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Contents

17

Design and convergence of new iterative methods with memory for solving nonlinear problems

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1 Introduction

One of the most common applied problems appearing in any scientific field is to calculate a solution of a nonlinear system of equations, i.e., the problem to obtain the solution $x^* \in \mathbb{R}^n$ of $F(x) = 0$, where F is a nonlinear function, $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, sufficiently differentiable in an open convex set D. Iterative methods have shown to be a good tool in order to approximate the solution of this kind of problems. Starting from an initial estimation, they generate a sequence of iterates that approximates the unknown x^*. There are many works in the literature devoted to the design of new iterative methods for the scalar case. Good overviews can be found in [2] and [4]. However, for the multidimensional case the amount of works is more limited.

Iterative schemes can be compared taking into account different criteria: their order of convergence p, the number of functional evaluations d done on each iteration of the method, the computational cost, among others. In addition, the efficiency of a method can be measured by relating these criteria through indices such as the efficiency index presented by Ostrowski in [1]. This index, defined by $I = p^{1/d}$, provides a relationship between the order of convergence and the number of functional evaluations of a method, being a high value an indicator that the corresponding method is suitable for solving problems efficiently.

Focusing on the number of previous iterates needed, iterative schemes are classified as methods with or without memory, being the first ones those that use more than one previous iterate to obtain the following estimation. Many researchers have shown that an adequate introduction of memory on iterative methods for solving nonlinear equations produces an increasing of its order of convergence without adding new functional evaluations. The extension of these results to nonlinear systems is a research area still in development, see for example [5, 6].

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2 Inclusion of memory on Traub-Steffensen's family

The starting point of this work is the Traub-Steffensen's family of iterative methods [2],

$$x^{(k+1)} = x^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1} F(x^{(k)}), \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where $w^{(k)} = x^{(k)} + bF(x^{(k)})$, and b is an arbitrary constant, $b \neq 0$. Note that $b = 1$ in (1) reduces to the well-known Steffensen's method [3].

Traub-Steffensen's family converges quadratically with independence on the value of b, as shows its error equation

$$e^{(k+1)} = C_2(I + bF'(x^*))(e^{(k)})^2 + \mathcal{O}((e^{(k)})^3), \tag{2}$$

where $e^{(k)} = x^{(k)} - x^*$ is the error on each iteration, I denotes the identity matrix and $C_j = \frac{1}{j!}[F'(x^*)]^{-1} F^{(j)}(x^*)$, $j \geq 2$. Let us also remark that (1) is a derivative-free scheme that only uses information from the current iteration, so defines a family of iterative methos without memory.

Following the same derivative-free iterative structure than (1), we can found in the literature the Kurchatov's iterative scheme [7]

$$x^{(k+1)} = x^{(k)} - [2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1} F(x^{(k)}), \qquad k = 1, 2, \ldots, \tag{3}$$

that is a quadratically convergent method with memory, as in the Kurchatov's divided difference operator, $[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]$, the current and the previous iterates are used.

By studying the error equation (2), we develop two iterative methods with memory using the iterative structure (1) and reaching higher order of convergence after the inclusion of previous iterates. The inclusion of memory is made by a properly approximation of the parameter b and the use of the Kurchatov's divided difference operator.

First, we can observe from equation (2) that $b = -[F'(x^*)]^{-1}$ provides a method of the family that has order of convergence three. This value of the parameter cannot be used because of the unknown x^* but an approximation of it can be made. In this sense, we propose the use of the Kurchatov's divided difference operator as an approximation of $F'(x^*)$. Then, the following approximation for the parameter $b := B^{(k)}$ is made:

$$B^{(k)} = -[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1}. \tag{4}$$

The replacement of parameter (4) in (1) gives rise to a method with memory that we have denoted by M3. The order of convergence of the method is set in the following result.

Theorem 1 *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D and let us denote by x^* the solution of $F(x) = 0$, such that F' is continuous and nonsigular in x^*. Let us suppose that $x^{(0)}$ and $x^{(1)}$ are close enough to x^*. Then, the sequence of iterates $\{x^{(k)}\}$ generated by method M3 converges to x^* with order of convergence three.*

Consequently, Method M3 is a scheme with memory belonging to the Traub-Steffensen's family that reaches higher order of convergence without adding new additional functional evaluations of the nonlinear function.

The second method developed in this work is obtained from the composition of the iterative expression of Traub-Steffensen's family, resulting in the multipoint scheme

$$\begin{aligned} y^{(k)} &= x^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1} F(x^{(k)}) \\ x^{(x+1)} &= y^{(k)} - [w^{(k)}, y^{(k)}; F]^{-1} F(y^{(k)}), \qquad k = 0, 1, 2, \ldots, \end{aligned} \tag{5}$$

being $w^{(k)} = x^{(k)} + bF(x^{(k)})$, $b \neq 0$. The next result shows the error equation of family (5) and its order of convergence.

Theorem 2 *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D and let us denote by x^* the solution of $F(x) = 0$, such that F' is continuous and nonsigular in x^*. When the initial estimation $x^{(0)}$ is close enough to x^*, the uniparametric family of iterative methods (5) has order of convergence three for any value of b and with error equation*

$$e^{(k+1)} = C_2(I + bF'(x^*))C_2(I + bF'(x^*))(e^{(k)})^3 + \mathcal{O}((e^{(k)})^4), \tag{6}$$

where $e^{(k)} = x^{(k)} - x^*$ and $C_j = \frac{1}{j!}[F'(x^*)]^{-1}F^{(j)}(x^*)$, $j \geq 2$.

The previous composition allows the design of multipoint methods without memory with cubic order of convergence. Althought the order is higher than in the original family, the number of functional evaluations in (5) is also higher, so an increase in the order of convergence is necessary in order to design methods more efficiently.

From the error equation (6), the term $I + bF'(x^*)$ shows that the same approximation for the parameter as previously can be made in order to grow up the order of the family. Then, when we set parameter (4) in the iterative expresion (5), and we denote the resulting method with memory by M5. Under the same conditions than in the previous theorems, method M5 has order of convergence five, as shows the result below.

Theorem 3 *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D. Let us denote by x^* the solution of $F(x) = 0$, such that F' is continuous and nonsigular in x^*. If $x^{(0)}$ and $x^{(1)}$ are close enough to x^*, method M5 converges to x^* with order of convergence five.*

After the theoretical analysis of the convergence of the proposed methods, in Section 3 we focus on the numerical results in order to compare computationally the performance of the methods.

3 Numerical results

In this section, the designed methods with memory M3 and M5 are tested numerically in order to show their performance for solving a nonlinear problem. The numerical results obtained for the proposed schemes are also compared with the results obtained for Kurchatov's method. The nonlinear system solved in the numerical implementation is the following system of ten nonlinear equations:

$$x_i - \cos\left(2x_i - \sum_{j=1}^{10} x_j\right) = 0, \qquad i = 1, 2, \ldots, 10.$$

The numerical tests are made using the software Matlab R2018b with variable precission arithmetics with 2000 digitis of matissa. As M3, M5 and Kurchatov's method are schemes with memory, two initial estimations are required. However, for the computational implementation we use, instead of $x^{(1)}$, an initial value for the parameter $B^{(0)} = -0.01I$ and an initial estimation $x^{(0)}$ to start the iterations of the methods, where I denotes the identity matrix. The iterative process begins with an initial $x^{(0)}$ and ends when the difference between two consecutive iterations $||x^{(k+1)} - x^{(k)}||$ or the value of the function in an iterate $||F(x^{(k+1)})||$ is lower than 10^{-50}, with a maximum of 50 iterations.

Table 1 summarizes the results for each method when different initial estimations $x^{(0)}$ are considered. For each method, we show the number of iterations required to reach the convergence (*iter*) and the values of the stopping criteria when the iterative process finishes. As is was expected, in all cases method M3 and M5 need less iterations, so they converge fastly. This results confirm that the proposed methods in this work have higher order of convergence than Kurchatov's method.

$\mathbf{x^{(0)}}$	**Method**	iter	$\|\|\mathbf{x^{(k+1)}} - \mathbf{x^{(k)}}\|\|$	$\|\|\mathbf{F}(\mathbf{x^{(k+1)}})\|\|$
$\begin{pmatrix} 0.9 \\ \vdots \\ 0.9 \end{pmatrix}$	Kurchatov	7	9.301e–36	2.685e–53
	M3	5	7.201e–45	2.058e–62
	M5	3	4.453e–31	1.129e–65
$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$	Kurchatov	8	1.537e–40	4.392e–58
	M3	5	4.198e–41	1.2e–58
	M5	3	2.589e–22	6.553e–57
$\begin{pmatrix} 0.8 \\ \vdots \\ 0.8 \end{pmatrix}$	Kurchatov	8	2.847e–39	8.157e–57
	M3	5	5.624e–38	6.37e–55
	M5	4	6.551e–49	1.302e–82

Table 1: Numerical results for the nonlinear system

4 Conclusions

From the quadratically convergent Traub-Steffensen's family, two new iterative schemes with memory are presented in this work with orders three and five. We show that the inclusion of memory on the original scheme and the composition of iterative structures allow to increase the order of convergence from two up to five. In addition, the numerical experiments performed in

Section 3 confirm the theoretical results of Section 2, as methods M3 and M5 converge fastly than Kurchatov's method.

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