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A multistep Steffensen-type method for solving nonlinear systems of equations

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This paper is devoted to the semilocal analysis of a high-order Steffensen-type method with frozen divided differences. The methods are free of bilinear operators and derivatives, which constitutes the main limitation of the classical high-order iterative schemes. Although the methods are more demanding, a semilocal convergence analysis is presented using weaker conditions than the classical Steffensen method.

KEYWORDS
frozen divided differences, high-order, semilocal convergence, Steffensen-type methods

1 | INTRODUCTION

Newton's method is the most popular tool to approximate the solution of a nonlinear equation $F(x) = 0\,(F : D \subseteq X \to X$, X is a Banach space and D is a non-empty open convex subset of X). Steffensen's method should be considered as a very good alternative, if we are not interested in the computation of derivatives, but having the same order of convergence. Moreover, our aim is to study iterative methods with a generic number of steps k with the purpose of offering an alternative to choose an iterative method with the desired order of convergence taking into account simultaneously their efficiency. Some papers have been published for this purpose in the unidimensional case proposing optimal derivative free iterative methods; see, for instance, Cordero et al[1] and Zheng et al.[2] But we concentrate in the most general case, we deal with Banach spaces with the aim of setting a semilocal convergence study.

In this paper, we study the following k-step method that increases the order of a uniparametric Steffensen-type method, for $n = 1, 2, \dots$:

$$\begin{aligned}
x_n^{(0)} &= x_n,\\
x_n^{(1)} &= x_n^{(0)} + \alpha\Gamma_n F(x_n),\\
x_n^{(2)} &= x_n^{(1)} - \Gamma_n F(x_n^{(1)}),\\
&\vdots\\
x_n^{(k)} &= x_n^{(k-1)} - \Gamma_n F(x_n^{(k-1)}),\\
x_{n+1} &= x_n^{(k)}
\end{aligned} \tag{1}$$

where $\Gamma_n = [x_n, x_n + F(x_n); F]^{-1} \in L(X, X)$ or $\Gamma_n = [x_n - F(x_n), x_n + F(x_n); F]^{-1}$.

An advantage of these methods is that, as the matrix that appears at each subiteration is the same, the computational time used to solve the associated linear systems is smaller. This happens because, at each iteration, only one LU decomposition is computed. In most cases, the computational cost of solving a linear system is more expensive than that of the evaluations of the operator. The maximum efficiency for a family of Newton-like methods with frozen derivatives, that is the number k of sub-iterations, depends on the problem, but it can be computed before solving it.[3] A similar analysis for methods free of derivatives can be found in Grau-Sánchez et al.[4]

We consider only one parameter, since we can prove that the only possibility to obtain order three with the two-step method with two parameters is to consider the first parameter ± 1 and the second 1; we refer Amat et al[5] for the Newton-type version of this scheme.

The objective of the present paper is to analyze theoretically and numerically the new family of iterative methods. We are able to obtain a new semilocal convergence analysis using weak conditions. The fact of obtaining the semilocal convergence study for a generic number of steps is an important task that requires a more intricate deployment of conditions and of course a nontrivial development for obtaining the whole process. However, the final result can be very useful having into account that it offers a possibility of taking the iterative method that fits with the needs of a problem and compare with similar procedures of different convergence order. We study the order of convergence and the efficiency of the family. We include also an application related to a nonlinear conservative system.

This area of research has many activities; we can find some recent results in the following incomplete list.[6-17]

This paper is organized as follows. In Section 2, we study the semilocal convergence for the family (1) using the first choice of Γ_n, ω-conditions for the divided differences, and constructing adequate functions for bounding the iterates. Section 3 is devoted to develop some applications for nonlinear systems of equations with maximum efficiency. We analyze three types of applications. The first one is related to a nondifferentiable operator considering high-order methods. In the second one, we consider a special case of a nonlinear conservative system and approximate its solution using different approximations by divided differences. We would like to emphasize the type of divided difference considered is important in order to ensure the theoretical errors. The third one is related to the use of this type of schemes in the approximation of boundary value problems via the shooting method. We point out the importance of the use of good approximations using divided differences of the Jacobian matrix in all the iterations. Finally, in Appendix A, we have included the analogous semilocal convergence analysis for the family (1) but using the second choice of Γ_n.

2 | SEMILOCAL CONVERGENCE STUDY FOR BANACH SPACES

Along the paper, let $U(v, \rho)$ and $\bar{U}(v, \rho)$ stand, respectively, for the open and closed balls in X with center $v \in X$ and of radius $\rho > 0$.

It is convenient for the semilocal convergence of our method to introduce some parameters and scalar functions.

Let $\gamma_0 > 0, \theta > 0, \eta > 0, \alpha \in \mathbb{R}$ be parameters.

Define

$$\begin{aligned}
R^* &:= \sup\{t \geq 0 : \bar{U}(x_0, t) \subseteq D\},\\
A &:= \{(s, t) : s \in [0, R^*], t \in [0, (1+\theta)R^*]\}.
\end{aligned}$$

L et also $\omega_0 : A \to [0, \frac{1}{\gamma_0})$, $\omega : \mathbb{R}_+ \cup \{0\} \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ be continuous and nondecreasing functions.

Moreover, define

$$
\begin{aligned}
R_0 &:= \sup\{(s,t) \in A : \gamma_0\omega_0(s,t) < 1\},\\
b_0 &:= |\alpha|\gamma_0\omega_0(\gamma_0\eta|\alpha|,\eta) + |\alpha+1|,\\
\eta_0 &:= \gamma_0(b_0 + |\alpha|)\eta,\\
\gamma &:= \gamma(s) = \gamma_0 max\{b_0, \omega_0(s, s+\eta), \sqrt{b_0\omega_0(\eta_0,(1+|\alpha|\gamma_0)\eta}\},\\
\delta_0 &:= \delta_0(s) = \delta_{0,k}(s) = \omega_0(s,s+\eta)\gamma^{k-1}, \quad \text{for} \quad k = 1,2,\ \dots,\\
\gamma_1 &:= \gamma_1(s) = \frac{\gamma_0}{1-\gamma_0\omega_0(s,(1+\theta)s)},\\
\lambda_0 &:= |\alpha|\gamma_1\delta_0,\\
b_1 &:= |\alpha+1|\delta_0 + \omega(2R, 2R+\delta_0\eta)\lambda_0,\\
\lambda &:= \lambda(s) = max\{b_1, \gamma_1 b_1, \gamma_1(\omega(2s, 2s+\delta_0\eta))\}.
\end{aligned}
$$

The semilocal convergence analysis is based on the following conditions:

(A.1) $F : D \subseteq X \to X$ is a nonlinear operator with a divided difference

$$[\cdot,\cdot;F] : D \times D \to L(X,X)$$

satisfying

$$[x,y;F](x-y) = F(x) - F(y)$$

for each $x, y \in D\,(x \neq y)$.

(A.2) There exists $x_0 \in D$ such that $\Gamma_0 = [x_0, x_0 + F(x_0); F]^{-1} \in L(X,X)$ and, for each $x, y \in D$,

$$||[x,y;F] - [x_0, x_0 + F(x_0); F)]|| \leq \omega_0(||x - x_0||, ||y - x_0 - F(x_0)||).$$

(A.3) For each $x, y, v, w \in U := D \cap U(x_0, R_0)$

$$||[x,y;F] - [v,w;F)]|| \leq \omega(||x - v||, ||y - w||).$$

(A.4) There exist $\theta > 0, \gamma_0 > 0, \eta > 0$ such that for each $x \in U$

$$
\begin{aligned}
||[x,x_0;F]|| &\leq \theta,\\
||\Gamma_0|| &\leq \gamma_0,\\
||F(x_0)|| &\leq \eta.
\end{aligned}
$$

(A.5) For each $s \in [0, R_0]$

$$
\begin{aligned}
\gamma &= \gamma(s) < 1,\\
\lambda &= \lambda(s) < 1,
\end{aligned}
$$

(A.6) Equation

$$\left(\frac{\lambda}{1-\lambda} + \frac{\gamma}{1-\gamma} + |\alpha|\gamma_0 + \lambda_0\right)\eta - t = 0$$

has at least one positive zero. Denote by R the smallest such zero, and $R < R_0$.

(A.7)

$$\bar{U}(x_0, R_1) \subset D,$$

where $R_1 = (1+\theta)R_0 + \eta$.

We have the following result for the family (1).

Theorem 1. *Suppose that the conditions* $(A.1)$ *to* $(A.7)$ *hold. Then method (1) is well defined, remains in* $U(x_0, R)$*, and converges to a solution* x^* *of the equation* $F(x) = 0$ *in* $\bar{U}(x_0, R)$*.*

Proof. We shall show sequence $\{x_n\}$ is complete and remains in $\bar{U}(x_0, R)$. Let $x \in \bar{U}(x_0, R_0)$, then we have that

$$||x + F(x) - x_0|| \leq ||x - x_0|| + ||[x, x_0; F](x - x_0)|| + ||F(x_0)|| \leq (1+\theta)R_0 + \eta = R_1,$$

so, $x + F(x) \in D$.

By conditions $(A.1)$ and $(A.2)$, iterates $x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(k)}$ are well defined. We can write by the first substep of method (1) that

$$F(x_0^{(1)}) = F(x_0^{(1)}) - F(x_0^{(0)}) - \Gamma_0^{-1}(x_0^{(1)} - x_0^{(0)}) + (\alpha+1)F(x_0^{(0)}),$$
$$= ([x_0^{(1)}, x_0^{(0)}; F] - [x_0^{(0)}, x_0^{(0)} + F(x_0^{(0)}); F])(x_0^{(1)} - x_0^{(0)}) + (\alpha+1)F(x_0^{(0)}).$$

Notice that

$$||x_0^{(0)} - (x_0^{(0)} + F(x_0^{(0)}))|| = ||F(x_0^{(0)})|| \le \eta < R,$$

and

$$||x_0^{(1)} - x_0^{(0)}|| = ||\alpha\Gamma_0 F(x_0)|| \le |\alpha|||\Gamma_0||||F(x_0)|| \le |\alpha|\gamma_0\eta < R,$$

so, $x_0^{(0)} + F(x_0^{(0)}) \in U(x_0, R)$ and $x_0^{(1)} \in U(x_0, R)$.

Thus, using $(A.2)$ to $(A.4)$, we get in turn that

$$\begin{aligned}||F(x_0^{(1)})|| &\le \omega_0(||x_0^{(1)} - x_0^{(0)}||, ||F(x_0^{(0)})||)||x_0^{(1)} - x_0^{(0)}|| + |\alpha+1|||F(x_0^{(0)})|| \\ &\le \omega_0(||\Gamma_0|||\alpha|||F(x_0^{(0)})||, ||F(x_0^{(0)})||)||\Gamma_0|||\alpha|||F(x_0^{(0)})|| + |\alpha+1|||F(x_0^{(0)})|| \\ &\le \omega_0(\gamma_0|\alpha|\eta, \eta)\gamma_0|\alpha|\eta + |\alpha+1|\eta = b_0\eta,\end{aligned}$$

so

$$||x_0^{(2)} - x_0^{(1)}|| = ||\Gamma_0 F(x_0^{(1)})|| \le ||\Gamma_0||||F(x_0^{(1)})|| \le \gamma_0 b_0\eta \le \gamma\eta \tag{2}$$

by the definition of γ, and

$$||x_0^{(2)} - x_0|| \le ||x_0^{(2)} - x_0^{(1)}|| + ||x_0^{(1)} - x_0|| \le \gamma_0 b_0\eta + \gamma_0|\alpha|\eta = \eta_0 < R,$$

by the definition of η_0 and $(A.6)$, so, $x_0^{(2)} \in U(x_0, R)$.

Similarly, for the second substep of (1), we can write

$$F(x_0^{(2)}) = F(x_0^{(2)}) - F(x_0^{(1)}) - \Gamma_0^{-1}(x_0^{(2)} - x_0^{(1)}),$$
$$= ([x_0^{(2)}, x_0^{(1)}; F] - [x_0^{(0)}, x_0^{(0)} + F(x_0^{(0)}); F])(x_0^{(2)} - x_0^{(1)}),$$

leading by the definition of γ to

$$\begin{aligned}||F(x_0^{(2)})|| &\le \omega_0(||x_0^{(2)} - x_0^{(0)}||, ||x_0^{(1)} - x_0^{(0)} - F(x_0^{0})||)||x_0^{(2)} - x_0^{(1)}|| \\ &\le \omega_0(\eta_0, |\alpha|\gamma_0\eta + \eta)\gamma_0 b_0\eta \le \frac{\gamma^2\eta}{\gamma_0},\end{aligned}$$

so

$$\begin{aligned}||x_0^{(3)} - x_0^{(2)}|| &= ||\Gamma_0 F(x_0^{(2)})|| \le ||\Gamma_0|| \quad ||F(x_0^{(2)})|| \\ &\le \gamma_0\omega_0(\eta_0, (1 + |\alpha|\gamma_0)\eta)\gamma_0 b_0\eta \le \gamma^2\eta,\end{aligned}$$

and

$$\begin{aligned}||x_0^{(3)} - x_0|| &\le ||x_0^{(3)} - x_0^{(2)}|| + ||x_0^{(2)} - x_0|| \\ &\le \omega_0(\eta_0, (1 + |\alpha|\gamma_0)\eta)\gamma_0^2 b_0\eta + \eta_0 < \gamma^2\eta + \eta_0 < R,\end{aligned}$$

so, by $(A.6)$, $x_0^{(3)} \in U(x_0, R)$.

Moreover, we have again by the definition of γ that

$$\begin{aligned}||F(x_0^{(3)})|| &\le \omega_0(||x_0^{(3)} - x_0||, ||x_0^{(2)} - x_0 - F(x_0^{(0)})||)||x_0^{(3)} - x_0^{(2)}|| \\ &\le \omega_0(R, R + \eta)\gamma^2\eta \le \frac{\gamma^3}{\gamma_0}\eta,\end{aligned}$$

so

$$||x_0^{(4)} - x_0^{(3)}|| \le \gamma_0\omega_0(R, R + \eta)\gamma^2\eta \le \gamma^3\eta$$

and

$$\begin{aligned}||x_0^{(4)} - x_0|| &\le ||x_0^{(4)} - x_0^{(3)}|| + ||x_0^{(3)} - x_0^{(2)}|| + ||x_0^{(2)} - x_0^{(1)}|| + ||x_0^{(1)} - x_0||\\ &\le \gamma^3\eta + \gamma^2\eta + \gamma\eta + |\alpha|\gamma_0\eta\\ &= \gamma\eta\frac{1-\gamma^3}{1-\gamma} + |\alpha|\gamma_0\eta < \frac{\gamma\eta}{1-\gamma} + |\alpha|\gamma_0\eta < R,\end{aligned}$$

so, $x_0^{(4)} \in U(x_0, R)$.

Then, in an analogous way,

$$||F(x_0^{(i)})|| \le \frac{\gamma^i}{\gamma_0}\eta, ||x_0^{(k)} - x_0^{(k-1)}|| \le \gamma^{k-1}\eta, \quad \text{for} \quad i = 1, 2, \ldots k,$$

and

$$||x_0^{(k)} - x_0|| \le \left(\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0\right)\eta < R.$$

Hence, $x_1 = x_0^{(k)} \in U(x_0, R)$ and is well defined.

We can write

$$\begin{aligned}F(x_1) &= F(x_0^{(k)}) - F(x_0^{(k-1)}) - \Gamma_0^{-1}(x_0^{(k)} - x_0^{(k-1)})\\ &= ([x_0^{(k)}, x_0^{(k-1)}; F] - \Gamma_0^{-1})(x_0^{(k)} - x_0^{(k-1)}),\end{aligned}$$

so

$$\begin{aligned}||F(x_1)|| &\le \omega_0(||x_0^{(k)} - x_0^{(0)}||, ||x_0^{(k-1)} - x_0^{(0)} - F(x_0^{(0)})||)||x_0^{(k)} - x_0^{(k-1)}||\\ &\le \omega_0(R, R+\eta)\gamma^{k-1}\eta = \delta_0\eta.\end{aligned}$$

Suppose that $x_m \in U(x_0, R)$. Next, we show that $\Gamma_m^{-1} \in L(X, X)$. We have in turn the estimate

$$\begin{aligned}||\Gamma_0|| \quad ||\Gamma_m^{-1} - \Gamma_0^{-1}|| &\le \gamma_0\omega_0(||x_m - x_0||, ||x_m + F(x_m) - x_0 - F(x_0)||)\\ &\le \gamma_0\omega_0(R, R + ||F(x_m) - F(x_0)||)\\ &\le \gamma_0\omega_0(R, R + ||[x_m, x_0; F|| \ \ ||x_m - x_0||)\\ &\le \gamma_0\omega_0(R, (1+\theta)R) < 1,\end{aligned}$$

since $R < R_0$.

It follows from the preceding estimate and the Banach lemma on invertible operators[18] that $\Gamma_m^{-1} \in L(X, X)$ and

$$||\Gamma_m|| \le \frac{\gamma_0}{1 - \gamma_0\omega_0(R, (1+\theta)R)} = \gamma_1.$$

By the definition of the method (1), we have that

$$||x_1^{(1)} - x_1^{(0)}|| \le |\alpha|||\Gamma_1||||F(x_1^{(0)})|| \le |\alpha|\gamma_1\delta_0\eta = \lambda_0\eta.$$

Then we can write

$$\begin{aligned}F(x_1^{(1)}) &= F(x_1^{(1)}) - F(x_1^{(0)}) - \Gamma_1^{-1}(x_1^{(1)} - x_1^{(0)}) + (\alpha+1)F(x_1^{(0)})\\ &= ([x_1^{(1)}, x_1^{(0)}; F] - [x_1, x_1 + F(x_1); F])(x_1^{(1)} - x_1^{(0)}) + (\alpha+1)F(x_1^{(0)}),\end{aligned}$$

leading to

$$\begin{aligned}||F(x_1^{(1)})|| &\le \omega(\|x_1^{(1)} - x_1\|, \|x_1^{(0)} - x_1 + F(x_1)\|)\|x_1^{(1)} - x_1^{(0)}\| + |\alpha+1|\|F(x_1^{(0)})\|\\ &\le \omega(2R, 2R + \delta_0\eta)\lambda_0\eta + |\alpha+1|\delta_0\eta = b_1\eta,\end{aligned}$$

so

$$||x_1^{(2)} - x_1^{(1)}|| = ||\Gamma_1 F(x_1^{(1)})|| \le \gamma_1 b_1\eta = \lambda\eta.$$

Notice that we have by $(A.6)$

$$||x_1 - x_0|| = ||x_0^{(k)} - x_0|| \le \left(\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0\right)\eta < R,$$

$$\begin{aligned}||x_1^{(1)} - x_0|| &\le ||x_1^{(1)} - x_1^{(0)}|| + ||x_1^{(0)} - x_0||\\ &\le \lambda_0\eta + ||x_1 - x_0|| \le \lambda_0\eta + \left(\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0\right)\eta < R\end{aligned}$$

and

$$\begin{aligned}||x_1^{(2)} - x_0|| &\leq ||x_1^{(2)} - x_1^{(1)}|| + ||x_1^{(1)} - x_0|| \\ &\leq \lambda\eta + \lambda_0\eta + ||x_1 - x_0|| \leq \lambda\eta + \lambda_0\eta + \left(\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0\right)\eta < R\end{aligned}$$

so $x_1, x_1^{(1)}, x_1^{(2)} \in U(x_0, R)$.

Similarly, we have that

$$\begin{aligned}||F(x_1^{(2)})|| &= ||F(x_1^{(2)}) - F(x_1^{(1)}) - \Gamma_1^{-1}(x_1^{(2)} - x_1^{(1)})|| \\ &= ||([x_1^{(2)}, x_1^{(1)}; F] - [x_1, x_1 + F(x_1); F])(x_1^{(2)} - x_1^{(1)})|| \\ &\leq \omega(||(x_1^{(2)} - x_0) + (x_0 - x_1)||, ||(x_1^{(1)} - x_0) + (x_0 - x_1) - F(x_1)||)||(x_1^{(2)} - x_1^{(1)})|| \\ &\leq \omega(2R, 2R + \delta_0\eta)\lambda\eta \leq \frac{\lambda^2\eta}{\gamma_1},\end{aligned}$$

leading to

$$||x_1^{(3)} - x_1^{(2)}|| = ||\Gamma_1 F(x_1^{(2)})|| \leq \gamma_1\omega(2R, 2R + \delta_0\eta)\lambda\eta \leq \lambda^2\eta$$

and

$$\begin{aligned}||x_1^{(3)} - x_0|| &\leq ||x_1^{(3)} - x_1^{(2)}|| + ||x_1^{(2)} - x_0|| \\ &\leq \lambda^2\eta + \lambda\eta + \lambda_0\eta + (\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0)\eta < R,\end{aligned}$$

so, $x_1^{(3)} \in U(x_0, R)$.

Therefore, we get in an analogous way that

$$||F(x_1^{(i)})|| \leq \frac{\lambda^i}{\gamma_1}\eta, ||x_1^{(k)} - x_1^{(k-1)}|| \leq \lambda^{k-1}\eta$$

and

$$x_1^{(i)} \in U(x_0, R), \quad \text{for} \quad i = 1, 2, \dots, k.$$

Notice that in view of the estimates on consecutive distances and the definition of λ and γ_1, we deduce that sequence $\{x_n\}$ is complete in a Banach space X and then it converges to some $x^* \in \bar{U}(x_0, R)$.

Finally, notice that sequence $\{F(x_n)\}$ is bounded from above by sequence $\{||x_n - x_{n-1}||\}$, so

$$||F(x^*)|| = \lim_{n\to\infty} ||F(x_n)|| \leq \lim_{n\to\infty} ||x_n - x_{n-1}|| = 0.$$

Hence, we deduce that $F(x^*) = 0$. □

Concerning the uniqueness of the solution, we have the following result.

Theorem 2. *Suppose the hypotheses of Theorem 1 hold. Then the point x^* is the only solution of the equation $F(x) = 0$ in $\bar{U}(x_0, R_2)$ where*

$$R_2 = \sup\{t \in [R, R^*] : \gamma_0\omega_0(t, R + \eta) < 1\}.$$

Proof. The existence of the solution of equation $F(x) = 0, x^* \in \bar{U}(x_0, R)$ has been shown in Theorem 1.

Let $y^* \in \bar{U}(x_0, R_1)$ be a solution of equation $F(x) = 0$.

Using $(A.2)$ and $(A.4)$, we get in turn for $M = [y^*, x^*; F]$

$$\begin{aligned}||\Gamma_0(M - \Gamma_0^{-1})|| &\leq \gamma_0\omega_0(||y^* - x_0||, ||x^* - x_0 - F(x_0)||) \\ &\leq \gamma_0\omega_0(R_2, R + \eta) < 1.\end{aligned}$$

It follows that $M^{-1} \in L(X, X)$. Then, from the identity

$$0 = F(y^*) - F(x^*) = M(y^* - x^*),$$

we conclude that $y^* = x^*$. □

Remark 1. The convergence of these type of methods usually involves a stronger condition than $(A.3)$ in the literature given by

$$||[x, y; F] - [v, w; F]|| \leq \omega_1(||x - v||, ||y - w||)$$

for each $x, y, v, w \in D$, where ω_1 is a function like ω.

Notice that in general for each pair (s, t),

$$\omega(s, t) \leq \omega_1(s, t)$$

since $U \subseteq D$ and

$$\omega_0(t, s) \leq \omega_1(t, s).$$

Moreover, the latest inequality have been used by us to refine convergence results for other simpler methods. The same is now true, if we use the first of the above inequalities. Notice that $(A.2)$, ie, the function ω_0 and the definition of γ_0 help us to define R_0, which in turn helps us define function ω. This way, the iterates are being located in U, which is a more precise location than D used in earlier studies.

3 | APPLICATION FOR NONLINEAR SYSTEMS OF EQUATIONS

The main goal of this section is to solve a nonlinear system of equations, given by

$$F(x) = 0, \tag{3}$$

where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is a nonlinear operator with D a nonempty open convex domain. We are interesting to approximate a solution of (3) with the maximum of efficiency by means of the iterative process (1). Thus, we would choose particular values of α and k. We consider the computational efficiency index,[19] given by

$$CE = \rho^{1/\mu},$$

where the R-order of convergence and the operational cost of doing an step of the algorithm (1) are denoted by ρ and μ, respectively. Once we have chosen the value of α and the number of steps k making optimum efficiency of the family of iterative processes (1), then, from Theorem 1, we solve the nonlinear system raised previously, (3).

3.1 | *R*-order of convergence

According to Traub,[20] it is known that we can obtain one-point iterative methods with a higher R-order of convergence from one-point iterative methods of the form

$$\begin{cases} x_0 \in D, \\ x_{n+1} = \mathcal{G}(x_n), \quad n \geq 0, \end{cases} \tag{4}$$

if we use the following modification of (4):

$$\begin{cases} x_0 \in D, \\ y_n = \mathcal{G}(x_n), \quad n \geq 0, \\ x_{n+1} = y_n - [F'(x_n)]^{-1} F(y_n), \end{cases} \tag{5}$$

if we suppose that method (4) has an R-order of convergence of at least ρ, then we remember[20] that method (5) has an R-order of convergence of at least $\rho + 1$.

Similar uniparametric methods to ones presented in this paper but using derivatives have been studied for a Newton-type method.[21-23] It is shown that the order of convergence is four for $\alpha = \pm 1$ and order three for $\alpha \neq \pm 1$. Moreover, in Amat et al,[5] a k-step method is studied similar to the present work but using derivatives and an interesting dynamical study is performed. In our study, we analyze these methods when we approximate the derivatives by divided differences. The fact is that for $\alpha = -1$, the resulting iterative methods preserve the order of convergence for any divided differences, but for $\alpha = 1$, the order of convergence reached depend on the problem itself.

Specifically, we know, see Grau-Sánchez et al,[24] that the fact of preserving the order of convergence does not depend on the method but on the systems of equations and if the associated divided difference verifies a particular condition that can be seen in theorem 4.1 of Grau-Sánchez et al.[24] In this case, we have that the method

$$\begin{cases} x_0 \in D, \\ y_n = \mathcal{G}(x_n), \quad k \geq 0, \\ x_{n+1} = y_n - [x_n, x_n + F(x_n); F]^{-1} F(y_n), \end{cases} \tag{6}$$

has also an R-order of convergence of at least $\rho + 1$.

We can calculate the R-order of convergence of family of iterative processes (1) from the previous results for different α values.

In first place, for $\alpha = -1$, if we consider $k = 1$, then we obtain Steffensen's method:

$$\begin{cases} x_0 \in D, \\ x_{n+1} = x_n - [x_n, x_n + F(x_n); F]^{-1} F(x_n), \end{cases}$$

which has quadratic convergence,[20] ie, R-order at least 2. Then, applying recursively Traub's result,[20] we obtain that, for $\alpha = -1$ and k steps, the family of iterative processes (1) has R-order of convergence $k + 1$.

In second place, for $\alpha = 1$, if we consider $k = 2$, then we obtain the iterative process given by

$$\begin{cases} x_0 \in D, \\ y_n = x_n + [x_n, x_n + F(x_n); F]^{-1} F(x_n) \\ x_{n+1} = y_n - [x_n, x_n + F(x_n); F]^{-1} F(y_n). \end{cases}$$

As it is know,[5] this iterative process has R-order of convergence at least three when it uses derivatives, and by Grau-Sánchez,[24] the order is preserved for an adequate divided difference operator verifying the mentioned condition cited in theorem 4.1 of Grau-Sánchez et al.[24] So, as before, applying recursively Traub's result,[20] we also obtain that the family of iterative processes (1) has R-order of convergence $k + 1$.

To finish our study of the R-order of the family of iterative processes (1), if we consider $\alpha \in \mathbb{R} - \{-1, 1\}$, as for $k = 1$, we have an iterative process with at least R-order of convergence 1, applying Traub's result, we obtain that the family of iterative processes (1) has R-order of convergence k.

3.2 | Operational cost

From now on, for computing the operational cost of doing an iteration of the algorithm (1), we note that the practical application of these iterative processes is performed from the following algorithm, depending on the chosen number of steps.

$$\begin{cases} x_n = x_n^{(0)}, \\ [x_n, x_n + F(x_n); F]\,(x_n^{(1)} - x_n^{(0)}) = \alpha F(x_n^{(0)}), \\ [x_n, x_n + F(x_n); F]\,(x_n^{(2)} - x_n^{(1)}) = -F(x_n^{(1)}), \\ \quad \vdots \\ [x_n, x_n + F(x_n); F]\,(x_n^{(k-1)} - x_n^{(k-2)}) = -F(x_n^{(k-2)}), \\ [x_n, x_n + F(x_n); F]\,(x_n^{(k)} - x_n^{(k-1)}) = -F(x_n^{(k-1)}) \\ x_{n+1} = x_n^{(k)}, \qquad n \geqslant 0, \end{cases} \tag{7}$$

In order to compute the operational cost of doing an iteration of this algorithm, we have $m(m-1)(2m-1)/6$ products and $m(m-1)/2$ quotients in the LU decomposition for the $[x_n, x_n + F(x_n); F]$ matrix and $m(m-1)$ products and m quotients in the resolution of two triangular linear systems. Taking into account that after k steps we have solved two triangular linear systems k times and only one LU decomposition, we obtain the following operational cost of doing an iteration of this algorithm with k steps for a nonlinear system of m equations:

$$\mu(k, m) = \frac{1}{3}(m^3 + 3km^2 - m). \tag{8}$$

3.3 | Efficiency

From the previous study, we have obtained

$$CE(k,m)=\begin{cases}(k+1)^{\frac{3}{m^3+3km^2-m}} & \text{if} \quad \alpha=-1,\\ (k+1)^{\frac{3}{m^3+3km^2-m}} & \text{if} \quad \alpha=1,\\ k^{\frac{3}{m^3+3km^2-m}} & \text{if} \quad \alpha\neq -1,1.\end{cases}$$

Then, obviously, for $\alpha = -1, 1$, we obtain the maximum efficiency. We can observe in Figure 1 different values for the computational efficiency of method defined by (1) for problems with size $m = 1, 3, 4$ on the left hand and $m = 5, 10, 15$ on the right hand. As can be seen in the graphics only in the unidimensional case the most efficient method is Steffensen one. For problems with two, three, four, and five unknowns, the most efficient method is for $k = 2$, which is a method of convergence order 3. However, as the size of the problem increases, we need to perform more steps in the method in order to reach the maximum efficiency. While for a problem of five unknowns, the maximum efficiency is reached for the method with two steps and for a problem of 15 unknowns the most efficient method perform five steps per iteration. Obviously, when the size of the problem increases, the efficiency indices tend to one.

3.4 | Numerical examples

We analyze three types of applications. The first one is related to a nondifferentiable operator considering high-order methods. In the second one, we consider a special case of a nonlinear conservative system and approximate its solution by using different approximations by divided differences. We would like to emphasize that the type of divided difference considered is important in order to ensure the theoretical errors. The third one is related to the use of this type of schemes in the approximation of boundary value problems via the shooting method. We point out the importance of the use of good approximations using divided differences of the Jacobian matrix in all the iterations. In our computations, we work by using program Matlab 2016b working in variable precision arithmetic with 50 digits of mantissa.

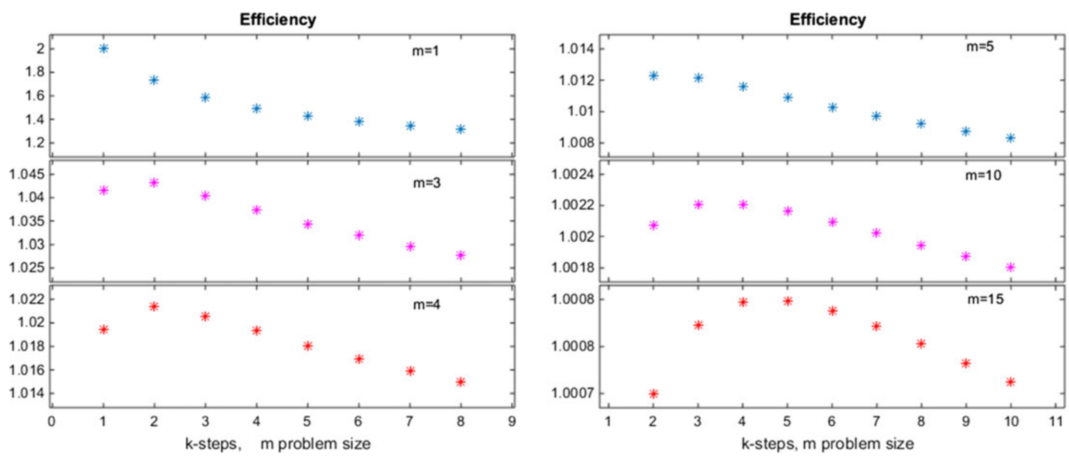
3.4.1 | A nondifferentiable operator

We consider the nondifferentiable system: $\begin{cases}|x^2-1|+y-1=0,\\ y^2+x-2=0.\end{cases}$

The associated nonlinear operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$F(x_1,x_2)=\begin{pmatrix}F_1(x_1,x_2)\\ F_2(x_1,x_2)\end{pmatrix}$$

where $F_1(x_1,x_2)=|x_1^2-1|+x_2-1$ and $F_2(x_1,x_2)=x_2^2+x_1-2$.



(a) $1 \le m \le 4$ (b) $5 \le m \le 15$

FIGURE 1 Efficiencies for different number of steps and different size problems [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 Residual error for each iteration

n	Secant $\|x^* - x_n\|$	Steffensen $\|x^* - x_n\|$	2-Steffensen $\|x^* - x_n\|$
1	6.4212e-02	1.0543e-01	3.5273e-02
2	5.4136e-03	1.1768e-02	2.1745e-04
3	1.4941e-04	2.9579e-04	2.1155e-11
4	1.0157e-04	1.2942e-07	3.3359e-32
5	3.2626e-09	2.5312e-14	1.1520e-94
6	4.5322e-09	9.5713e-28	4.9828e-282

This operator is nondifferentiable, but the methods converge with their order of convergence when we use the proposed iterative methods (1), resulting the most efficient method the corresponding to $k = 2$. So we can see in Table 1 the performance for the method with one and two steps. Moreover, we compare with secant method that is the classical method that can be used when we have a nondifferentiable problem. We use starting guess $x_0 = [1.3, 1.25]$ and obtain the distance to the exact solution $x^* = [1, 1]$ in each iteration.

We can see the super linear convergence for secant method, the quadratic and third order of convergence for Steffensen methods with one and two steps, respectively.

3.4.2 | On the importance of the type of divided differences

Now, we consider the special case of a nonlinear conservative system described by the equation

$$\frac{d^2x(t)}{dt^2} + \Psi(x(t)) = 0 \tag{9}$$

with the boundary conditions

$$x(0) = x(1) = 0. \tag{10}$$

After that, we use a discretization process to transform problem (9)-(10) into a finite-dimensional problem and look for an approximated solution of this problem when a particular function $\Psi(u)$ is considered. So we transform problem (9)-(10) into a system of nonlinear equations by approximating the second derivative by a standard numerical formula.

Now, we introduce the points $t_j = jh, j = 0, 1, \ldots, m + 1$, where $h = \frac{1}{m+1}$ and m is an appropriate integer. A scheme is then designed for the determination of numbers x_j as approximations of the values $x(t_j)$, which is the true solution at the nodes t_j. A standard approximation for the second derivative at these points is

$$x_j'' \approx \frac{x_{j-1} - 2x_j + x_{j+1}}{h^2}, \quad j = 1, 2, \ldots, m.$$

A natural way to obtain such a scheme is to demand that the x_j satisfy at each interior mesh point t_j the difference equation

$$x_{j-1} - 2x_j + x_{j+1} + h^2\Psi(x_j) = 0. \tag{11}$$

Since x_0 and x_{m+1} are determined by the boundary conditions, the unknowns are $x_1, x_2, \ldots, x_m$.

A further discussion is simplified by the use of matrix and vector notation. Introducing the vectors

$$\mathbf{x} = (x_1, x_2, \ldots, x_m)^t, \qquad v_{\mathbf{x}} = (\Psi(x_1), \Psi(x_2), \ldots, \Psi(x_m))^t$$

and the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & \ldots & 0 \\ 0 & 1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -2 \end{pmatrix},$$

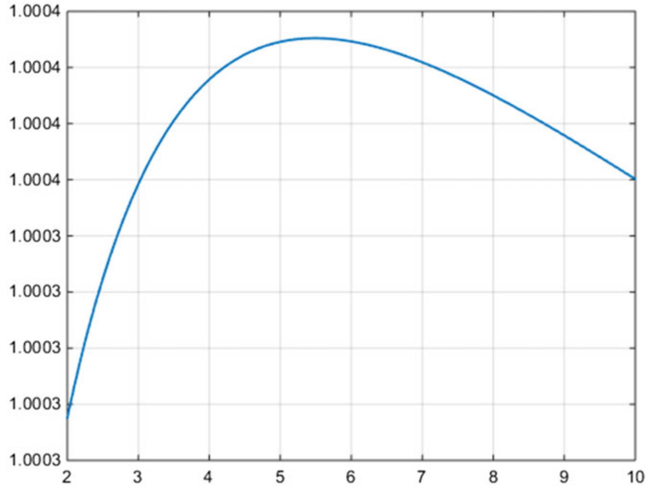


FIGURE 2 Efficiencies for $m = 20$ [Colour figure can be viewed at wileyonlinelibrary.com]

the system of equations, arising from demanding that (11) holds for $j = 1, 2, \dots, m$, can be written compactly in the form

$$F(\mathbf{x}) \equiv A\mathbf{x} + h^2 v_{\mathbf{x}} = 0, \tag{12}$$

where F is a function from $\mathbb{R}^m$ into $\mathbb{R}^m$.

From now on, the focus of our attention is to solve a particular system of form (12). We choose $m = 20$ and the infinity norm. For this size of systems, we plot the graphic of efficiency, see Figure 2, having that the best case is between 5 and 6, we use iterative methods with values of $k = 2, 3, 4, 5$.

If we now choose, for instance, the law $\Psi(u) = 1 + u^3$ for the heat generation in problem (9)-(10), then the vector $v_{\mathbf{x}}$ of (12) is given by

$$v_{\mathbf{x}} = (v_1, v_2, \dots, v_{20})^t, \qquad v_i = 1 + x_i^3, \qquad i = 1, 2, \dots, 20. \tag{13}$$

Then, we apply iterative method 1 to solve this problem by using different divided difference operators $[x, y; F]_{ij}$, $i, j = 1, \dots, n$, defined as follows:

$$\frac{1}{y_j - x_j}\left(F_i(y_1, \dots, y_{j-1}, y_j, x_{j+1}, \dots, x_n) - F_i(y_1, \dots, y_{j-1}, x_j, x_{j+1}, \dots, x_n)\right);$$

this is the classical first-order approximation of the Jacobian $F'(x)$ and will be denoted in the numerical experience as $dd1$ and $[x - F(x), x + F(x); F]$ as approximation of second order for the derivative, we denote in the numerical results as $dd2$.

In order to obtain the numerical results, we have used variable arithmetic precision with 100 digits, with different number of steps k, considering $\alpha = -1, 1/2, 1, 2$. By taking starting guess $x_0 = (1, \dots, 1)$ in Table 2, one can check the computational convergence order denoted by p, the number of iterations needed, denoted by it, in order to reach the stopping criterion $\|x_{n+1} - x_n\| < 10^{-30}$. Finally, we include in the numerical experience the norm value of the function at the approximation of the solution, $\|F(x_{n+1})\|$. As it can be seen at the solution for the parameter $\alpha = 1$ and divided differences of order one, $dd1$, the convergence order fells down one unit.

So we perform a new computational experience for cases where maximum order is reached; these are $\alpha = -1$ with divided differences given by $dd1$ and $dd2$ and $\alpha = 1$ with divided differences given by $dd2$, with the aim of studying the computational time, denoted by CT, for reaching the solution under the criterion established before. We also obtain the total operational cost, TOC, multiplying the value obtained in (8) by the number of iterations performed, and the total computational efficiency, defined by $TCE = (k+1)^{\frac{1}{TOC}}$. As can be seen in Table 3, when we analyzed deeply the cases for maximum efficiency, that is when the difference divided used allow us to preserve the convergence order, we notice that although the operational cost for an iteration of this k-step method gives us maximum efficiency for k around 5 and 6, in this example, if we have into account the total number of iterations performed the maximum efficiency, it is obtained for $k = 8$ and $\alpha = -1$; similar results are obtained for divided differences $dd1$ and $dd2$; moreover although $dd2$ perform one more functional evaluation by iteration, similar computational times are obtained.

TABLE 2 Numerical results for different values of α

I.M.		dd1				dd2			
k	α	$\|x_{n+1}-x_n\|$	*it*	*p*	$\|F(x_{n+1})\|$	$\|x_{n+1}-x_n\|$	*it*	*p*	$\|F(x_{n+1})\|$
2	2	7.5e-41	6	1.9	1.1e-84	6.4e-41	6	1.9	8.1e-85
	1/2	3.8e-38	7	1.9	1.2e-78	6.7e-38	7	1.9	3.5e-78
	−1	5.6e-50	5	2.9	1.2e-108	3.4e-85	5	2.9	8.1e-109
	1	3.8e-35	7	1.9	2.7e-74	5.6e-68	7	2.9	4.4e-106
3	2	1.8e-61	5	2.9	4.3e-63	4.7e-61	5	2.9	1.1e-62
	1/2	7.2e-53	5	3.0	1.7e-54	7.1e-53	5	2.9	1.7e-54
	−1	5.1e-38	4	3.9	8.0e-40	5.1e-38	4	3.9	1.2e-39
	1	6.6e-60	5	3.0	1.5e-61	4.9e-88	5	4.0	1.1e-89
4	2	3.4e-42	4	3.9	8.2e-44	5.1e-42	4	3.9	1.2e-43
	1/2	1.3e-31	4	4.1	3.2e-33	1.7e-31	4	4.1	3.2e-33
	−1	6.9e-62	4	5.0	5.4e-64	6.9e-62	4	5.0	1.6e-63
	1	1.7e-36	4	4.2	1.1e-58	1.7e-36	4	5.0	4.2e-38
5	2	6.3e-71	4	5.0	1.4e-72	1.2e-70	4	5.0	3.1e-72
	1/2	2.8e-61	4	4.9	6.7e-63	3.6e-61	4	4.9	8.5e-63
	−1	1.1e-103	4	5.9	9.9e-105	3.7e-103	4	5.9	8.8e-105
	1	4.1e-63	4	5.0	8 9.8e-65	2.5e-72	4	5.9	6.74e-74

TABLE 3 Comparing results for values of maximum efficiency

I.M.		dd1				dd2			
k	α	*it*	*TOC*	*CT*	*TCE*	*it*	*TOC*	*CT*	*TCE*
2	−1	5	17300	7.32	1.000064	5	17300	7.23	1.000064
	1					7	24220	7.25	1.000045
3	−1	4	15440	5.82	1.000090	6	23160	8.77	1.000060
	1					5	19300	7.28	1.000072
4	−1	4	17040	5.98	1.000094	4	17040	6.03	1.000094
	1					4	17040	5.94	1.000094
5	−1	4	18640	6.24	1.000096	5	18640	6.25	1.000096
	1					4	18640	6.12	1.000096
7	−1	4	21840	7.02	1.000095	4	21840	6.41	1.000095
	1					4	21840	6.33	1.000095
8	−1	3	17580	5.88	**1.000125**	3	17580	6.06	1.000125
	1					3	23440	7.33	1.000094
9	−1	3	18780	5.04	1.000123	3	18780	4.89	1.000123
	1					3	18780	4.94	1.000123
10	−1	3	19980	5.03	1.000120	3	19980	5.2	1.000120
	1					3	19980	5.08	1.000120

3.4.3 | On the importance of approximating well the Jacobian matrix by divided differences in all the iterations

We consider the following boundary problem

$$y''(t) = f(t, y(t), y'(t)), \qquad y(a) = \alpha, \qquad y(b) = \beta, \tag{14}$$

choose a discretization of $[a, b]$ with N subintervals,

$$t_j = a + \frac{T}{N} j, \qquad T = b - a, \qquad j = 0, 1, \ldots, N,$$

and propose the use of the multiple shooting method for solving it. First, in each interval $[t_j, t_{j+1}]$, we compute the function

$y(t;s_0,s_1,\ldots,s_{j-1})$ recursively, by solving the initial value problems

$$y''(t)=f(t,y(t),y'(t)),\qquad y(t_j)=y(t_j;s_0,s_1,\ldots,s_{j-1}),\qquad y'(t_j)=s_j,$$

whose solution is denoted by $y(t;s_0,s_1,\ldots,s_j)$.

To approximate a solution of problem (14), we approximate a solution of the nonlinear system of equations $F(s)=0$, where $F:\mathbb{R}^N\to\mathbb{R}^N$ and

$$\begin{cases} F_1(s_0,s_1,\ldots,s_{N-1}) = s_1-y'(t_1;s_0)\\ F_2(s_0,s_1,\ldots,s_{N-1}) = s_2-y'(t_2;s_0,s_1)\\ \vdots\\ F_{N-1}(s_0,s_1,\ldots,s_{N-1}) = s_{N-1}-y'(t_{N-1};s_0,s_1,\ldots,s_{N-2})\\ F_N(s_0,s_1,\ldots,s_{N-1}) = \beta-y(t_N;s_0,s_1,s_{N-2},s_{N-1}). \end{cases}$$

For this, we consider the classical Steffensen's method and different k-step methods defined in (1) and compare their numerical performance. In our study, we consider the usual divided difference of first order. So, for $\mathbf{u},\mathbf{v}\in\mathbb{R}^N$, such that $\mathbf{u}\neq\mathbf{v}$, we consider $[\mathbf{u},\mathbf{v};F]=\left([\mathbf{u},\mathbf{v};F]_{ij}\right)_{i,j=1}^N\in\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)$, where

$$[\mathbf{u},\mathbf{v};F]_{ij}=\frac{1}{u_j-v_j}\quad(F_i(u_1,\ldots,u_j,v_{j+1},\ldots,v_N)-F_i(u_1,\ldots,u_{j-1},v_j,\ldots,v_N)).$$

For the initial slope $\vec{s}_0=\left(s_0^0,s_1^0,\ldots,s_{N-1}^0\right)$, to apply the methods, we consider

$$\begin{cases} s_0^0 = \frac{\beta-\alpha}{b-a}=\frac{y(t_N)-y(t_0)}{t_N-t_0},\\ s_1^0 = \frac{y(t_N)-y(t_1;s_0)}{t_N-t_1},\\ s_2^0 = \frac{y(t_N)-y(t_2;s_0,s_1)}{t_N-t_2},\\ \vdots\\ s_{N-1}^0 = \frac{y(t_N)-y(t_{N-1};s_0,s_1,\ldots,s_{N-2})}{t_N-t_{N-1}}. \end{cases}$$

We approximate the following boundary value problem (Stoer and Bulirsch[25], p554):

$$\begin{aligned} y''(t)&=\lambda\sinh(\lambda y(t)),\\ y(0)&=0,\quad y(1)=1, \end{aligned}$$

for $\lambda=2.5$ and $N=4$ subintervals. The exact solution is

$$y(t)=\frac{2}{\lambda}\operatorname{arg\,sinh}\left(\frac{s}{2}\frac{\mathrm{sn}(\lambda t,1-s^2/4)}{\mathrm{cn}(\lambda t,1-s^2/4)}\right)$$

where s is the correct initial slop. In this case,

$$s=y'(0)=0.3713363932677645,$$

and $\mathrm{sn}(\cdot,\cdot)$ and $\mathrm{cn}(\cdot,\cdot)$ are Jacobi elliptic functions.

We can compare with Newton's method since the operator is differentiable. In this application, the iterative schemes are used as black box inside the multiple shooting method. The most efficient method in our family in this application is the two-step method.

Newton

n	$\|\|F(\vec{s}_n)\|\|_\infty$	$\|\|y(t) - y_n\|\|_\infty$	$\|\|y'(t) - y'_n\|\|_\infty$
0	3.0	$6.3 \cdot 10^{-1}$	1.398
1	$5.7 \cdot 10^{-1}$	$3.3 \cdot 10^{-1}$	$2.2 \cdot 10^{-1}$
2	$4.9 \cdot 10^{-2}$	$4.5 \cdot 10^{-2}$	$3.2 \cdot 10^{-2}$
3	$7.3 \cdot 10^{-4}$	$6.6 \cdot 10^{-4}$	$4.7 \cdot 10^{-4}$
4	$1.6 \cdot 10^{-7}$	$1.4 \cdot 10^{-7}$	$1.0 \cdot 10^{-7}$
5	$7.0 \cdot 10^{-15}$	$8.1 \cdot 10^{-15}$	$1.3 \cdot 10^{-14}$

2-step Steffensen's method (1), with $\Gamma_n = [x_n, x_n + a_n F(x_n); F]^{-1}$ and $\alpha = 1$

n	$\|\|F(\vec{s}_n)\|\|_\infty$	$\|\|y(t) - y_n\|\|_\infty$	$\|\|y'(t) - y'_n\|\|_\infty$	$\max\{\|dF/ds - J\|_{i,j}\}$
0	3.0	$6.3 \cdot 10^{-1}$	1.398	$1.2 \cdot 10^{-6}$
1	$4.9 \cdot 10^{-2}$	$4.5 \cdot 10^{-2}$	$3.2 \cdot 10^{-2}$	$8.5 \cdot 10^{-7}$
2	$1.6 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$	$1.0 \cdot 10^{-6}$	$8.043 \cdot 10^{-6}$
3	$3.0 \cdot 10^{-17}$	$1.1 \cdot 10^{-17}$	$4.4 \cdot 10^{-17}$	$- - -$

Steffensen

n	$\|\|F(\vec{s}_n)\|\|_\infty$	$\|\|y(t) - y_n\|\|_\infty$	$\|\|y(t) - y_n\|\|_\infty$	$\max\{\|dF/ds - J\|_{i,j}\}$
0	3.0	$6.3 \cdot 10^{-1}$	1.398	$1.8 \cdot 10^{0}$
1	$6.7 \cdot 10^{-1}$	$5.5 \cdot 10^{-1}$	$5.2 \cdot 10^{-1}$	$5.5 \cdot 10^{-2}$
2	$4.0 \cdot 10^{-2}$	$1.4 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	$1.3 \cdot 10^{-2}$
3	$1.3 \cdot 10^{-4}$	$1.3 \cdot 10^{-4}$	$1.1 \cdot 10^{-4}$	$2.2 \cdot 10^{-5}$
4	$2.7 \cdot 10^{-9}$	$1.5 \cdot 10^{-9}$	$1.1 \cdot 10^{-9}$	$4.5 \cdot 10^{-4}$
5	$3.7 \cdot 10^{-13}$	$3.7 \cdot 10^{-13}$	$3.1 \cdot 10^{-13}$	$1.9 \cdot 10^{-1}$
6	$4.0 \cdot 10^{-14}$	$2.0 \cdot 10^{-14}$	$1.5 \cdot 10^{-14}$	$1.8 \cdot 10^{0}$
7	$1.1 \cdot 10^{-14}$	$1.2 \cdot 10^{-14}$	$1.5 \cdot 10^{-14}$	$4.0 \cdot 10^{0}$
8	$1.5 \cdot 10^{-14}$	$3.0 \cdot 10^{-15}$	$1.1 \cdot 10^{-14}$	$1.4 \cdot 10^{1}$
9	$1.8 \cdot 10^{-14}$	$1.8 \cdot 10^{-15}$	$1.0 \cdot 10^{-14}$	$1.3 \cdot 10^{0}$
10	$1.3 \cdot 10^{-14}$	$1.1 \cdot 10^{-14}$	$2.4 \cdot 10^{-14}$	$- - -$

Only Newton's method using Jacobians and our two-step Steffensen's method obtain the desired order. We have considered $\{a_n\}$, such that $\|a_n F(x_n)\| \leq 10^{-6}$ for all n. Steffensen's method has difficulties produced from the bad approximation of the Jacobian, denoted by J, in some iterations.

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CONFLICT OF INTEREST

There are no conflict of interest to this work.

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APPENDIX A

The semilocal convergence analysis for the family (1) is presented, but for Γ_n given by the second choice of Γ_n in an analogous way to Theorem 1, there are some differences. To simplify the notation of the parameters, we use the same symbols but different definitions.

Define

$$
\begin{aligned}
R_0 &:= \sup\{(s,t) \in A : \gamma_0\omega_0(s,t) < 1\},\\
b_0 &:= |\alpha|\gamma_0\omega_0(\gamma_0\eta|\alpha| + \eta, \eta) + |\alpha + 1|,\\
\eta_0 &:= \gamma_0(b_0 + |\alpha|)\eta,\\
\gamma &:= \gamma(s) = \gamma_0 max\{b_0, \omega_0(s+\eta, s+\eta), \sqrt{b_0\omega_0(\eta_0 + \eta, (1+|\alpha|\gamma_0)\eta}\},\\
\delta_0 &:= \delta_0(s) = \delta_{0,k}(s) = \omega_0(s+\eta, s+\eta)\gamma^{k-1}, \quad \text{for} \quad k = 1, 2, \ldots,\\
\gamma_1 &:= \gamma_1(s) = \frac{\gamma_0}{1 - \gamma_0\omega_0((1+\theta)s, (1+\theta)s)},\\
\lambda_0 &:= |\alpha|\gamma_1\delta_0,\\
b_1 &:= |\alpha|\delta_0 + \omega(2R + \delta_0\eta, 2R + \delta_0\eta),\\
\lambda &:= \lambda(s) = max\{b_1, \gamma_1 b_1, \gamma_1(\omega(2s + \delta_0\eta, 2s + \delta_0\eta))\}.
\end{aligned}
$$

The semilocal convergence analysis is based on the following conditions:

(A.1) $F : D \subseteq X \to X$ is a nonlinear operator with a divided difference

$$[\cdot,\cdot;F] : D \times D \to L(X,X)$$

satisfying

$$[x,y;F](x-y) = F(x) - F(y)$$

for each $x, y \in D\,(x \neq y)$.

(A.2) There exists $x_0 \in D$ such that $\Gamma_0 = [x_0 - F(x_0), x_0 + F(x_0)]^{-1} \in L(X,X)$ and, for each $x, y \in D$,

$$||[x,y;F] - [x_0, x_0 + F(x_0); F)]|| \leq \omega_0(||x - x_0 + F(x_0)||, ||y - x_0 - F(x_0)||).$$

(A.3) For each $x, y, v, w \in U := D \cap U(x_0, R_0) - F(x_0)$

$$||[x,y;F] - [v,w;F)]|| \leq \omega(||x-v||, ||y-w||).$$

(A.4) There exist $\theta > 0, \gamma_0 > 0, \eta > 0$ such that for each $x \in U$

$$
\begin{aligned}
||[x,x_0;F]|| &\leq \theta,\\
||\Gamma_0|| &\leq \gamma_0,\\
||F(x_0)|| &\leq \eta.
\end{aligned}
$$

(A.5) For each $s \in [0, R_0]$

$$
\begin{aligned}
\gamma = \gamma(s) < 1,\\
\lambda = \lambda(s) < 1,
\end{aligned}
$$

(A.6) Equation

$$(\frac{\lambda}{1-\lambda} + \frac{\gamma}{1-\gamma} + |\alpha|\gamma_0 + \lambda_0)\eta - t = 0$$

has at least one positive zero. Denote by R the smallest such zero, and $R < R_0$.

(A.7)

$$\bar{U}(x_0, R_1) \subset D,$$

where $R_1 = (1+\theta)R_0 + \eta$.

We have the following result for the family (1).

Proposition 1. *Suppose that the conditions $(A.1)$ to $(A.7)$ hold. Then method (1) is well defined, remains in $U(x_0, R)$, and converges to a solution x^* of the equation $F(x) = 0$ in $\bar{U}(x_0, R)$.*

Proof. We shall show sequence $\{x_n\}$ is complete and remains in $\bar{U}(x_0, R)$. Let $x \in \bar{U}(x_0, R_0)$, then we have that

$$||x + F(x) - x_0|| \leq ||x - x_0|| + ||[x, x_0; F](x - x_0)|| + ||F(x_0)|| \leq (1+\theta)R_0 + \eta = R_1,$$

so $x + F(x) \in D$.

By conditions $(A.1)$ to $(A.2)$, iterates $x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(k)}$ are well defined. We can write by the first substep of method (1) that

$$\begin{aligned} F(x_0^{(1)}) &= F(x_0^{(1)}) - F(x_0^{(0)}) - \Gamma_0^{-1}(x_0^{(1)} - x_0^{(0)}) + (\alpha+1)F(x_0^{(0)}), \\ &= ([x_0^{(1)}, x_0^{(0)}; F] - [x_0^{(0)} - F(x_0^{(0)}), x_0^{(0)} + F(x_0^{(0)}); F])(x_0^{(1)} - x_0^{(0)}) + (\alpha+1)F(x_0^{(0)}). \end{aligned}$$

Notice that

$$||x_0^{(0)} - (x_0^{(0)} \pm F(x_0^{(0)}))|| = ||F(x_0^{(0)})|| \le \eta < R,$$

and

$$||x_0^{(1)} - x_0^{(0)}|| = ||\alpha\Gamma_0 F(x_0)|| \le |\alpha|||\Gamma_0||||F(x_0)|| \le |\alpha|\gamma_0\eta < R,$$

so, $x_0^{(0)} + F(x_0^{(0)}) \in U(x_0, R)$ and $x_0^{(1)} \in U(x_0, R)$.

Thus, using $(A.2)$ to $(A.4)$, we get in turn that

$$\begin{aligned} ||F(x_0^{(1)})|| &\le \omega_0(||x_0^{(1)} - x_0^{(0)} + F(x_0^{(0)})||, ||F(x_0^{(0)})||)||x_0^{(1)} - x_0^{(0)}|| + |\alpha+1|||F(x_0^{(0)})|| \\ &\le \omega_0((||\Gamma_0|||\alpha| + 1)||F(x_0^{(0)})||, ||F(x_0^{(0)})||)||\Gamma_0|||\alpha|||F(x_0^{(0)})|| + |\alpha+1|||F(x_0^{(0)})|| \\ &\le \omega_0((\gamma_0|\alpha| + 1)\eta, \eta)\gamma_0|\alpha+1|\eta + |\alpha|\eta = b_0\eta, \end{aligned}$$

so

$$\begin{aligned} ||x_0^{(2)} - x_0^{(1)}|| &= ||\Gamma_0 F(x_0^{(1)})|| \\ &\le ||\Gamma_0||||F(x_0^{(1)})|| \\ &\le \gamma_0 b_0 \eta \le \gamma\eta \end{aligned} \tag{A1}$$

by the definition of γ, and

$$||x_0^{(2)} - x_0|| \le ||x_0^{(2)} - x_0^{(1)}|| + ||x_0^{(1)} - x_0|| \le \gamma_0 b_0\eta + \gamma_0|\alpha|\eta = \eta_0 < R,$$

by the definition of η_0 and $(A.6)$, so, $x_0^{(2)} \in U(x_0, R)$.

Similarly, for the second substep of (1), we can write

$$\begin{aligned} F(x_0^{(2)}) &= F(x_0^{(2)}) - F(x_0^{(1)}) - \Gamma_0^{-1}(x_0^{(2)} - x_0^{(1)}), \\ &= ([x_0^{(2)}, x_0^{(1)}; F] - [x_0^{(0)} - F(x_0^{(0)}), x_0^{(0)} + F(x_0^{(0)}); F])(x_0^{(2)} - x_0^{(1)}), \end{aligned}$$

leading by the definition of γ to

$$\begin{aligned} ||F(x_0^{(2)})|| &\le \omega_0(||x_0^{(2)} - x_0^{(0)} + F(x_0^{(0)})||, ||x_0^{(1)} - x_0^{(0)} - F(x_0^{0})||)||x_0^{(2)} - x_0^{(1)}|| \\ &\le \omega_0(\eta_0 + \eta, |\alpha|\gamma_0\eta + \eta)\gamma_0 b_0\eta \le \frac{\gamma^2\eta}{\gamma_0}, \end{aligned}$$

so

$$\begin{aligned} ||x_0^{(3)} - x_0^{(2)}|| = ||\Gamma_0 F(x_0^{(2)})|| &\le ||\Gamma_0|| \quad ||F(x_0^{(2)})|| \\ &\le \gamma_0\omega_0(\eta_0 + \eta, (1 + |\alpha|\gamma_0)\eta)\gamma_0 b_0\eta \le \gamma^2\eta, \end{aligned}$$

and

$$\begin{aligned} ||x_0^{(3)} - x_0|| &\le ||x_0^{(3)} - x_0^{(2)}|| + ||x_0^{(2)} - x_0|| \\ &\le \omega_0(\eta_0 + \eta, (1 + |\alpha|\gamma_0)\eta)\gamma_0^2 b_0\eta + \eta_0 < \gamma^2\eta + \eta_0 < R, \end{aligned}$$

so, by $(A.6)$, $x_0^{(3)} \in U(x_0, R)$.

Moreover, we have again by the definition of γ that

$$\begin{aligned} ||F(x_0^{(3)})|| &\le \omega_0(||x_0^{(3)} - x_0 + F(x_0^{(0)})||, ||x_0^{(2)} - x_0 - F(x_0^{(0)})||)||x_0^{(3)} - x_0^{(2)}|| \\ &\le \omega_0(R + \eta, R + \eta)\gamma^2\eta \le \frac{\gamma^3}{\gamma_0}\eta, \end{aligned}$$

so

$$||x_0^{(4)} - x_0^{(3)}|| \le \gamma_0\omega_0(R + \eta, R + \eta) \le \gamma^3\eta$$

and

$$\begin{aligned}||x_0^{(4)} - x_0|| &\le ||x_0^{(4)} - x_0^{(3)}|| + ||x_0^{(3)} - x_0^{(2)}|| + ||x_0^{(2)} - x_0^{(1)}|| + ||x_0^{(1)} - x_0|| \\ &\le \gamma^3\eta + \gamma^2\eta + \gamma\eta + |\alpha|\gamma_0\eta \\ &= \gamma\eta\frac{1-\gamma^3}{1-\gamma} + |\alpha|\gamma_0\eta < \frac{\gamma\eta}{1-\gamma} + |\alpha|\gamma_0\eta < R,\end{aligned}$$

so, $x_0^{(4)} \in U(x_0, R)$.

Then, in an analogous way

$$||F(x_0^{(i)})|| \le \frac{\gamma^i}{\gamma_0}\eta, ||x_0^{(k)} - x_0^{(k-1)}|| \le \gamma^{k-1}\eta, \quad \text{for} \quad i = 1, 2, \ldots k,$$

and

$$||x_0^{(k)} - x_0|| \le (\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0)\eta < R.$$

Hence, $x_1 = x_0^{(k)} \in U(x_0, R)$ and is well defined.

We can write

$$\begin{aligned}F(x_1) &= F(x_0^{(k)}) - F(x_0^{(k-1)}) - \Gamma_0^{-1}(x_0^{(k)} - x_0^{(k-1)}) \\ &= ([x_0^{(k)}, x_0^{(k-1)}; F] - \Gamma_0^{-1})(x_0^{(k)} - x_0^{(k-1)}),\end{aligned}$$

so

$$\begin{aligned}||F(x_1)|| &\le \omega_0(||x_0^{(k)} - x_0^{(0)} + F(x_0^{(0)})||, ||x_0^{(k-1)} - x_0^{(0)} - F(x_0^{(0)})||)||x_0^{(k)} - x_0^{(k-1)}|| \\ &\le \omega_0(R+\eta, R+\eta)\gamma^{k-1}\eta = \delta_0\eta.\end{aligned}$$

Suppose that $x_m \in U(x_0, R)$. Next, we show that $\Gamma_m^{-1} \in L(X, X)$. We have in turn the estimate

$$\begin{aligned}||\Gamma_0|| \quad ||\Gamma_m^{-1} - \Gamma_0^{-1}|| &\le \gamma_0\omega_0(||x_m - x_0 + F(x_0) - F(x_m)||, ||x_m + F(x_m) - x_0 - F(x_0)||) \\ &\le \gamma_0\omega_0(R + ||F(x_m) - F(x_0)||, R + ||F(x_m) - F(x_0)||) \\ &\le \gamma_0\omega_0(R + ||[x_m, x_0; F|| \quad ||x_m - x_0||, R + ||[x_m, x_0; F|| \quad ||x_m - x_0||) \\ &\le \gamma_0\omega_0((1+\theta)R, (1+\theta)R) < 1,\end{aligned}$$

since $R < R_0$.

It follows from the preceding estimate and the Banach lemma on invertible operators[18] that $\Gamma_m^{-1} \in L(X, X)$ and

$$||\Gamma_m|| \le \frac{\gamma_0}{1 - \gamma_0\omega_0((1+\theta)R, (1+\theta)R)} = \gamma_1.$$

By the definition of the method (1), we have that

$$||x_1^{(1)} - x_1^{(0)}|| \le |\alpha|||\Gamma_1||||F(x_1^{(0)})|| \le |\alpha|\gamma_1\delta_0\eta = \lambda_0\eta.$$

Then we can write

$$\begin{aligned}F(x_1^{(1)}) &= F(x_1^{(1)}) - F(x_1^{(0)}) - \Gamma_1^{-1}(x_1^{(1)} - x_1^{(0)}) + (\alpha+1)F(x_1^{(0)}) \\ &= ([x_1^{(1)}, x_1^{(0)}; F] - [x_1 - F(x_1), x_1 + F(x_1); F])(x_1^{(1)} - x_1^{(0)}) + (\alpha+1)F(x_1^{(0)}),\end{aligned}$$

leading to

$$\begin{aligned}||F(x_1^{(1)})|| &\le \omega(\|x_1^{(1)} - x_1 + F(x_1)\|, \|x_1^{(0)} - x_1 - F(x_1)\|)\|x_1^{(1)} - x_1^{(0)}\| + |\alpha+1|\|F(x_1^{(0)})\| \\ &\le \omega(2R + \delta_0\eta, 2R + \delta_0\eta)\lambda_0\eta + |\alpha+1|\delta_0\eta = b_1\eta,\end{aligned}$$

so

$$||x_1^{(2)} - x_1^{(1)}|| = ||\Gamma_1 F(x_1^{(1)})|| \le \gamma_1 b_1\eta = \lambda\eta.$$

Notice that we have by $(A.6)$

$$||x_1 - x_0|| = ||x_0^{(k)} - x_0|| \le \left(\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0\right)\eta < R,$$

$$\begin{aligned}||x_1^{(1)} - x_0|| &\le ||x_1^{(1)} - x_1^{(0)}|| + ||x_1^{(0)} - x_0|| \\ &\le \lambda_0\eta + ||x_1 - x_0|| \le \lambda_0\eta + (\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0)\eta < R\end{aligned}$$

and

$$\begin{aligned}||x_1^{(2)} - x_0|| &\leq ||x_1^{(2)} - x_1^{(1)}|| + ||x_1^{(1)} - x_0|| \\ &\leq \lambda\eta + \lambda_0\eta + ||x_1 - x_0|| \leq \lambda\eta + \lambda_0\eta + \left(\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0\right)\eta < R\end{aligned}$$

so $x_1, x_1^{(1)}, x_1^{(2)} \in U(x_0, R)$.

Similarly, we have that

$$\begin{aligned}||F(x_1^{(2)})|| &= \|F(x_1^{(2)}) - F(x_1^{(1)}) - \Gamma_1^{-1}(x_1^{(2)} - x_1^{(1)})\| \\ &= \|([x_1^{(2)}, x_1^{(1)}; F] - [x_1 - F(x_1), x_1 + F(x_1); F])(x_1^{(2)} - x_1^{(1)})\| \\ &\leq \omega(\|(x_1^{(2)} - x_0) + (x_0 - x_1) + F(x_1)\|, \|(x_1^{(1)} - x_0) + (x_0 - x_1) - F(x_1)\|)\|(x_1^{(2)} - x_1^{(1)})\| \\ &\leq \omega(2R + \delta_0\eta, 2R + \delta_0\eta)\lambda\eta \leq \frac{\lambda^2\eta}{\gamma_1},\end{aligned}$$

leading to

$$||x_1^{(3)} - x_1^{(2)}|| = ||\Gamma_1 F(x_1^{(2)})|| \leq \gamma_1\omega(2R + \delta_0\eta, 2R + \delta_0\eta)\lambda\eta \leq \lambda^2\eta$$

and

$$||x_1^{(3)} - x_0|| \leq ||x_1^{(3)} - x_1^{(2)}|| + ||x_1^{(2)} - x_0|| \leq \lambda^2\eta + \lambda\eta + \lambda_0\eta + (\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0)\eta < R,$$

so, $x_1^{(3)} \in U(x_0, R)$.

Therefore, we get in an analogous way that

$$||F(x_1^{(i)})|| \leq \frac{\lambda^i}{\gamma_1}\eta, \; ||x_1^{(k)} - x_1^{(k-1)}|| \leq \lambda^{k-1}\eta$$

and

$$x_1^{(i)} \in U(x_0, R), \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Notice that in view of the estimates on consecutive distances and the definition of λ and γ_1, we deduce that sequence $\{x_n\}$ is complete in a Banach space X and then it converges to some $x^* \in \bar{U}(x_0, R)$.

Finally, notice that sequence $\{F(x_n)\}$ is bounded from above by sequence $\{||x_n - x_{n-1}||\}$, so

$$||F(x^*)|| = \lim_{n\to\infty} ||F(x_n)|| \leq \lim_{n\to\infty} ||x_n - x_{n-1}|| = 0.$$

Hence, we deduce that $F(x^*) = 0$. □

Concerning the uniqueness of the solution, we have the following result.

Proposition 2. *Suppose the hypotheses of Proposition 1 hold. Then the point x^* is the only solution of the equation $F(x) = 0$ in $\bar{U}(x_0, R_2)$ where*

$$R_2 = \sup\{t \in [R, R^*] : \gamma_0\omega_0(t + \eta, R + \eta) < 1\}.$$

Proof. The existence of the solution of equation $F(x) = 0, x^* \in \bar{U}(x_0, R)$ has been shown in Theorem 1.

Let $y^* \in \bar{U}(x_0, R_1)$ be a solution of equation $F(x) = 0$.

Using $(A.2)$ and $(A.4)$, we get in turn for $M = [y^*, x^*; F]$

$$\begin{aligned}||\Gamma_0(M - \Gamma_0^{-1})|| &\leq \gamma_0\omega_0(||y^* - x_0 + F(x_0)||, ||x^* - x_0 - F(x_0)||) \\ &\leq \gamma_0\omega_0(R_2, R + \eta) < 1.\end{aligned}$$

It follows that $M^{-1} \in L(X, X)$. Then, from the identity

$$0 = F(y^*) - F(x^*) = M(y^* - x^*),$$

we conclude that $y^* = x^*$. □