \omega_0^2 X(t) = Y(t), \tag{1}$$

where ζ is the damping coefficient, $\omega_0 > 0$ is the frequency, ϵ is a small perturbation ($|\epsilon| << 1$) affecting the nonlinear term $X^2(t)\dot{X}(t)$ and the input term $Y(t)$ is a stationary zero-mean Gaussian stochastic process and mean square differentiable.

The key point of the perturbation technique is to consider that the solution $X(t)$ can be developed in the powers of ϵ. Commonly, when this technique is applied only the first order approximation is considered

$$\widehat{X}(t) = X_0(t) + \epsilon X_1(t). \tag{2}$$

Substituting expression (2) into Eq. (1), produces the next sequence of linear differential equations, that can be solved in cascade,

$$\begin{aligned} \ddot{X}_0(t) + 2\zeta\omega_0\dot{X}_0(t) + \omega_0^2 X_0(t) &= Y(t), \\ \ddot{X}_1(t) + 2\zeta\omega_0\dot{X}_1(t) + \omega_0^2 X_1(t) &= -X_0^2(t)\dot{X}_0(t). \end{aligned} \tag{3}$$

We are interested in analysing, from a probabilistic standpoint, the steady-state solution. Taking advantage of the linear theory, the system (3) can be solved using the convolution inte-

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gral [1,2]:

$$X_0(t) = \int_0^\infty h(s) Y(t-s)\, \mathrm{d}s, \tag{4}$$

and

$$X_1(t) = \int_0^\infty h(s) \left[-X_0^2(t-s) \dot{X}_0(t-s) \right] \mathrm{d}s, \tag{5}$$

where

$$h(t) = \left(\omega_0^2 - \zeta^2 \omega_0^2 \right)^{-\frac{1}{2}} \mathrm{e}^{-\zeta \omega_0 t} \sin \left[\left(\omega_0^2 - \zeta^2 \omega_0^2 \right)^{\frac{1}{2}} t \right]$$

is the impulse response function for the underdamped case $\zeta^2 < 1$.

From the approximate solution given in (2), we can calculate the main statistical functions of $\widehat{X}(t)$, namely, higher moments, $\mathbb{E}\left\{X^n(t)\right\}$, the variance, $\mathbb{V}\left\{X(t)\right\}$, the covariance, $\mathbb{C}\mathrm{ov}\left\{X(t_1), X(t_2)\right\}$ and the correlation function $\Gamma_{XX}(\tau)$.

2 Results and discussion

This section is addressed to illustrate, by means of an example, the theorical results. Let us consider $Y(t) = \xi_1 \cos(t) + \xi_2 \sin(t)$, where $\xi_1, \xi_2 \sim \mathrm{N}(0,1)$ the trigonometric stochastic process as excitation. Notice that $Y(t)$ satisfies the hypotheses specified in the previous section. Also, we take $\xi = 0.05$ and $\omega_0 = 1$. Replacing this data into Eq. (1) reads,

$$\ddot{X}(t) + 0.1\dot{X}(t) + \epsilon X^2(t)\dot{X}(t) + X(t) = \xi_1 \cos(t) + \xi_2 \sin(t),\ \ \xi_1, \xi_2 \sim \mathrm{N}(0,1). \tag{6}$$

Now we shall obtain the main statistical functions of $\widehat{X}(t)$. To compute the mean of the approximation, we use the expectation operator in (2). Applying (4) and (5) and using that we can interchange the expectation operator with the mean square integral, we obtain,

$$\mathbb{E}\{\widehat{X}(t)\} = \mathbb{E}\{X_0(t)\} + \epsilon \mathbb{E}\{X_1(t)\} = 0.$$

In addition, we observe that this happens with all moments of odd order. In our case we calculate the first five moments, therefore, $\mathbb{E}\{\widehat{X}^3(t)\} = 0$ and $\mathbb{E}\{\widehat{X}^5(t)\} = 0$. Now, due to the positiveness of the second and fourth order moments we will deduce appropriate bounds for ϵ. First, let us compute $\mathbb{E}\{\widehat{X}^2(t)\}$ up to the first order term of ϵ,

$$\begin{aligned}
\mathbb{E}\left\{\widehat{X}^2(t)\right\} &= \mathbb{E}\left\{X_0^2(t)\right\} + 2\epsilon \mathbb{E}\left\{X_0(t) X_1(t)\right\} \\
&= \int_0^\infty h(s) \int_0^\infty h(s_1) \Gamma_{YY}(s-s_1)\, \mathrm{d}s_1\, \mathrm{d}s - 2\epsilon \Bigg(\int_0^\infty h(s) \int_0^\infty h(s_1) \int_0^\infty h(s_2) \int_0^\infty h(s_3) \int_0^\infty h(s_4) \\
&\quad \cdot \Big(2\Gamma_{YY}(s_1 - s - s_2) \Gamma'_{YY}(s_3 - s_4) + \Gamma'_{YY}(s_1 - s - s_4) \Gamma_{YY}(s_2 - s_3) \Big) \mathrm{d}s_4\, \mathrm{d}s_3\, \mathrm{d}s_2\, \mathrm{d}s_1\, \mathrm{d}s \Bigg) \\
&= 100 - 200000\epsilon.
\end{aligned}$$

In this case, we derive the bound $\epsilon < 0.0005$. Since $\mathbb{E}\{\widehat{X}(t)\} = 0$, we can deduce that the variance is equal to $\mathbb{E}\{\widehat{X}^2(t)\}$. Secondly, from the expression of $\mathbb{E}\{\widehat{X}^4(t)\}$,

$$\mathbb{E}\left\{\widehat{X}^4(t)\right\} = \mathbb{E}\left\{X_0^4(t)\right\} + 4\epsilon \mathbb{E}\left\{X_0^3(t) X_1(t)\right\} = 30000 - \frac{1153800000000}{6409}\epsilon,$$

we can refine the above bound $\epsilon < 0.000166641$.

Now, we focus on the correlation function of the approximation $\widehat{X}(t)$,

$$\begin{aligned}\Gamma_{\widehat{X}\widehat{X}}(\tau) = & \int_0^\infty \int_0^\infty h(s)h(s_1)\Gamma_{YY}(\tau - s_1 + s)\,\mathrm{d}s\,\mathrm{d}s_1 \\ & -\epsilon \int_0^\infty h(s) \int_0^\infty h(s_1) \int_0^\infty h(s_2) \int_0^\infty h(s_3) \int_0^\infty h(s_4) \Big\{ 2\,\Gamma_{YY}(\tau - s - s_2 + s_1)\Gamma'_{YY}(s_3 - s_4) \\ & + \Gamma'_{YY}(\tau - s - s_4 + s_1)\Gamma_{YY}(s_2 - s_3) + \Gamma_{YY}(s_1 - s_2)\Gamma'_{YY}(\tau - s_4 + s + s_3) \\ & + 2\,\Gamma'_{YY}(s_1 - s_3)\Gamma_{YY}(\tau - s_4 + s + s_2) \Big\}\,\mathrm{d}s_4\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}s = 100(1 - 2000\epsilon)\cos(\tau).\end{aligned}$$

In this example, the covariance coincide with the correlation function, since $\mathbb{E}\{\widehat{X}(t)\} = 0$,

$$\mathbb{C}\mathrm{ov}\{\widehat{X}(t_1), \widehat{X}(t_2)\} = \Gamma_{\widehat{X}\widehat{X}}(\tau) = 100(1 - 2000\epsilon)\cos(\tau), \quad \tau = |t_1 - t_2|.$$

In Fig. 1 we have plotted the approximations of correlation functions for different values of ϵ. It should be noted that when epsilon is larger, the variability decreases.

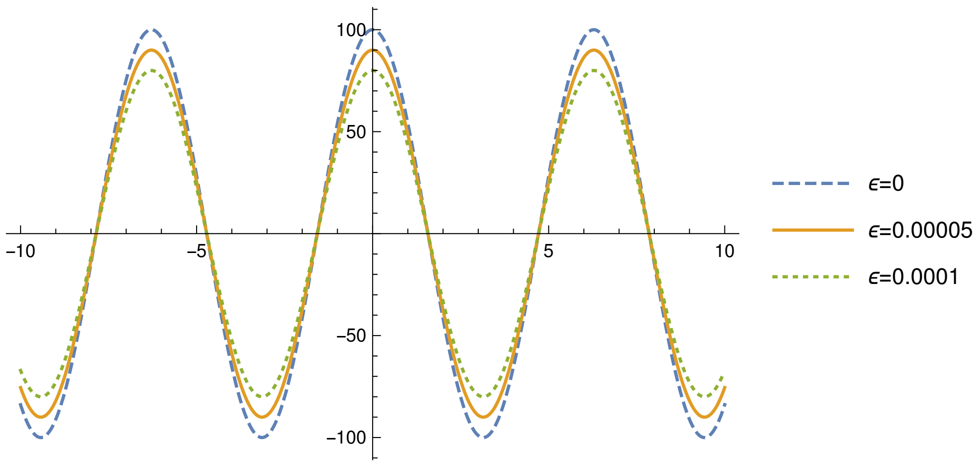

Figure 1: Correlation function $\Gamma_{\widehat{X}\widehat{X}}(\tau)$ of $X(t)$ for different values of ϵ.

Finally, we construct a reliable approximation of the probability density function using the Maximum Entropy Principle,

$$f_{\widehat{X}(t)}(x) = \mathrm{e}^{-1-2.243+2.552\cdot 10^{-8}x-0.004x^2-2.177\cdot 10^{-9}x^3-3.789\cdot 10^{-6}x^4+6.754\cdot 10^{-13}x^5}.$$

In Fig. 2 we show the graphical representation of $f_{\widehat{X}(t)}(x)$.

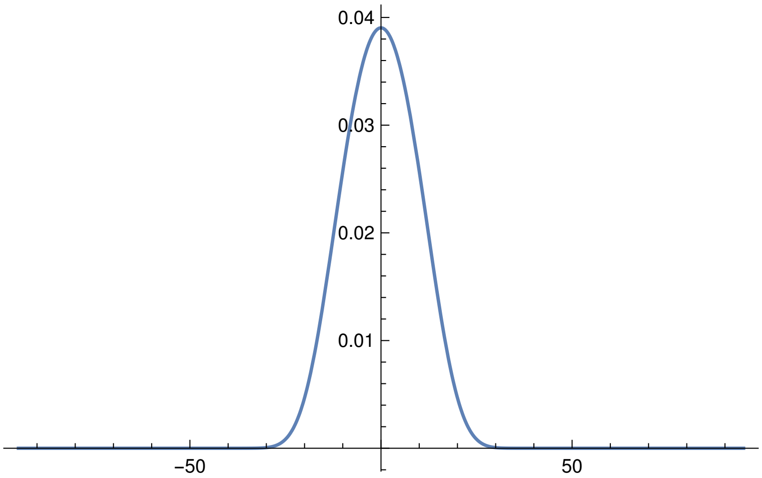

Figure 2: Approximation of PDF, $f_{\widehat{X}(t)}(x)$, for $\epsilon = 0.00005$ via the PME.

3 Conclusions

We have studied, from a probabilistic point of view, a random nonlinear oscillator where the term of perturbation affects the crossnonlinear term (position and velocity). Our main contribution has been the approximation of the probability density function taking advantage of the Principle of Maximum Entropy. In this manner, we provide a fuller probabilistic information of the solution.

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References

[1] Crandall, S. H., Perturbation Techniques for Random Vibration of Nonlinear Systems. *The Journal of the Acoustical Society of America*, 35:1700–1705, 1963.

[2] Soong, T.T., Random Differential Equations in Science and Engineering. New York, Academis Pres, 1973.

[3] El-Tawil, M.A., Al-Jihany, A.S., On the solution of stochastic oscillatory quadratic nonlinear equations using different techniques, a comparison study, *Topol. Methods Nonlinear Anal.* 31(2): 315–330, 2008.

[4] Cortés, J.-C., Romero, J.-V., Roselló, M.-D., Santamaría, C., Solving random diffusion models with nonlinear perturbations by the Wiener–Hermite expansion method. *Computers & Mathematics with Applications*. 61(8): 1946-1950, 2011.

[5] Calatayud, J., Cortés, J.-C., Jornet, M., The damped pendulum random differential equation: A comprehensive stochastic analysis via the computation of the probability density function. *Physica A: Statistical Mechanics and its Applications*. 512: 261-279, 2018.