Proceedings

of the

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada

Gijón (Asturias), Spain

June 14-18, 2021

CEDYA CMA 2020 GIJÓN

SēMA
Sociedad Española de Matemática Aplicada

Universidad de Oviedo

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Universidad de Oviedo
Servicio de Publicaciones de la Universidad de Oviedo
Campus de Humanidades. Edificio de Servicios. 33011 Oviedo (Asturias)
Tel. 985 10 95 03 Fax 985 10 95 07
http: www.uniovi.es/publicaciones
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ISBN: 978-84-18482-21-2

Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Different Approximations of the Parameter for Low-Order Iterative Methods with Memory

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Abstract

A technique for generating iterative methods for solving nonlinear equations with memory can be constructed from a method without memory that includes a parameter, provided the parameter is present in the error equation.

Generally, the parameter depends on the evaluation of the function and its derivatives in the solution. However, this information is not available. So this parameter is approximated using interpolation techniques, taking the current iterate x_k and the previous iterates $x_{k-1}, x_{k-2}, \dots$

In this paper we explore different interpolation techniques to obtain both the convergence order of the new methods and their stability characteristics.

1. Introduction

Many phenomena in applied sciences do not respond to a linear pattern. Nonlinearities are present in most fields, such as physics, fluid mechanics, economics or ecology, among others. In this case, these phenomena can be modeled by means of a nonlinear equation $f(x) = 0, f : I \subseteq \mathbb{R} \to \mathbb{R}$, or by means of a system of nonlinear equations $F(x) = 0, F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$. The desired solution x^* of these problems is a closed-form analytic expression. However, there are problems whose analytic solution is hardly available. Obtaining approximate solutions becomes an alternative, by applying numerical methods based on iterative algorithms.

Numerical methods for solving nonlinear equations can be sorted by different criteria. Single-step methods respond to the scheme $x_{k+1} = \phi(x_k)$, while multi-step methods are those that match with $y_k = \phi_1(x_k), x_{k+1} = \phi_2(x_k, y_k)$. A quantitative comparison between methods can be perfomed by the order of convergence p and the efficiency index [17] $I = p^{1/d}$, where d stands for the number of functional evaluations in each step. Kung-Traub's conjecture [15] states that there exists an upper bound for the order of convergence that is $p \leq 2^{d-1}$; thus, the iterative method is optimal when $p = 2^{d-1}$. There is an interesting overview of these methods in [12].

Kung-Traub's conjecture sets an upper bound for the order of convergence in numerical methods without memory. However, this restriction can be overcome by using iterative methods with memory. These kind of methods are defined as

$$x_{k+1} = \phi(x_k, x_{k-1}, \dots, x_{k-m}).$$

In other words, the current iterate is calculated taking into account the last $m + 1$ iterates. This idea was introduced by Traub [23], including memory from Steffensen's method. In the last years, many schemes of iterative methods with memory have been presented. A key overview can be found in [18, 19].

One technique for the design of a method with memory consists of the inclusion of an accelerating parameter in the expression of a method without memory. This technique has been widely adopted in the research of this kind of methods for both nonlinear equations [5, 6, 10], and nonlinear systems of equations [7, 16, 20].

Once the parameter has been included in the iterative expression, the next step is the analysis of the error equation. When the parameter is present in the lower term of this equation, the goal is the replacement of the parameter by an expression that cancels this error term. There are different techiques for the approximation of the parameter.

In this paper, we analyze the most common techniques of replacing the parameter, as well as other novel techniques. In [4] the authors introduced the general form of one-step iterative methods using the weight function technique given by

$$x_{k+1} = x_k - H(t_k), \quad k = 0, 1, 2, \dots, \tag{1.1}$$

where $t_k = f(x_k)/f'(x_k)$. Family (1.1) has quadratic convergence when $H(t)$ satisfies $H(0) = 0$, $H'(0) = 1$ and $|H''(0)| < \infty$. The error equation of members of family (1.1) is

$$e_{k+1} = \left(c_2 - \frac{H''(0)}{2}\right) e_k^2 + O(e_k^3), \tag{1.2}$$

where $e_k = x_k - x^*$ and $c_j = \frac{f^{(j)}(x^*)}{j!f'(x^*)}$, $j \geq 2$. Note that $H(t) = t + \alpha\frac{t^2}{2}$ satisfies the conditions of quadratic convergence of (1.1) for $H(t)$, resulting in

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \alpha\frac{f^2(x_k)}{2(f'(x_k))^2}, \tag{1.3}$$

and its error equation is

$$e_{k+1} = \left(c_2 - \frac{\alpha}{2}\right) e_k^2 + O(e_k^3). \tag{1.4}$$

For $\alpha = 2c_2$, the second order error term vanishes. However, the value of $c_2 = \frac{f''(x^*)}{2f'(x^*)}$ is not known. Therefore, some approximations of $f'(x^*)$ and $f''(x^*)$ must be applied.

2. The approximations of f and the convergence analysis

In order to obtain an approximation of f, we compare the approximation of different interpolatory structures. The most of papers apply Newton's interpolation polynomial of different degrees [11, 14, 24]. Let us denote by $N(t)$ the interpolation polynomial of Newton of second degree, whose expression is

$$N(t) = f(x_k) + f[x_{k-1}, x_k](t - x_k) + f[x_{k-2}, x_{k-1}, x_k](t - x_k)(t - x_{k-1}), \tag{2.1}$$

where $f[\cdot,\cdot]$ and $f[\cdot,\cdot,\cdot]$ are the divided differences of orders one and two. The lower degree of the polynomial in order to avoid that $N''(t)$ vanishes is two. Approximating

$$\begin{cases} f'(x^*) &= f'(x_k), \\ f''(x^*) &= N''(x_k), \end{cases}$$

the value of the parameter is

$$\alpha_k = 2\frac{f[x_{k-2}, x_{k-1}, x_k]}{f'(x_k)}. \tag{2.2}$$

Then, parameter α_k is replaced in (1.3), resulting in an iterative method with memory. Note that this method requires the knowledge of three previous iterates and two new functional evaluations.

The Taylor expansion of a function can also give an approximation for the value of α. From the regressive Taylor expansion at node x_{k-1} of order $O((x_{k-1} - x_k)^2)$ the parameter can be approximated by

$$\alpha_k = \frac{2}{(x_{k-1} - x_k)^2}\left(\frac{f(x_{k-1}) - f(x_k)}{f'(x_k)} - (x_{k-1} - x_k)\right). \tag{2.3}$$

In this case, the method requires the value of the two last iterates, and three evaluations of f.

Another option for the approximation of the parameter is the use od Padé's approximant. It has been applied for solving nonlinear equations [9, 21], but –up to our knowledge– it has not been used for methods with memory. Let $P(t)$ be the Padé's approximant

$$P(t) = \frac{a_0 + a_1(t - x_k)}{1 + a_2(t - x_k)}. \tag{2.4}$$

The values of a_0, a_1 and a_2 can be obtained when (2.4) satisfies

$$\begin{cases} P(x_k) &= f(x_k), \\ P(x_{k-1}) &= f(x_{k-1}), \\ P'(x_k) &= f'(x_k). \end{cases}$$

The approximation of the parameter in this case has the expression

$$\alpha_k = \frac{P''(x_k)}{f'(x_k)} = 2\frac{f'(x_k)\left(f(x_{k-1} - f(x_k) + f'(x_k)(x_k - x_{k-1})\right)}{\left(f(x_k) - f(x_{k-1})\right)(x_k - x_{k-1})}. \tag{2.5}$$

The resulting method only requires two iterates for the approximation of the parameter and three functional evaluations.

Theorem 2.1 gathers the analysis of the R-order of convergence of the previous methods.

Theorem 2.1 *Let x^* be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval I. If x_0 is close enough to x^* and α_0 is given, then the R-orders of method* (1.3) *replacing α_k by expressions* (2.2), (2.3) *and* (2.5) *are* $1 + \sqrt{2}$.

Table 1 collects the comparison of the main values of each technique.

Let us remark from Table 1 that every method has the same order of convergence, while the number of functional evaluations is lower for Taylor and Padé's approximant.

Technique	Newton	Taylor	Padé
Iterates	3	2	2
d	4	3	3
p	$1+\sqrt{2}$	$1+\sqrt{2}$	$1+\sqrt{2}$

Tab. 1 Quantitative comparison of the parameter approximation

3. Real multidimensional dynamical analysis

The dynamics of an iterative method analyses their stability in terms of the amount of initial guesses that converge to the expected solution. Some fundamentals about dynamics of iterative methods without memory can be found in [1, 13], while for the iterative methods with memory the basics are in [2, 3].

The fixed points, for real multidimensional dynamics, involves the definition of an auxilary function $G : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$G(z, x) = (x, g(z, x)),$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ is the iterative expression $x_{k+1} = g(x_{k-1}, x_k)$, $z = x_{k-1}$ and $x = x_k$. Therefore, the fixed points are defined as $G(z_F, x_F) = (z_F, x_F)$. Fixed points that does not match with the roots of f are named strange fixed points. They affect the unstability of the method. A T-periodic point is defined as $G^T(z_T, x_T) = G(z_T, x_T)$, satisfying $G^t(z_T, x_T) \neq (z_T, x_T), t < T$; note that for $T = 1$, the periodic point is a fixed point. The asymptotical behavior of T-periodic points is defined in [22]. Theorem 3.1 collects the asymptotical behavior for $T = 1$.

Theorem 3.1 *Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be C^2. Let μ_1, μ_2 be the eigenvalues of the Jacobian matrix G' on a fixed point (z_F, x_F). Then*

1. *If $|\mu_1| < 1$ and $|\mu_2| < 1$, then (z_F, x_F) is attracting.*
2. *If $|\mu_1| > 1$ and $|\mu_2| > 1$, then (z_F, x_F) is repelling.*
3. *If $|\mu_1| < 1$ and $|\mu_2| > 1$, or $|\mu_1| > 1$ and $|\mu_2| < 1$, then (z_F, x_F) is unstable.*

The attracting fixed points are denoted by (z^+, x^+). The basin of attraction of an attracting fixed point $\mathcal{A}(z^+, x^+)$ is the set of points that satisfy

$$\mathcal{A}(z^+, x^+) = \{(z, x) \in \mathbb{R}^2 : G^n(z, x) \to (z^+, x^+), n \to \infty\}.$$

The dynamical analysis is performed applying the expressions of α on (1.3) for the solution of $f(x) = x^2 - \lambda$.

In order to make a reasonable comparison, we are analysing the resulting methods of Taylor's and Padé's approximations of α. Note that these methods only require the two last iterates, while Newton's approximation requires three previuos iterates.

The comparison is performed via the representation of the basins of attraction, in a similar manner as described in [8]. In this particular case, the basins of $(z^+, x^+) = \sqrt{\lambda}(1, 1)$ are represented in orange, the basins of $(z^+, x^+) = -\sqrt{\lambda}(1, 1)$ are represented in blue, and the convergence to a different point than $(z^+, x^*) = \pm\sqrt{\lambda}(1, 1)$ is represented in black. The fixed attracting points are represented with white stars.

3.1. Taylor's approximation

Replacing (2.3) in (1.3), the auxiliary function is

$$T(z, x) = \left(x, \frac{3x^4 + 6x^2\lambda - \lambda^2}{8x^3}\right).$$

There are two fixed attracting points $(z^+, x^+) = \pm\sqrt{\lambda}(1, 1)$ and two unstable points $(z, x) = \pm\sqrt{\frac{\lambda}{5}}(1, 1)$.

Figure 1 represents the basins of attraction of $T(z, x)$ for different values of λ. Since $T(z, x)$ does not have dependence on the value of $z = x_{k-1}$, the dynamical planes are vertical bands. Note that every initial guess converge to an attracting fixed point, and bands are wider as the value of λ increases.

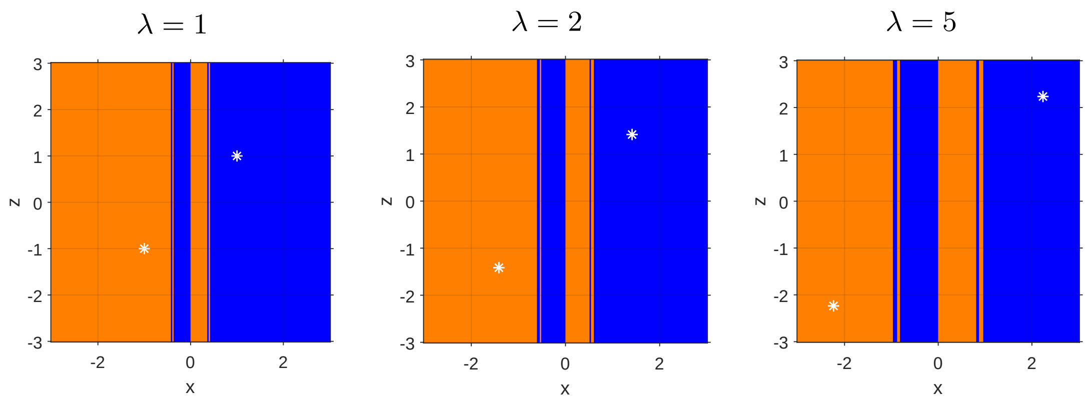

Fig. 1 Dynamical planes using Taylor's approximation of α

3.2. Padé's approximation

Replacing (2.5) in (1.3), the auxiliary function is

$$P(z,x) = \left(x, \frac{x^2 - \frac{(x^2-\lambda)^2}{x+z} + \lambda}{2x} \right).$$

There are two fixed attracting points $(z^+, x^+) = \pm\sqrt{\lambda}(1,1)$ and two unstable points $(z,x) = \left(-1 \pm \sqrt{1+\lambda}\right)(1,1)$.

Figure 2 represents the basins of attraction of $P(z,x)$ for different values of λ. In this case, $P(z,x)$ depends on both $z = x_{k-1}$ and $x = x_k$, so dynamical planes are not vertical bands. There are regions of convergence to the roots of f, but there are other regions that diverge or converge to another point, as black areas represent. Moreover, as λ increases, the width of black central region also does.

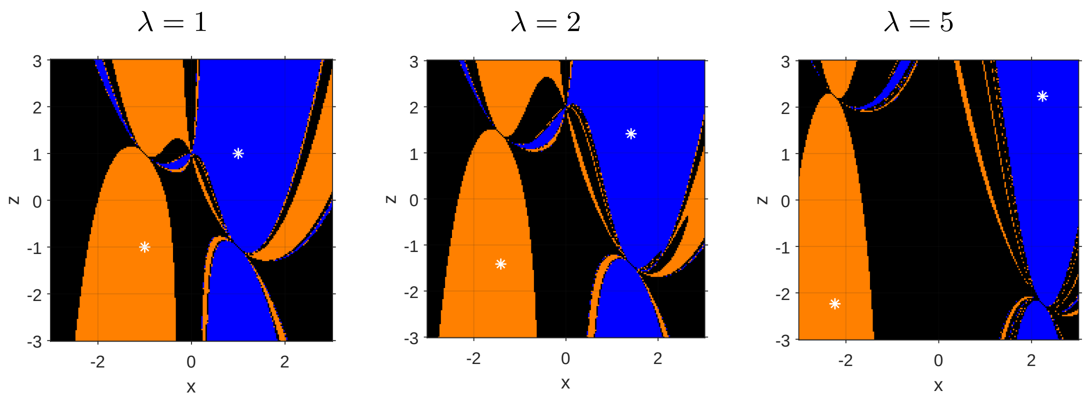

Fig. 2 Dynamical planes using Padé's approximation of α

4. Conclusions

Three new techniques have been introduced for the approximation of the self-accelerating parameter in a low-order iterative method. The order of convergence for the three cases have increased from 2 to $1+\sqrt{2}$. In order to make a reasonable comparison for the stability counterpart, two approximations that involve the same number of previous iterates have been taken. Taylor's approximation results in vertical dynamical planes, because of the independence of $T(z,x)$ with z. In addition, Taylor's approximation results in more stable dynamical planes than Padé's approximation.

Acknowledgements

The authors were supported by the internal research project ADMIREN of Universidad Internacional de La Rioja (UNIR). The first author was also partially supported by PGC2018-095896-B-C22 (MCIU/AEI/FEDER, UE).

References

[1] P. Blanchard. Complex analytic dynamics on the riemann sphere. *Bulletin of the American Mathematical Society*, 11:85–141, 1984.

[2] B. Campos, A. Cordero, J. R. Torregrosa, and P. Vindel. A multidimensional dynamical approach to iterative methods with memory. *Applied Mathematics and Computation*, 271:701–715, 2015.

[3] B. Campos, A. Cordero, J. R. Torregrosa, and P. Vindel. Stability of king's family of iterative methods with memory. *Journal of Computational and Applied Mathematics*, 318:504–514, 2017.

[4] F. I. Chicharro, A. Cordero, N. Garrido, and J. R. Torregrosa. Generating root-finder iterative methods of second order: Convergence and stability. *Axioms*, 8(2):55, 2019.

[5] F. I. Chicharro, A. Cordero, N. Garrido, and J. R. Torregrosa. Anomalies in the convergence of traub-type methods with memory. *Computational and Mathematical Methods*, 2:e1060, 2020.

[6] F. I. Chicharro, A. Cordero, N. Garrido, and J. R. Torregrosa. On the choice of the best members of the kim family and the improvement of its convergence. *Mathematical Methods in the Applied Sciences*, 43:8051–8066, 2020.

[7] F. I. Chicharro, A. Cordero, N. Garrido, and J. R. Torregrosa. On the improvement of the order of convergence of iterative methods for solving nonlinear systems by means of memory. *Applied Mathematics Letters*, 104:106277, 2020.

[8] F. I. Chicharro, A. Cordero, and J. R. Torregrosa. Drawing dynamical and parameters planes of iterative families and methods. *The Scientific World Journal*, 2013:780153, 2013.

[9] F. I. Chicharro, A. Cordero, and J. R. Torregrosa. Dynamics and fractal dimension of steffensen-type methods. *Algorithms*, 8:271–279, 2015.

[10] N. Choubey, A. Cordero, Jaiswal J. P., and J. R. Torregrosa. Dynamical techniques for analyzing iterative schemes with memory. *Complexity*, 1231341:1–13, 2018.

[11] A. Cordero, H. Ramos, and J. R. Torregrosa. Some variants of halley's method with memory and their applications for solving several chemical problems. *Journal of Mathematical Chemistry*, 58:751–774, 2020.

[12] A. Cordero and J. R. Torregrosa. *On the Design of Optimal Iterative Methods for Solving Nonlinear Equations*, pages 79–111. Springer International Publishing, 2016.

[13] R. L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley, New York, USA, 1964.

[14] F. W. Khdhr, F. Soleymani, R. K. Saeed, and A. Agkül. An optimized steffensen-type iterative method with memory associated with annuity calculation. *European Physical Journal Plus*, 134:146, 2019.

[15] H. T. Kung and J. F. Traub. Optimal order of one-point and multipoint iteration. *Journal of the Association for Computing Machinery*, 21:643–651, 1974.

[16] M. Narang, S. Bhatiua, A. S. Alshomrani, and V. Kanwar. General efficient class of steffensen type methods with memory for solving systems of nonlinear equations. *Journal of Computational and Applied Mathematics*, 352:23–39, 2019.

[17] A. M. Ostrowski. *Solution of Equations and Systems of Equations*. Academic Press, New York, USA, 1960.

[18] M. S. Petković, B. Neta, L. D. Petković, and J. Dzǔnic. *Multipoint Methods for Solving Nonlinear Equations*. Academic Press, The Netherlands, 2013.

[19] M. S. Petković, B. Neta, L. D. Petković, and J. Dzǔnic. Multipoint methods for solving nonlinear equations: a survey. *Applied Mathematics and Computation*, 226:635–660, 2014.

[20] M. S. Petković and J. R. Sharma. On some efficient derivative-free iterative methods with memory for solving systems of nonlinear equations. *Numerical Algorithms*, 71:457–474, 2016.

[21] P. Praks and D. Brkić. One-log call iterative solution of the colebrook equation for flow friction based on padé polynomials. *Energies*, 11:1825, 2018.

[22] R. C. Robinson. *An Introduction to Dynamical Systems: Continuous and Discrete*. American Mathematical Society, Rhode Island, USA, 2012.

[23] J. F. Traub. *Iterative Methods for the Solution of Equations*. Prentice-Hall, New York, USA, 1964.

[24] X. Wang and Q. Fan. A modified ren's method with memory using a simple self-accelerating parameter. *Mathematics*, 8:540, 2020.