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All Jordan canonical forms of irreducible totally nonnegative matrices

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ABSTRACT
Let $A \in \mathbb{R}^{n \times n}$ be an irreducible totally nonnegative matrix with rank r and principal rank p, that is, every minor of A is nonnegative and p is the size of the largest invertible principal submatrix of A. Using Number Theory, we calculate the number of Jordan canonical forms of irreducible totally nonnegative matrices associated with a realizable triple (n, r, p). Moreover, by using full rank factorizations of A and applying the Flanders theorem we obtain all these Jordan canonical forms. Finally, some algorithms associated with these results are given.

KEYWORDS
Totally nonnegative matrix; irreducible matrix; principal rank; Jordan canonical form

AMS CLASSIFICATION
15A03, 15A15, 65F40

1. Introduction

$A \in \mathbb{R}^{n \times n}$ is called totally nonnegative if all its minors are nonnegative and it is abbreviated as TN, see for instance [1–5]. Due to its wide variety of applications in algebra, computer aided geometric design, differential equations, economics, quantum theory and other fields, TN matrices have been studied by several authors who have obtained properties, the Jordan structure and characterizations by using the Neville elimination [5–10].

We recall that A is an *irreducible* matrix if there is not a permutation matrix P such that $PAP^T = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$, where O is an $(n-r) \times r$ zero matrix $(1 \le r \le n-1)$.

In general, the *rank* of a given matrix A, denoted by rank(A), is the size of the largest invertible square submatrix of A. The *principal rank* of A, denoted by p-rank(A), is the size of the largest invertible principal submatrix of A. Obviously the following inequality holds,

$$0 \le p\text{-rank}(A) \le \text{rank}(A) \le n. \tag{1}$$

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Fallat et al. [2–4] characterized all possible Jordan canonical forms of irreducible TN matrices using weighted planar diagrams associated with TN matrices. In [4, p.87] the authors denoted by ITN the irreducible TN matrices and they began the study using the triple $(n, \text{rank}, p\text{-rank})$ among the class of ITN matrices.

Definition 1.1 (p.709 [2]). A triple (n, r, p) is called *realizable* if there exists $A \in \mathbb{R}^{n \times n}$ ITN matrix such that $\text{rank}(A) = r$ and $p\text{-rank}(A) = p$.

From Definition 1.1, if A is an $n \times n$ ITN matrix with $\text{rank}(A) = r$ and $p\text{-rank}(A) = p$, then we say that A is *associated with* the realizable triple (n, r, p).

If A is TN with $p\text{-rank}(A) = p$, its characteristic polynomial is given by

$$q_A(\lambda) = \lambda^{n-p} \left(\lambda^p - c_1 \lambda^{p-1} + c_2 \lambda^{p-2} + \cdots + (-1)^p c_p \right),$$

where $c_p \neq 0$. Then, $n - p\text{-rank}(A)$ is the sum of the sizes of zero Jordan blocks of A, i.e., the algebraic multiplicity of the zero eigenvalue of A which we denote by $\text{am}(0) = n - p\text{-rank}(A)$. Furthermore, $n - \text{rank}(A) = \dim \text{Ker}(A)$ is the number of zero Jordan blocks of any matrix A, i.e., the geometric multiplicity of the zero eigenvalue of A which we denote by $\text{gm}(0) = n - \text{rank}(A)$. By [2, p.709], $p\text{-rank}(A)$ is the number of positive eigenvalues of A.

In [3, Theorem 3.3] it is shown that the nonzero eigenvalues of a singular ITN matrix A are positive and distinct. As a consequence, if $\text{rank}(A) = r$, $p\text{-rank}(A) = p$ and $\lambda_1, \ldots, \lambda_p, \ldots, \lambda_n$ are the eigenvalues of A, we have

$$\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0, \text{ and } \lambda_{p+1} = \lambda_{p+2} = \ldots = \lambda_n = 0.$$

In [11] the authors use the principal rank to study the dependence relations between rows and columns of an ITN matrix and they introduce the sequence of the first p-indices of linearly independent rows and columns of A. They consider the notation given in [1], that is, for $k, n \in \mathbb{N}$, $1 \leq k \leq n$, $\mathcal{Q}_{k,n}$ denotes the set of all increasing sequences of k natural numbers less than or equal to n. If $A \in \mathbb{R}^{n \times n}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathcal{Q}_{k,n}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathcal{Q}_{k,n}$, $A[\alpha|\beta]$ denotes the $k \times k$ submatrix of A lying in rows α_i and columns β_i, $i = 1, 2, \ldots, k$. The principal submatrix $A[\alpha|\alpha]$ is abbreviated as $A[\alpha]$.

Definition 1.2 (Definition 1 of [11]). Let $A \in \mathbb{R}^{n \times n}$ be a matrix with $p\text{-rank}(A) = p$. We say that the sequence of integers $\alpha = \{i_1, i_2, \ldots, i_p\} \in \mathcal{Q}_{p,n}$ is the *sequence of the first p-indices of A* if for $j = 2, \ldots, p$ we have

$$\det(A[i_1, i_2, \ldots, i_{j-1}, i_j]) \neq 0,$$
$$\det(A[i_1, i_2, \ldots, i_{j-1}, t]) = 0, \quad i_{j-1} < t < i_j.$$

We study the structure of zero Jordan blocks of A, that is, Jordan blocks corresponding to the zero eigenvalue in the Jordan canonical form of A. First, we consider some results to characterize ITN matrices.

Lemma 1.3 (Lemma 2.2 of [3]). *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a TN matrix with no zero rows or columns. Then A is irreducible if and only if $a_{ij} > 0$ for all $|i - j| \leq 1$.*

Theorem 1.4 (Theorem 10 of [4], Theorem 1 of [11]). *Let $A \in \mathbb{R}^{n \times n}$ be an ITN matrix with p-rank$(A) = p$, $1 \le p < n$. Then rank$(A^p) = $ p-rank$(A) = p$. In particular, the size of the largest zero Jordan block is at most p.*

Lemma 1.5 (Lemma 2.2 of [11]). *Let $A \in \mathbb{R}^{n \times n}$ be an ITN matrix with rank$(A) = r$ and p-rank$(A) = p$. Then*

$$p \le r \le n - \left\lceil \frac{n-p}{p} \right\rceil. \tag{2}$$

Lemma 1.5 implies that a triple (n, r, p) is realizable if equation (2) holds.

The paper is organized as follows: in the next section by using Number Theory and Combinatorics we prove Theorem 2.2 to get the number of Jordan canonical forms of ITN matrices associated with a realizable triple (n, r, p). Furthemore, from Theorem 2.2 we develop a Matlab algorithm to compute this number.

In Section 3 we study some properties of ITN matrices by using full rank factorizations and the Flanders theorem. Using the results of Section 3 we calculate all Jordan canonical forms of these matrices in Section 4 and we introduce a Matlab algorithm associated with these results.

2. The number of Jordan canonical forms of ITN matrices

A partition of a positive integer n is a nonincreasing finite sequence of positive integers whose sum is n. Two sums that differ only in the order of their summands are considered the same partition (see [12]). We denote by $p(n)$ the total number of partitions of n, by $p_k(n)$ the number of partitions of n into exactly k parts and we denote by $p_k^{(h)}(n)$ the number of partitions of n into exactly k parts, with largest part at most h.

The aim of this section is to obtain the number of zero Jordan canonical forms of an ITN matrix A associated with a realizable triple (n, r, p). By Section 1 we have that am$(0) = n - p$, gm$(0) = n - r$ and the size of the largest zero Jordan block is at most p. So, the number of zero Jordan canonical forms of A is the number of partitions of $n - p$ into exactly $n - r$ parts with the largest part at most p, and our purpose is to calculate $p_{n-r}^{(p)}(n-p)$.

Remark 1. We take $p(n) = 0$ for all negative values of n and $p(0)$ is defined to be 1. Note that $p_k(n) = 0$ holds if $k > n$. Thus, $p(n) = \sum_{k=1}^{n} p_k(n)$. Moreover, it is verified that

$$\begin{aligned} p_1(n) &= 1, \quad \text{for } n \ge 1, \\ p_2(2) &= p_2(3) = p_3(3) = 1, \end{aligned}$$

and for $n \ge 4$,

$$p_2(n) = \left\lfloor \frac{n}{2} \right\rfloor, \quad p_{n-2}(n) = 2, \quad p_{n-1}(n) = p_n(n) = 1.$$

Each partition of n into exactly k parts can be represented graphically by the Ferrers diagram with exactly k rows having each row the same number of dots as the k-th

term in the partition and all rows have at least one dot. The Ferrers diagram is used to prove the following well-known result in an easy way. We include the proof of this result because it is useful for Theorem 2.2.

Theorem 2.1. *Given $n > 1$ and $1 \le k \le n$, then*

$$p_k(n) = \sum_{j=1}^{\min\{k,n-k\}} p_j(n-k). \tag{3}$$

Proof. Since the partitions are exactly in k parts, all of them need k dots in order to complete the first column of its Ferrers diagram and the remaining $n - k$ dots can be distributed in the k rows arbitrarily in nonincreasing form; that is, we should fix $n - k$ dots in at most k rows. As a consequence,

$$p_k(n) = p_1(n-k) + p_2(n-k) + \cdots + p_k(n-k) = \sum_{j=1}^{k} p_j(n-k)$$

If $n - k < k$, by Remark 1 we have that $\sum_{j=1}^{k} p_j(n-k) = \sum_{j=1}^{n-k} p_j(n-k)$. Therefore,

$$p_k(n) = \sum_{j=1}^{\min\{k,n-k\}} p_j(n-k).$$

$\square$

It is known that an algorithm, due to C. F. Hindenburg's 18th-century dissertation *Infinitinomii Dignitatum Exponentis Indeterminati* (Göttingen, 1779), pp.73-91, generates all partitions of n into exactly k parts (see [13]). However, if we want that the largest part of a partition to be less than or equal to a number h, then equation (3) has to be updated. For example, by equation (3) and Remark 1 we obtain

$$p_7(17) = \sum_{j=1}^{7} p_j(10) = 1 + \left\lfloor \tfrac{10}{2} \right\rfloor + \sum_{j=3}^{7} p_j(10) = 1 + 5 + \sum_{j=1}^{3} p_j(7) + \sum_{j=1}^{4} p_j(6) +$$

$$\sum_{j=1}^{5} p_j(5) + \sum_{j=1}^{4} p_j(4) + \sum_{j=1}^{3} p_j(3) = 1 + 5 + 8 + 9 + 7 + 5 + 3 = 38.$$

However, if we want that the largest part of the partition to be less than or equal to $h = 3$, then the number of partitions is significantly reduced to only 3 as we will show in Example 2.3-b.

In 1976 George E. Andrews (see [12, Chapter 3]) studied some properties of the Restricted Partitions, that is partitions of a positive integer n in which the largest part is at most h and the number of parts is less than or equal to k. He denoted this kind of partition by $p(h, k, n)$ and using our notation we have that

$$p(h, k, n) = \sum_{j=1}^{k} p_j^{(h)}(n).$$

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These partitions are the coefficients of the generating function

$$G(h, k; q) = \sum_{n \geq 0} p(h, k, n) \, q^n$$

which is a polynomial in q of degree hk known as a Gaussian polynomial. Its coefficients are called the q-binomial coefficients or Gaussian binomial coefficients.

An interesting property is that $p(h, k, n) - p(h, k - 1, n)$ enumerates the number of partitions of n into exactly k parts, each part $\leq h$, that is

$$p(h, k, n) - p(h, k - 1, n) = p_k^{(h)}(n),$$

which is precisely what we want to obtain. So, we must calculate the Gaussian polynomials $G(h, k; q)$ and $G(h, k - 1; q)$, and subtract the coefficients of q^n of each polynomial. Since the rest of the q-binomial coefficients of $G(h, k; q)$ and $G(h, k - 1; q)$ are not necessary to solve our problem, we give Theorem 2.2 to calculate directly the value of $p_k^{(h)}(n)$.

Remark 2. Note that

(1) if n is negative then $p_k^{(h)}(n) = 0$,

(2) if $k > n$ then $p_k^{(h)}(n) = 0$ for all $h \geq 1$,

(3) if $k = n$ then $p_n^{(h)}(n) = 1$ for all $h \geq 1$,

(4) if $k < n$ and $h = 1$ then $p_k^{(h)}(n) = 0$,

(5) if $k = 0$ then $p_k^{(h)}(n) = 0$,

(6) if $k = 1$ then $\begin{cases} \text{if } h < n \text{ then } p_1^{(h)}(n) = 0, \\ \\ \text{if } h \geq n \text{ then } p_1^{(h)}(n) = 1. \end{cases}$

Theorem 2.2. *Given $n > 1$, $1 < k < n$ and $1 < h$, with $k, h \in \mathbb{Z}_+$, the number of partitions of n into exactly k parts with the largest part at most h is given by*

(1) If $kh < n$ then $p_k^{(h)}(n) = 0$.

(2) If $kh = n$ then $p_k^{(h)}(n) = 1$.

(3) If $kh > n$ and

(3.1) $n - k + 1 \leq h$ then

$$p_k^{(h)}(n) = p_k(n) = \sum_{j=1}^{\min\{k, n-k\}} p_j(n - k)$$

(3.2) $n - k + 1 > h$ and

(3.2.1) $k(h - 1) = n$ then

$$p_k^{(h)}(n) = 1 + p_{k-1}^{(h)}(n - h)$$

(3.2.2) $k(h - 1) < n$. Let $i = n - k(h - 1)$, then

$$p_k^{(h)}(n) = 1 + p_{k-(i+1)}^{(h)}(n - (i + 1)h)$$

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(3.2.3) $k(h-1) > n$, *and* $t_0 = \left\lfloor \frac{n+k}{2} \right\rfloor - 1$
(3.2.3.1) if $s_0 \leq t_0$, *with* $k - 1 \leq s_0 \leq n - (h+1)$, *then*

$$p_k^{(h)}(n) = \sum_{j=1}^{\min\{k,n-k\}} p_j(n-k) - \sum_{s_0=k-1}^{n-(h+1)} p_{k-1}(s_0).$$

(3.2.3.2) Otherwise, $p_k^{(h)}(n) =$

$$= \sum_{j=1}^{\min\{k,n-k\}} p_j(n-k) - \sum_{s_0=k-1}^{t_0} p_{k-1}(s_0) - \sum_{s_0=t_0+1}^{n-(h+1)} p_{k-1}^{(n-s_0)}(s_0).$$

Proof. (1) Obvious.
(2) Obvious.
(3) Now, suppose that $kh > n$.
(3.1) Following the proof of Theorem 2.1, since $n - k + 1 \leq h$, any distribution of the $n - k$ dots in k rows gives partitions with the largest part less than or equal to h. Then, $p_k^{(h)}(n) = p_k(n)$ and

$$p_k^{(h)}(n) = p_k(n) = \sum_{j=1}^{\min\{k,n-k\}} p_j(n-k)$$

(3.2) Now, $n - k + 1 > h$
(3.2.1) Since $k(h-1) = n$, we have only one partition whose largest part is $h - 1$. In fact, the k entries of the partition are equal to $h - 1$. If we complete the first row with h dots, we obtain all possible partitions of $n - h$ in $k - 1$ dots with the largest part h. Thus,

$$p_k^{(h)}(n) = 1 + p_{k-1}^{(h)}(n-h)$$

(3.2.2) Since $k(h-1) < n$, if we want to obtain a partition of n in exactly k parts with the largest part less than or equal to h, it is necessary that almost the first term of the partition is equal to h. Moreover, if $n - k(h-1) = i$, we need that the first i terms of any partition are equal to h. The rest of terms $n - ih$ are distributed in $k - i$ parts with the largest part less than or equal to h. Thus,

$$p_k^{(h)}(n) = p_{k-i}^{(h)}(n-ih).$$

As

$$(k-i)(h-1) = k(h-1) - i(h-1) = (n-i) - i(h-1) = n - ih$$

applying the previous result we have

$$p_k^{(h)}(n) = 1 + p_{k-(i+1)}^{(h)}(n-(i+1)h).$$

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(3.2.3) Finally, $k(h-1) > n$. We obtain the number of partitions of n in exactly k parts, $p_k(n)$, removing those that have terms greater than h, from $n - k + 1$ to $h + 1$. Then, the number of partitions that we remove is

$$p_{k-1}^{(n-k+1)}(k-1) \; + \; p_{k-1}^{(n-k)}(k) + p_{k-1}^{(n-k-1)}(k+1) \; + \; \cdots$$

$$\cdots \; + \; p_{k-1}^{(n-(h+1))}(n-h-1) = \sum_{s_0=k-1}^{n-(h+1)} p_{k-1}^{(n-s_0)}(s_0).$$

(3.2.3.1) If for each s_0, with $k - 1 \le s_0 \le n - (h+1)$, it is satisfied that

$$s_0 - (k-1) + 1 \le n - s_0 \quad \longleftrightarrow \quad s_0 \le \left\lfloor \frac{n+k}{2} \right\rfloor - 1 = t_0.$$

From (3.1) we have

$$\sum_{s_0=k-1}^{n-(h+1)} p_{k-1}^{(n-s_0)}(s_0) = \sum_{s_0=k-1}^{n-(h+1)} p_{k-1}(s_0).$$

Thus,

$$p_k^{(h)}(n) = \sum_{j=1}^{\min\{k,n-k\}} p_j(n-k) - \sum_{s_0=k-1}^{n-(h+1)} p_{k-1}(s_0).$$

(3.2.3.2) Otherwise, $p_k^{(h)}(n) =$

$$= \sum_{j=1}^{\min\{k,n-k\}} p_j(n-k) - \sum_{s_0=k-1}^{t_0} p_{k-1}(s_0) - \sum_{s_0=t_0+1}^{n-(h+1)} p_{k-1}^{(n-s_0)}(s_0).$$

$\square$

Example 2.3. Calculate $p_4^{(8)}(9)$, $p_7^{(3)}(17)$, $p_6^{(4)}(20)$ and $p_8^{(7)}(50)$.

(a) By (3.1) we have

$$p_4^{(8)}(9) = p_4(9) = \sum_{j=1}^{4} p_j(5) = 6.$$

(b) By (3.2.2) with $i = 3$ we have

$$p_7^{(3)}(17) = 1 + p_3^{(3)}(5) = 1 + p_3(5) = 1 + \sum_{j=1}^{2} p_j(2) = 1 + 2 = 3.$$

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(c) By (3.2.2) with $i = 2$, and by (3.2.3.1) with $t_0 = 6$ we obtain

$$p_6^{(4)}(20) = 1 + p_3^{(4)}(8) = 1 + \sum_{j=1}^{3} p_j(5) - \sum_{s_0=2}^{3} p_2(s_0).$$

Thus,

$$p_6^{(4)}(20) = 1 + p_1(5) + p_2(5) + p_3(5) - p_2(2) - p_2(3) = 4.$$

(d) By (3.2.2) with $i = 2$, and by (3.2.3.2) with $t_0 = 16$ we have

$$p_8^{(7)}(50) = 1 + p_5^{(7)}(29) = 1 + \sum_{j=1}^{5} p_j(24) - \sum_{s_0=4}^{21} p_4^{(29-s_0)}(s_0) =$$

$$= 1 + \sum_{j=1}^{5} p_j(24) - \sum_{s_0=4}^{16} p_4(s_0) - \sum_{s_0=17}^{21} p_4^{(29-s_0)}(s_0).$$

Applying Theorem 2.2 to each term of the last summation

$$p_8^{(7)}(50) = 1 + \sum_{j=1}^{5} p_j(24) - \sum_{s_0=4}^{16} p_4(s_0) - \sum_{s_0=17}^{21} p_4^{(29-s_0)}(s_0)$$

$$= 1 + 333 - 155 - (p_4^{(12)}(17) + p_4^{(11)}(18) + p_4^{(10)}(19) + p_4^{(8)}(20) + p_4^{(8)}(21))$$

$$= 1 + 33 - 155 - (37 + 40 + 38 + 33 + 20) = 11.$$

Now, consider a realizable triple (n, r, p) and an ITN matrix A associated with this triple. The number of zero Jordan canonical forms of A is $p_{n-r}^{(p)}(n-p)$. Since (n, r, p) is a realizable triple equation (2) holds, therefore $(n - r)p \geq n - p$ and we only consider the third item of Theorem 2.2. We introduce the following algorithm, which first checks if the triple (n, r, p) is realizable and after that, it computes $p_{n-r}^{(p)}(n-p)$.

Algorithm 1 $c = \text{numberpartitions}(n, r, p)$

1: $j = 1$; $A(j, 1:11) = [n - p, \ n - r, \ p, \ (n - r) * p, \ \text{zeros}(1, 7)]$; $w = \text{size}(A, 1)$;
2: **if** $r < p \ || \ r > n - \text{ceil}((n - p)/p)$ **then**
3:      It is not a realizable triple
4:      $A(j, 10) = 0$;
5: **else if** $r == p \ || \ r == n - \text{ceil}((n - p)/p)$ **then**
6:      $A(j, 10) = 1$;
7: **end if**
8: $A(j + 1, 1:11) = \text{zeros}(1, 11)$;
9: **while** $A(j, 1) > 0$ **do**
10:      **if** $A(j, 4) > A(j, 1)$ **then**
11:          $A(j, 5) = A(j, 1) - A(j, 2) + 1$;
12:      **end if**
13:      **if** $A(j, 5) <= A(j, 3)$ **then**
14:          $[T, h] = \text{triangle}(A(j, 1) + A(j, 3), A(j, 1) + A(j, 3) - A(j, 2), A(j, 3))$; $A(j, 10) = h$;
15:          **if** $j + 1 > w$ **then**
16:              $A(j + 1, 1:11) = \text{zeros}(1, 11)$; $w = w + 1$;
17:          **end if**
18:      **else**
19:          $A(j, 6) = A(j, 2) * (A(j, 3) - 1)$;
20:          **if** $A(j, 6) <= A(j, 1)$ **then**
21:              $A(j, 7) = A(j, 1) - A(j, 6)$; $A(j, 10) = 1$;
22:              $A(w + 1, 1 \ : \ 11) = [A(j, 1) - (A(j, 7) + 1) * A(j, 3), A(j, 2) - (A(j, 7) + 1), \ A(j, 3), (A(j, 2) - (A(j, 7) + 1)) * A(j, 3), \text{zeros}(1, 7)]$;
23:              $w = w + 1$;
24:          **else**
25:              $A(j, 8) = \text{floor}((A(j, 1) + A(j, 2))/2) - 1$; $A(j, 9) = A(j, 1) - (A(j, 3) + 1)$;
26:              $l = \text{min}(A(j, 8), A(j, 9))$; $v = \text{max}([A(j, 2), \ A(j, 1) - A(j, 2), \ l])$;
27:              $[T, h] = \text{triangle}(v + 2 * A(j, 2), v + A(j, 2), A(j, 2))$;
28:              $b = \text{min}([A(j, 2), \ A(j, 1) - A(j, 2)])$;
29:              **if** $A(j, 9) <= A(j, 8)$ **then**
30:                  $A(j, 10) = T(A(j, 1) - A(j, 2), 1 \ : \ b) * \text{ones}(b, 1) - \text{ones}(1, A(j, 1) - (A(j, 2) + A(j, 3) - 1)) * T(A(j, 2) - 1 : A(j, 9), \ A(j, 2) - 1)$;
31:                  **if** $j + 1 > w$ **then**
32:                      $A(j + 1, 1:11) = \text{zeros}(1, 11)$; $w = w + 1$;
33:                  **end if**
34:              **else**
35:                  $(j, 10) = T(A(j, 1) - A(j, 2), 1 \ : \ b) * \text{ones}(b, 1) - \text{ones}(1, A(j, 8) - A(j, 2) + 2) * T(A(j, 2) - 1 : A(j, 8), A(j, 2) - 1)$;
36:                  **for** $i = 1 : A(j, 9) - A(j, 8)$ **do**
37:                      $A(w + i, 1:11) = [A(j, 8) + i, A(j, 2) - 1, A(j, 1) - (A(j, 8) + i), (A(j, 2) - 1) * (A(j, 1) - (A(j, 8) + i)), \text{zeros}(1, 6), \ A(j, 11) + 1]$;
38:                  **end for**
39:                  $w = \text{size}(A, 1)$;
40:              **end if**
41:          **end if**
42:      **end if**
43:      $j = j + 1$;
44: **end while**
45: **for** $q = 1 : j$ **do**
46:      $A(q, 10) = A(q, 10) * (-1)^{A(q, 11)}$;
47: **end for**
48: $c = \text{ones}(1, j) * A(1 : j, 10)$;

We need the following procedure for Algorithm 1, which constructs a lower triangular matrix as a consequence of Remark 1 and Theorem 2.1.

Algorithm 2 $[T, h] = \text{triangle}(n, r, p)$

1: $T = \text{zeros}(r - p, r - p)$; $T(1:3, 1:3) = \text{tril}(\text{ones}(3,3))$;
2: **for** $i = 4 : r - p$ **do**
3:     $T(i, 1:2) = [1, \text{floor}([i/2])]$; $T(i, i-2 : i) = [2, \ 1, \ 1]$;
4: **end for**
5: **for** $i = 6 : r - p$ **do**
6:     **for** $j = 3 : i - 3$ **do**
7:        $q = 0$; $t = \min([j, i - j])$;
8:        **for** $h = 1 : t$ **do**
9:           $q = q + T(i - j, h)$;
10:        **end for**
11:        $T(i, j) = q$;
12:     **end for**
13: **end for**
14: $u = \min([n - r, \ r - p])$; $h = 0$;
15: **for** $j = 1 : u$ **do**
16:     $h = h + T(r - p, j)$;
17: **end for**

Example 2.4. We apply Algorithm 1 to the corresponding triples associated with the four cases of Example 2.3, taking into account the conversion $\widehat{n} = n - p$, $k = n - r$ and $h = p$, where $\widehat{n}$ is renamed the number n in Section 2. That is,

(a) $p_4^{(8)}(9)$ corresponds to the triple $(17, 13, 8)$.

(b) $p_7^{(3)}(17)$ corresponds to $(20, 13, 3)$.

(c) $p_6^{(4)}(20)$ corresponds to $(24, 18, 4)$.

(d) $p_8^{(7)}(50)$ with $(57, 49, 7)$.

3. Properties of ITN matrices by full rank factorizations and the Flanders theorem

We have computed the number of zero Jordan canonical forms associated with a realizable triple (n, r, p) in Section 2. Now, we study some properties related to full rank factorizations of ITN matrices to obtain these Jordan structures. We recall the following definition [14].

Definition 3.1. Let $A \in \mathbb{R}^{n \times n}$ be a matrix with $\text{rank}(A) = r$. We say that $A = FG$ is a full rank factorization of A if $F \in \mathbb{R}^{n \times r}$ and $G \in \mathbb{R}^{r \times n}$.

Note that a full rank factorization of any singular matrix is not unique. For ITN matrices we use the following factorization in echelon form.

Definition 3.2. Let A be an $n \times n$ ITN matrix with $\text{rank}(A) = r$. We say that $A = LU$ is the *full rank factorization in echelon form* of A, if $L \in \mathbb{R}^{n \times r}$ is a lower echelon matrix and $U \in \mathbb{R}^{r \times n}$ is an upper echelon matrix.

We recall [15, Section 1] that $U \in \mathbb{R}^{r \times n}$ with $\text{rank}(U) = r$ is an upper echelon matrix if it satisfies the following conditions:

(1) The first nonzero entry in each row is called leading entry for that row.
(2) Each leading entry is to the right of the leading entry in the row above it.

L is a lower echelon matrix if its transpose is an upper echelon matrix.

The full rank factorization in echelon form of A can be obtained using the quasi-Neville elimination process [16]. It is a variant of the Neville elimination process for singular matrices which consists of leaving the zero row in its position and continuing the elimination process with the matrix obtained by deleting the zero rows. It allows one to obtain the unique TN matrices L and U without zero rows and columns.

In general, the spectral relations between two matrices FG and GF have been proved by Flanders [17]. In the particular case of full rank factorizations, the next theorem [14, Theorem 4.1] gives the spectral relations between $A = LU$ and $A_1 = UL$.

Theorem 3.3. *Let $A \in \mathbb{R}^{n \times n}$ be an ITN matrix with $\mathrm{rank}(A) = r$. Let $A = LU$ be the full rank factorization in echelon form of A. If $A_1 = UL \in \mathbb{R}^{r \times r}$ then*

(1) A and A_1 have the same elementary divisors with nonzero roots.
(2) If $s_1 \geq s_2 \geq \cdots \geq s_m > 0$ and $s_1' \geq s_2' \geq \cdots \geq s_m' \geq 0$ are the sizes of zero Jordan blocks of A and A_1 respectively, then $s_i - s_i' = 1$ for all i.

Definition 3.4 ([18])**.** The *Segre characteristic* of a matrix $A \in \mathbb{R}^{n \times n}$ relative to its eigenvalue λ is the non-increasing ordered sequence of sizes of Jordan blocks of A associated with λ.

By Theorem 3.3, the Segre characteristic of A relative to 0 is the sequence $\{s_1, s_2, \ldots, s_m\}$.

Definition 3.5 ([18])**.** The *Weyr characteristic* of a matrix $A \in \mathbb{R}^{n \times n}$ relative to its eigenvalue λ is the conjugate partition of the Segre characteristic of A relative to λ.

If $\{s_1, s_2, \ldots, s_m\}$ is the Segre characteristic of A relative to 0 and $\{w_1, w_2, \ldots, w_{s_1}\}$ is the Weyr characteristic of A relative to 0, then w_i is the number of s_j's which are greater than or equal to i. Note that

$$w_1 = \dim Ker(A) = m$$
$$w_i = \dim Ker(A^i) - \dim Ker(A^{i-1}), \quad i = 2, 3, \ldots, s_1.$$

By [18] we consider the Ferrers diagram to represent the Weyr and the Segre characteristics of A relative to 0. The number of dots in row j is w_j, while s_i is the number of dots in column i.

Remark 3. If $A = LU$ is the full rank factorization in echelon form of a singular ITN matrix $A \in \mathbb{R}^{n \times n}$ and $A_1 = UL \in \mathbb{R}^{r \times r}$, then by Theorem 3.3 we have that

$$\mathrm{rank}(A_1) = \mathrm{rank}(A^2)$$
$$\Downarrow \tag{4}$$
$$w_2 = \dim \mathrm{Ker}(A^2) - \dim \mathrm{Ker}(A) = \dim \mathrm{Ker}(A_1).$$

If $A_1 \in \mathbb{R}^{r \times r}$ is singular, we can obtain the full rank factorization $A_1 = L_1 U_1$ and by Theorem 3.3 the matrix $A_2 = U_1 L_1$ satisfies that

$$\mathrm{rank}(A_2) = \mathrm{rank}(A_1^2) = \mathrm{rank}(A^3)$$
$$\Downarrow \tag{5}$$
$$w_3 = \dim \mathrm{Ker}(A^3) - \dim \mathrm{Ker}(A^2) = \dim \mathrm{Ker}(A_2).$$

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Proceeding in this way, we can construct a sequence of matrices $A_1, A_2, \ldots, A_q$, with A_q nonsingular. For $i = 2, 3, \ldots, q$, we have that

$$\operatorname{rank}(A_i) = \operatorname{rank}(A_{i-1}^2) = \operatorname{rank}(A_{i-2}^3) = \cdots = \operatorname{rank}(A_1^i) = \operatorname{rank}(A^{i+1})$$
$$\Downarrow$$
$$w_{i+1} = \dim Ker(A^{i+1}) - \dim Ker(A^i) = \dim Ker(A_i). \tag{6}$$

If we calculate the rank of each matrix A_i, $i = 1, 2, \ldots, q-1, q$, by (6) we can obtain the Weyr characteristic of A relative to 0.

The following results give properties related to the positivity, the principal rank and the irreducibility of $A_1 = UL$, when LU is the full rank factorization in echelon form of the ITN matrix A. These results can be extended to matrices A_i, for all $i > 1$.

Theorem 3.6. *Let* $A \in \mathbb{R}^{n \times n}$ *be an ITN matrix such that* $\operatorname{rank}(A) = r$ *and* p-$rank(A) = p$. *Let* $A = LU$ *the full rank factorization in echelon form of* A. *If* $A_1 = UL \in \mathbb{R}^{r \times r}$ *then,* A_1 *is a TN matrix with* p-$rank(A_1) = p$.

Proof. A_1 is a TN matrix because it is product of TN matrices. Now, we are going to prove that p-$\operatorname{rank}(A_1) = p$-$\operatorname{rank}(A) = p$.

Suppose that $\{i_1, i_2, i_3, \ldots, i_p\}$ is the sequence of the first p-indices of A (see Definition 1.2), then

$$\det A[i_1, i_2, i_3, \ldots, i_p] > 0.$$

By the Binet-Cauchy formula, we have that

$$\det A[i_1, i_2, i_3, \ldots, i_p] = \sum_{\forall \gamma \in \mathcal{Q}_{p,r}} \det L[i_1, i_2, i_3, \ldots, i_p \,|\, \gamma] \det U[\gamma \,|\, i_1, i_2, i_3, \ldots, i_p] > 0,$$

then, there exists at least a sequence $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ such that

$$\det L[i_1, i_2, i_3, \ldots, i_p \,|\, \gamma_1, \gamma_2, \ldots, \gamma_p] > 0$$
$$\det U[\gamma_1, \gamma_2, \ldots, \gamma_p \,|\, i_1, i_2, i_3, \ldots, i_p] > 0.$$

Therefore,

$$\det A_1[\gamma_1, \gamma_2, \ldots, \gamma_p] = \sum_{\forall \rho \in \mathcal{Q}_{p,n}} \det U[\gamma_1, \gamma_2, \ldots, \gamma_p \,|\, \rho] \det L[\rho \,|\, \gamma_1, \gamma_2, \ldots, \gamma_p]$$
$$= \det U[\gamma_1, \gamma_2, \ldots, \gamma_p \,|\, i_1, i_2, i_3, \ldots, i_p] \det L[i_1, i_2, i_3, \ldots, i_p \,|\, \gamma_1, \gamma_2, \ldots, \gamma_p] +$$
$$\sum_{\substack{\forall \rho \in \mathcal{Q}_{p,n} \\ \rho \neq \{\gamma_1, \gamma_2, \ldots, \gamma_p\}}} \det U[\gamma_1, \gamma_2, \ldots, \gamma_p \,|\, \rho] \det L[\rho \,|\, \gamma_1, \gamma_2, \ldots, \gamma_p] > 0$$

that is, p-$\operatorname{rank}(A_1) \geq p$.

Now, we suppose without loss of generality that p-$\operatorname{rank}(A_1) = p + 1$. As a conse-

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quence, there exists a sequence $\xi = \{\xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1}\}$ such that

$$\det A_1[\xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1}] > 0.$$

By the Binet-Cauchy formula, we have that

$$\det A_1[\xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1}] = \\ \sum_{\forall \rho \in \mathcal{Q}_{p,n}} \det U[\xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1} \mid \rho] \det L[\rho \mid \xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1}] > 0.$$

That is, there exists at least a sequence $\rho = (\rho_1, \rho_2, \ldots, \rho_{p+1})$ such that

$$\det U[\,\xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1} \mid \rho_1, \rho_2, \ldots, \rho_{p+1}] \; > \; 0$$
$$\det L[\rho_1, \rho_2, \ldots, \rho_{p+1} \mid \xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1}] \; > \; 0.$$

Therefore,

$$\det A[\rho_1, \rho_2, \ldots, \rho_{p+1}] = $$
$$= \sum_{\forall \gamma \in \mathcal{Q}_{p,r}} \det L[\rho_1, \rho_2, \ldots, \rho_{p+1} \mid \gamma] \det U[\gamma \mid \rho_1, \rho_2, \ldots, \rho_{p+1}]$$
$$= \det L[\rho_1, \rho_2, \ldots, \rho_{p+1} \mid \xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1}] \det U[\xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1} \mid \rho_1, \rho_2, \ldots, \rho_{p+1}]$$
$$+ \sum_{\substack{\forall \gamma \in \mathcal{Q}_{p,r} \\ \gamma \neq \{\xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1}\}}} \det L[\rho_1, \rho_2, \ldots, \rho_{p+1} \mid \gamma] \det U[\gamma \mid \rho_1, \rho_2, \ldots, \rho_{p+1}] > 0,$$

which implies that p-rank$(A) \geq p + 1$. This contradicts the initial hypothesis, then p-rank$(A_1) = p$. $\qquad\square$

Theorem 3.7. *Let* $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ *be an ITN matrix such that* rank$(A) = r$ *and* p-rank$(A) = p$*. Let* $A = LU$ *the full rank factorization in echelon form of* A*. If* $A_1 = (a_{ij}^{(1)}) = UL \in \mathbb{R}^{r \times r}$*, then* A_1 *is an irreducible TN matrix.*

Proof. By Theorem 3.6, we know that A_1 is a TN matrix. Moreover, since L and U are TN matrices without zero rows and columns then, A_1 also has no zero rows and columns.

By Lemma 1.3 if we prove that $a_{ij}^{(1)} > 0$, for all $i, j = 1, 2, \ldots, r$ with $|i - j| \leq 1$ we will have that A_1 is irreducible.

(1) Since A is ITN, by Lemma 1.3, $a_{11} > 0$ and therefore $a_{11}^{(1)} > 0$.
(2) Suppose that $a_{12}^{(1)} = 0$. Since A is ITN, suppose that a_{1j} is the last nonzero element of its first row, with $2 \leq j \leq n$. Then,

(2.1) if $j = n$ we have that

$$a_{12}^{(1)} = U(1,:)L(:,2) = [a_{11}\ a_{12}\ \ldots\ a_{1n}]\begin{bmatrix} 0 \\ l_{22} \\ \vdots \\ l_{j2} \\ \vdots \\ l_{n2} \end{bmatrix} = \sum_{i=1}^{n} a_{1i}l_{i2} = 0,$$

which implies that $l_{i2} = 0$, for $i = 1, 2, \ldots, n$. This is a contradiction because L has no zero columns.

(2.2) if $2 \leq j < n$, then

$$a_{12}^{(1)} = U(1,:)L(:,2) = [a_{11}\ a_{12}\ \ldots\ a_{1j}\ 0\ \ldots\ 0]\begin{bmatrix} 0 \\ l_{22} \\ \vdots \\ l_{j2} \\ \vdots \\ l_{n2} \end{bmatrix} = \sum_{i=2}^{j} a_{1i}l_{i2} = 0,$$

that is, $l_{i2} = 0$, for $i = 2, 3, \ldots, j$. Therefore, rows $2, 3, \ldots, j$ of A are linear combinations of the first row of A. Concretely, $A(j,:) = \alpha_j A(1,:)$, then $a_{j,j+1} = 0$, which contradicts the irreducibility of A.

We conclude that $a_{12}^{(1)} > 0$. Analogously by columns we can demostrate that $a_{21}^{(1)} > 0$. Since A_1 is TN we have that $a_{22}^{(1)} > 0$.

(3) Now, suppose that the submatrix $A_1[1, 2, \ldots, r-1]$ is an ITN matrix and $a_{r-1,r}^{(1)} = 0$, with $3 \leq r \leq n$. Since A_1 is TN then, $a_{j,r}^{(1)} = 0$, for $j = 1, 2, \ldots, r-2$, that is,

$$A_1(1 : r - 1, r) = U(1 : r - 1, :)L(:, r) = O_{(r-1) \times 1}.$$

We have that $a_{r,r+1} > 0$ because A is ITN. If a_{rj} is the last nonzero entry of the r-th row of A, with $r + 1 \leq j \leq n$, then $a_{is} = 0$ for $i = 1, 2, \ldots, r$ and $s = j + 1, j + 2, \ldots, n$.

Let $2 \leq t_1 \leq t_2 \leq \ldots \leq t_{r-1} \leq t_r = j \leq n$ be the indices of the last nonzero elements of A in rows $1, 2, \ldots, r-1, r$, respectively, with $t_q \geq q + 1$, for $q = 1, 2, \ldots, r$. Then, these are also the indices of the last nonzero elements of

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U in rows $1, 2, \ldots, r-1, r$. As a consequence, from

$$A_1(1 : r-1, r) = U(1 : r-1, :)L(:, r) = U(1 : r-1, :) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{rr} \\ l_{r+1,r} \\ \vdots \\ l_{nr} \end{bmatrix} = O_{(r-1) \times 1}$$

we conclude that $l_{sr} = 0$ for $s = 1, 2, \ldots, t_{r-1}$. Therefore, we have

$$A(j, :) = \sum_{i=1}^{r-1} \alpha_i A(i, :) = \sum_{i=1}^{r-1} \alpha_i [a_{i1} \cdots a_{it_i} \, 0 \cdots 0], \quad t_i \leq j.$$

This implies that $a_{j,j+1} = 0$, which contradicts the irreducibility of A. Thus, $a_{r-1,r}^{(1)} > 0$, with $2 \leq r \leq n-1$. Analogously by columns we can prove that $a_{r,r-1}^{(1)} > 0$ and since A_1 is TN we conclude that $a_{r,r}^{(1)} > 0$.

The obtained results prove that A_1 is irreducible. $\qquad\square$

Summarizing, we have shown that if $A \in \mathbb{R}^{n \times n}$ is an ITN matrix with $\text{rank}(A) = r$ and $p\text{-rank}(A) = p$, and $A = LU$ is the full rank factorization in echelon form of A then, matrix $A_1 = UL \in \mathbb{R}^{r \times r}$ is also an ITN matrix such that $p\text{-rank}(A_1) = p$.

4. Jordan canonical forms of ITN matrices

In this section we give a procedure to obtain all Jordan canonical forms associated with a realizable triple. Consider an ITN matrix A associated with the realizable triple (n, r_1, p). If $p = 1$ then by equation (2) we have that $r_1 = 1$. In this case, A is associated with the realizable triple $(n, 1, 1)$, which implies that $\text{am}(0) = \text{gm}(0) = n-1$. Therefore, A has $n-1$ zero Jordan blocks of size 1×1. Now we study the case $p > 1$.

Let $A = LU$ be the full rank factorization in echelon form of A and $A_1 = UL$. By Theorems 3.6 and 3.7, A_1 is an ITN matrix with $p\text{-rank}(A_1) = p\text{-rank}(A) = p$, but we do not know the value of $\text{rank}(A_1)$, that is, A_1 is an ITN matrix associated with the realizable triple (r_1, r_2, p), where the value of r_2 is unknown. Taking into account that the relation (2) for n, r_1 and p is

$$p \leq r_1 \leq n - \left\lceil \frac{n-p}{p} \right\rceil$$

the next theorem obtains the corresponding relations between r_1, r_2 and p, with all possible values for r_2.

Theorem 4.1. *Let (n, r_1, p) be a realizable triple and let $A \in \mathbb{R}^{n \times n}$ be an ITN matrix associated with this triple. We consider $A = LU$ the full rank factorization in echelon*

form of A. If $A_1 = UL \in \mathbb{R}^{r_1 \times r_1}$, *then* A_1 *is an ITN matrix associated with the realizable triple* (r_1, r_2, p) *where*

$$\max\{p, 2r_1 - n\} \le r_2 \le r_1 - \left\lceil \frac{r_1 - p}{p - 1} \right\rceil.$$

Proof. As we have already seen, A_1 is an ITN matrix associated with the triple (r_1, r_2, p). To obtain all possible values of r_2 note that $p \le r_2$ and, by Theorem 3.3, the number of eigenvectors of A_1 corresponding to 0 is less than or equal to the number of eigenvectors of A corresponding to the same eigenvalue, that is, $n - r_1 \ge r_1 - r_2$ which implies that $r_2 \ge 2r_1 - n$. Then,

$$\max\{p, 2r_1 - n\} \le r_2.$$

By expression (6) $\operatorname{rank}(A_1^i) = \operatorname{rank}(A^{i+1})$, for $i = 1, 2, \ldots$; then we have that $\operatorname{rank}(A_1^{p-1}) = \operatorname{rank}(A^p) = p$, and $\dim \operatorname{Ker}(A_1^{p-1}) = r_1 - p$, that is, the maximum size of zero Jordan blocks of A_1 is less than or equal to $p - 1$. So, $(p-1)(r_1 - r_2) \ge r_1 - p$ and therefore

$$r_2 \le r_1 - \left\lceil \frac{r_1 - p}{p - 1} \right\rceil.$$

$\square$

Remark 4. Consider Theorem 4.1,

(1) If $r_2 = p$, A_1 is associated with the triple (r_1, p, p) and $\operatorname{am}(0) = \operatorname{gm}(0) = r_1 - p$, which implies that A_1 has $r_1 - p$ zero Jordan blocks of size 1×1. Since $\operatorname{rank}(A) = r_1$ by (4) we have that A has $r_1 - p$ zero Jordan blocks of size 2×2 and $n + p - 2r_1$ zero Jordan blocks of size 1×1.

(2) If $r_2 > p$, we repeat the previous process, that is, we obtain the full rank factorization in echelon form of A_1, $A_1 = L_1 U_1$ and construct the ITN matrix $A_2 = U_1 L_1$ associated with the triple (r_2, r_3, p). Since $\operatorname{rank}(A_2) = \operatorname{rank}(A_1^2) = \operatorname{rank}(A^3)$ reasoning similarly to Theorem 4.1 we obtain that r_3 satisfies the following inequalities

$$\max\{p, 2r_2 - r_1\} \le r_3 \le r_2 - \left\lceil \frac{r_2 - p}{p - 2} \right\rceil.$$

Now, for each value of r_3, we study the different possibilities. This process follows with each r_i until all obtained triples have the same rank and principal rank.

(3) For each item of this process we obtain the new full rank factorization in echelon form $A_{i-1} = L_{i-1} U_{i-1}$ and a new matrix $A_i = U_{i-1} L_{i-1}$. This matrix is associated with the realizable triple (r_i, r_{i+1}, p) and the following inequalities hold

$$\max\{p, 2r_i - r_{i-1}\} \le r_{i+1} \le r_i - \left\lceil \frac{r_i - p}{p - i} \right\rceil. \tag{7}$$

Note that the different zero Jordan canonical forms of an ITN matrix associated with a realizable triple can be directly obtained from the values of r_i and the new realizable

triples following the above process taking into account equation (7). It is not necessary to calculate the full rank factorizations in echelon form. The next procedure computes the process.

Procedure 1. *Given the realizable triple* (n, r_1, p), *where*

$$1 \leq r_1 \leq n - 1, \qquad p \leq r_1 \leq n - \left\lceil \frac{n-p}{p} \right\rceil$$

the following steps compute all Jordan canonical forms associated with this triple.

Let $r_0 = n$, *for* $i = 1, 2, \ldots$,

    Step 1. Obtain the triples (r_i, r_{i+1}, p) *where*

$$\max\{p, 2r_i - r_{i-1}\} \leq r_{i+1} \leq r_i - \left\lceil \frac{r_i - p}{p - i} \right\rceil$$

    For each r_{i+1} *do*
    Step 2. If $r_{i+1} = p \longrightarrow$ *end.*
    Step 3. If $r_{i+1} > p \longrightarrow$ *go to Step 1.*

From each sequence of realizable triples obtained by this procedure, we calculate the Weyr and the Segre characteristics of an ITN matrix associated with the initial realizable triple (n, r_1, p). The following example shows this process.

Example 4.2. Obtain all possible zero Jordan canonical forms of an ITN matrix associated with the realizable triple $(17, 13, 8)$.

By applying Algorithm 1 to Example 2.4-a, we have that the number of the zero Jordan canonical forms that we can obtain is 6. Now, we calculate these zero Jordan canonical forms.

From the triple $(17, 13, 8)$ we have $n = r_0 = 17$, $r_1 = 13$, $p = 8$, $\mathrm{am}(0) = n - p = 9$ and $\mathrm{gm}(0) = n - r_1 = 4$.

Applying Procedure 1 to the triple $(17, 13, 8)$ we obtain the triples $(13, r_2, 8)$, where r_2 satisfies the inequalities of Theorem 4.1. Then, $9 \leq r_2 \leq 12$:

If $r_2 = 9$, since it is greater that $p = 8$, we apply again Procedure 1 obtaining the triple $(r_2, r_3, p) = (9, 8, 8)$. Now, $r_3 = p = 8$ and therefore, we finish to apply Procedure 1. The sequence of realizable triples obtained is the following,

$$
\begin{array}{ccccc}
A & & A_1 & & A_2 \\
(17, 13, 8) & \longrightarrow & (13, 9, 8) & \longrightarrow & (9, 8, 8) \\
\dim \mathrm{Ker}(A) = 4 & & \dim \mathrm{Ker}(A_1) = 4 & & \dim \mathrm{Ker}(A_2) = 1.
\end{array}
$$

By (4) and (5) the Weyr characteristic of A relative to the eigenvalue 0 is $(w_1, w_2, w_3) = (4, 4, 1)$ and its conjugated sequence $(s_1, s_2, s_3, s_4) = (3, 2, 2, 2)$ is the Segre characteristic of A relative to 0. Using the Ferrers diagram in French notation we represent the

Weyr and Segre characteristic of A relative to the eigenvalue 0,

$$
\begin{array}{cccc}
\bullet & & & & w_3 = 1 \\
\bullet & \bullet & \bullet & \bullet & w_2 = 4 \\
\bullet & \bullet & \bullet & \bullet & w_1 = 4 \\
s_1 = 3 & s_2 = 2 & s_3 = 2 & s_4 = 2 &
\end{array}
$$

If $r_2 = 10$, applying again Procedure 1 we obtain that r_3 satisfies $8 \leq r_3 \leq 9$:
• If $r_3 = 8$ we have

$$
\begin{array}{ccccc}
A & & A_1 & & A_2 \\
(17, 13, 8) & \longrightarrow & (13, 10, 8) & \longrightarrow & (10, 8, 8) \\
\dim \mathrm{Ker}(A) = 4 & & \dim \mathrm{Ker}(A_1) = 3 & & \dim \mathrm{Ker}(A_2) = 2.
\end{array}
$$

In this case, the Weyr and the Segre characteristics of A relative to 0 are $(4, 3, 2)$ and $(3, 3, 2, 1)$, respectively. Graphically,

$$
\begin{array}{cccc}
\bullet & \bullet & & & w_3 = 2 \\
\bullet & \bullet & \bullet & & w_2 = 3 \\
\bullet & \bullet & \bullet & \bullet & w_1 = 4 \\
s_1 = 3 & s_2 = 3 & s_3 = 2 & s_4 = 1 &
\end{array}
$$

• If $r_3 = 9$, reapplying Procedure 1 we obtain the triple $(9, 8, 8)$. So, we have

$$
\begin{array}{ccccccc}
A & & A_1 & & A_2 & & A_3 \\
(17, 13, 8) & \longrightarrow & (13, 10, 8) & \longrightarrow & (10, 9, 8) & \longrightarrow & (9, 8, 8) \\
\dim \mathrm{Ker}(A) = 4 & & \dim \mathrm{Ker}(A_1) = 3 & & \dim \mathrm{Ker}(A_2) = 1 & & \dim \mathrm{Ker}(A_3) = 1
\end{array}
$$

Now, the Weyr and the Segre characteristics of A relative to 0 are $(4, 3, 1, 1)$ and $(4, 2, 2, 1)$, respectively. The corresponding Ferrers diagram in French notation is

$$
\begin{array}{cccc}
\bullet & & & & w_4 = 1 \\
\bullet & & & & w_3 = 1 \\
\bullet & \bullet & \bullet & & w_2 = 3 \\
\bullet & \bullet & \bullet & \bullet & w_1 = 4 \\
s_1 = 4 & s_2 = 2 & s_3 = 1 & s_4 = 1 &
\end{array}
$$

If $r_2 = 11$ we proceed as in the previous cases obtaining the following two sequences of realizable triples and the corresponding Jordan structures.

$$
\begin{array}{ccccccc}
A & & A_1 & & A_2 & & A_3 \\
(17, 13, 8) & \longrightarrow & (13, 11, 8) & \longrightarrow & (11, 9, 8) & \longrightarrow & (9, 8, 8) \\
\dim \mathrm{Ker}(A) = 4 & & \dim \mathrm{Ker}(A_1) = 2 & & \dim \mathrm{Ker}(A_2) = 2 & & \dim \mathrm{Ker}(A_3) = 1
\end{array}
$$

The Weyr and the Segre characteristics of A relative to 0 are $(4, 2, 2, 1)$ and $(4, 3, 1, 1)$, respectively.

$$
\begin{array}{ccccccc}
A & & A_1 & & A_2 & \\
(17, 13, 8) & \longrightarrow & (13, 11, 8) & \longrightarrow & (11, 10, 8) & \longrightarrow \\
\dim \mathrm{Ker}(A) = 4 & & \dim \mathrm{Ker}(A_1) = 2 & & \dim \mathrm{Ker}(A_2) = 1 &
\end{array}
$$

$$
\begin{array}{ccccc}
& A_3 & & A_4 & \\
\longrightarrow & (10, 9, 8) & \longrightarrow & (9, 8, 8) & \\
& \dim \mathrm{Ker}(A_3) = 1 & & \dim \mathrm{Ker}(A_4) = 1.
\end{array}
$$

Now, the Weyr and the Segre characteristics of A relative to 0 are $(4, 2, 1, 1, 1)$ and $(5, 2, 1, 1)$, respectively.

Finally, if $r_2 = 12$, by Procedure 1 we obtain the following sequence of realizable triples

$$
\begin{array}{ccccccc}
A & & A_1 & & A_2 & \\
(17, 13, 8) & \longrightarrow & (13, 12, 8) & \longrightarrow & (12, 11, 8) & \longrightarrow \\
\dim \mathrm{Ker}(A) = 4 & & \dim \mathrm{Ker}(A_1) = 1 & & \dim \mathrm{Ker} r(A_2) = 1 &
\end{array}
$$

$$
\begin{array}{ccccccc}
& A_3 & & A_4 & & A_5 & \\
\longrightarrow & (11, 10, 8) & \longrightarrow & (10, 9, 8) & \longrightarrow & (9, 8, 8) & \\
& \dim \mathrm{Ker}(A_3) = 1 & & \dim \mathrm{Ker}(A_4) = 1 & & \dim \mathrm{Ker}(A_5) = 1.
\end{array}
$$

The Weyr and the Segre characteristics of A relative to 0 are $(4, 1, 1, 1, 1, 1)$ and $(6, 1, 1, 1)$, respectively.

The described process is given by the tree digraph that appears in Figure 1. In that tree, each branch represents the sequence of realizable triples obtained when we apply Procedure 1 to $(17, 13, 8)$.

From the described Procedure 1 we give the following algorithm which first checks if the triple (n, r, p) is realizable and after that, its computes the different zero Jordan canonical forms corresponding to this triple.
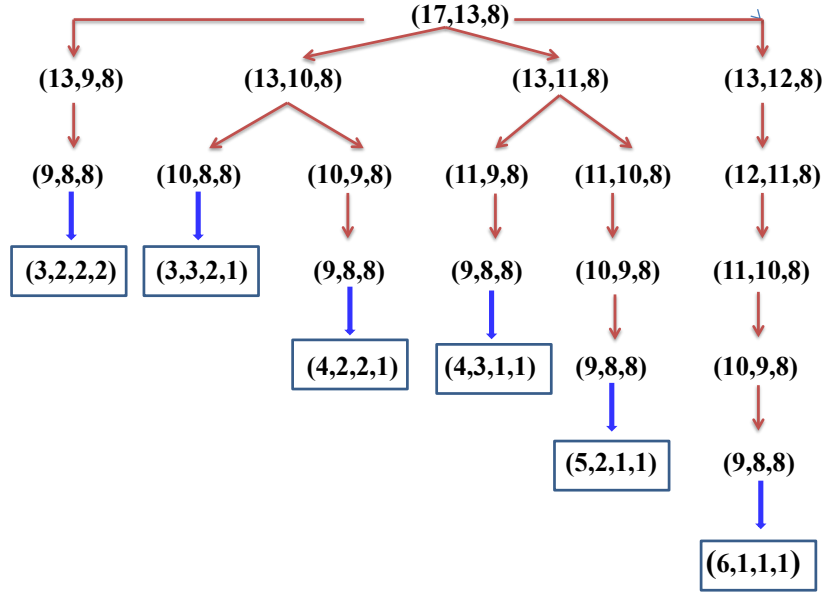
Figure 1. Digraph associated with the triple $(17, 13, 8)$

Algorithm 3 Program(n, r, p)

1: **if** $r < p \ || \ r > n - \text{ceil}((n - p)/p)$ **then**
2:     It is not a realizable triple
3: **else if** $r == p$ **then**
4:     $w = ones(1, n - p) = 1;$
5: **else**
6:     $M = [n, r, p, 1, 0, 0]; j = 1; d = 1;$
7:     **while** $j <= d$ **do**
8:       **if** $M(j, 5) == 0$ **then**
9:         $M(j, 5) == 1; a = M(j, 1); b = M(j, 2); i = M(j, 4);$
10:         $t = \max([p, 2 * b - a]); s = b - \text{ceil}((b - p)/(p - i));$
11:         **for** $k = t : s$ **do**
12:           $d = d + 1;$
13:           **if** $k > p$ **then**
14:             $M(d, :) = [b, k, p, i + 1, 0, j];$
15:           **else**
16:             $M(d, :) = [b, k, p, i + 1, 1, j]$
17:           **end if**
18:         **end for**
19:       **end if**
20:       $j = j + 1;$
21:     **end while**
22:     $k = 0; i = d - k;$
23:     **while** $i > 1$ **do**
24:       **if** $M(i, 5) == 1$ **then**
25:         $H = \text{zeros}(1, 3);$
26:         **for** $j = M(i, 4) : -1 : 1$ **do**
27:           $H(j, 1 : 3) = M(i, 1 : 3); M(i, 5) = 0; i = M(i, 6);$
28:         **end for**
29:         $g = size(H, 1); v = (H(:, 1) - H(:, 2))'; z = ones(g, 1);$
30:         **for** $w = 1 : v(1)$ **do**
31:           $x = v > w - 1; s(w) = x * z;$
32:         **end for**
33:         w
34:         $v = 0; \ M(1, 5) = 1$
35:       **end if**
36:       $k = k + 1; i = d - k;$
37:     **end while**
38: **end if**

Example 4.3. Applying Algorithm 3 to the realizable triple $(17, 13, 8)$ we compute the 6 zero Jordan canonical forms that we have already obtained by Procedure 1 in Example 4.2.

The Segre characteristic
$s_1 = (6, 1, 1, 1)$
$s_2 = (5, 2, 1, 1)$
$s_3 = (4, 3, 1, 1)$
$s_4 = (4, 2, 2, 1)$
$s_5 = (3, 3, 2, 1)$
$s_6 = (3, 2, 2, 2)$

Acknowledgements

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