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Symmetry relations between dynamical planes

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1 Introduction

There is a large number of problems in Science and Engineering that can not be solved analytically. One example is the solution of the nonlinear equation $f(x) = 0$. A way to proceed with this kind of problems is the application of iterative methods. Many techniques regarding this topic can be found in the literature [1], and so many classifications based on their own features. Some criteria to sort the iterative methods are based on its order of convergence p, the number of functional evaluations per step d or the optimality of the method [2] when $p = 2^{d-1}$, amongst others. However, a key point of an iterative scheme is the stability of the method for every initial guess. This study can be performed with a dynamical analysis.

During the last years, the analysis of the stability by means of complex dynamics has been widely extended between the authors of this topic. In order to understand where and why the iterative method fails or succeeds, a deep study on complex dynamics needs to be performed [3, 4]. However, a smoother analysis can be performed if the objective is only the knowledge of where the iterative scheme fails or succeeds. In both cases, the complex dynamics analysis is assisted by graphical tools. The main representation is the dynamical plane [5, 6]. The dynamical plane represents, for a set of initial guesses in the complex plane, the final state of the orbit of each initial guess.

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. Moreover, this rational function represents the fixed point operator of an iterative method when it is applied on a polynomial $p(z)$. In [7] it was stated that the dynamical planes of the introduced iterative methods satisfied the property

$$R(\bar{z}) = \overline{R(z)}. \tag{1}$$

One consequence can be found by means of symmetry . If $R(z)$ satisfies (1), then the dynamical planes are symmetric about polar axis. Regarding the computational cost side, a symmetry directly involves the reduction of the number of operations that need to be performed to obtain the dynamical planes.

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2 Symmetries in one-point iterative methods of order two

Any iterative method of one point of order two can be expressed as [8]

$$z_{k+1} = z_k - H(t(z_k))\frac{f(z_k)}{f'(z_k)}, \tag{2}$$

where $H(t)$ is a function of variable $t = f(z)/f'(z)$ that satisfies $H(0) = 1$.

Proposition 1 *Let $f(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and $H(t) = \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. If $f(z)$ and $H(t)$ satisfy*

1. $f(\bar{z}) = \overline{f(z)}$ *and* $f'(\bar{z}) = \overline{f'(z)}$,
2. $H(\bar{t}) = \overline{H(t)}$,

then $R(\bar{z}) = \overline{R(z)}$.

On the one hand, some functions $f(z)$ that satisfy the previous property are the polynomials of real coefficients $p(z)$

$$p(z) = \sum_{i=0}^{n} p_i z^i, \quad p_i \in \mathbb{R}.$$

In addition, some functions $H(t)$ that satisfy both $H(\bar{t}) = \overline{H(t)}$ and $H(0) = 1$ are the rational functions of real coefficients, such that

$$H(t) = \frac{a_0 + \sum_{i=1}^{n} a_i t^i}{a_0 + \sum_{j=1}^{m} b_j t^j}, a_i, b_j \in \mathbb{R}.$$

Some examples can be found in the literature, such that the methods of

- Newton, $H(t) = 1$.
- Kanwar-Tomar [9], $H(t) = \frac{1}{1+\beta t}, \beta \in \mathbb{R}$.
- Kou-Li [10], $H(t) = 1 + \frac{\lambda t}{(1+\beta t)(1+2\beta t)}, \lambda, \beta \in \mathbb{R}$.

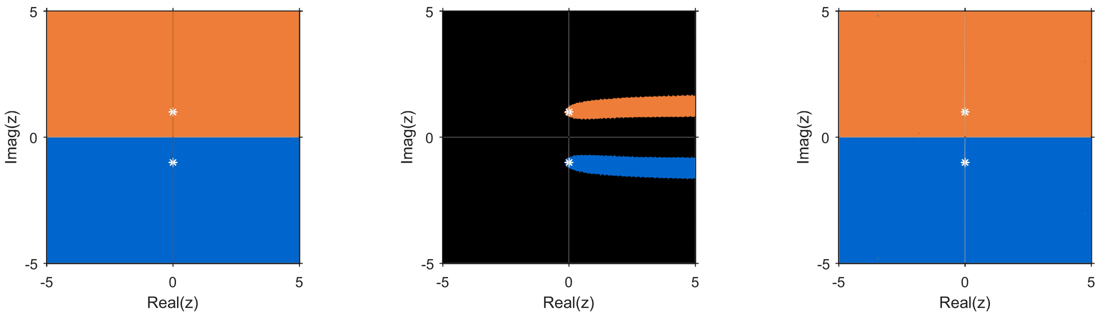
2.1 Application of $p(z) = z^2 + 1$

The polynomial $p(z) = z^2+1$ has two complex roots. The application of the iterative methods of Newton, Kanwar-Tomar and Kou-Li on $p(z)$ satisfy the symmetry property, since $p(\bar{z}) = \overline{p(z)}$. The corresponding fixed point operators of Newton's, Kanwar-Tomar's and Kou-Li's methods are

$$R(z) = \frac{z^2-1}{2z}, R(z) = \frac{\beta\left(z^3+z\right)+z^2-1}{\beta z^2+\beta+2z}, R(z) = z - \frac{\left(z^2+1\right)\left(\frac{\lambda\left(z^3+z\right)}{(\beta z^2+\beta+z)(\beta z^2+\beta+2z)}+1\right)}{2z},$$

respectively.

Figure 1 represents the dynamical planes of these methods when $p(z) = z^2 + 1$, for $\beta = 5, \lambda = -1/2$. The initial guesses in blue converge to the root $z_1 = -i$, while the orange corresponding ones converge to the root $z_2 = i$. In the three dynamical planes, the symmetry property can be observed. The knowledge of the semiplane $Im(z) > 0$ involves directly the knowledge of the behavior of the semiplane $Im(z) < 0$.



(a) Newton (b) Kanwar-Tomar (c) Kou-Li

Figure 1: Dynamical planes of methods of order two applied on $p(z) = z^2 + 1$.

2.2 Application of $p(z) = z^2 - 1$

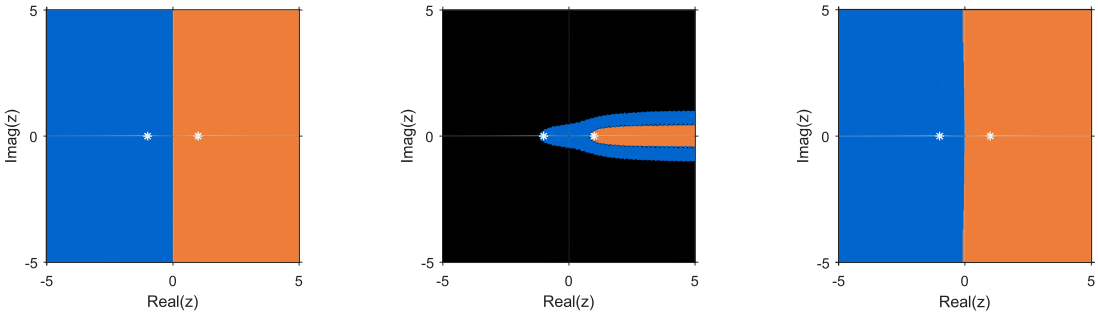
The polynomial $p(z) = z^2 - 1$ has two real roots, and satisfies the condition $p(\bar{z}) = \overline{p(z)}$, since $p(z)$ is a polynomial of real coefficients. In this particular case, the fixed point operators of Newton', Kanwar-Tomar' and Kou-Li's methods on $p(z)$ are

$$R(z) = \frac{z^2+1}{2z}, R(z) = \frac{\beta\left(z^2-1\right)z + z^2 + 1}{\beta\left(z^2-1\right) + 2z}, R(z) = z - \frac{\left(z^2-1\right)\left(\frac{\lambda\beta z\left(z^2-1\right)}{\left(\beta(z^2-1)+z\right)\left(\beta(z^2-1)+2z\right)} + 1\right)}{2z},$$

respectively.

Figure 2 represents the dynamical planes of these methods when $p(z) = z^2 - 1$, for $\beta = 5, \lambda = -1/2$. The initial guesses in blue converge to the root $z_1 = -1$, while the orange corresponding ones converge to the root $z_2 = 1$.

As in the previous case, there exists a conjugated symmetry in the dynamical planes. Obtaining the dynamical plane of the semiplane $Im(z) > 0$ is enough to know the aspect of the semiplane $Im(z) < 0$.



(a) Newton (b) Kanwar-Tomar (c) Kou-Li

Figure 2: Dynamical planes of methods of order two applied on $p(z) = z^2 - 1$.

3 Conclusions

The representation of the dynamical plane associated to an iterative method when a nonlinear function is applied gives information about the initial guesses where the method fails or succeeds. Its obtention requires a computational cost that depends on the number of initial guesses, amongst other parameters. If the nonlinear function $f(z)$ and the weight function $H(t)$ that implements the iterative method satisfy the described properties, the computational cost of representing the dynamical plane can be reduced.

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