MODELLING FOR ENGINEERING & HUMAN BEHAVIOUR 2019

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Instituto Universitario de Matemática Multidisciplinar
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Analysis of finite dimensional linear control systems subject to uncertainties via probabilistic densities

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1 Introduction

Control Theory is a branch of Mathematics that studies the behaviour of a dynamical system with controllers, one or more, applied through actuators. Furthermore, its main objective is to develop control models for controlling such systems using a control action in an optimum manner, that is ensuring the stability. Applications of Control Theory, in irrigation systems, can be found since the ancient Mesopotamia more than 2000 years B.C. But it was not until the 1868 that the first definitive mathematical description of Control Theory was established in the works by J.C. Maxwell, [1]. From this moment Control Theory gained importance, becoming nowadays a fundamental tool to develop new technologies.

A control problem consists in finding controls, say $u(t)$, such that the solution of a model, $x(t;u)$, coincides or gets close to a target value x^1 at a final time instant T, i.e., $x(T;u)=x^1$. Generally, an optimal control problem is defined via a set of differential equations, ordinary or partial, describing the states which depend on the control variables that minimize a particular cost function of the form

$$J(v)=\frac{1}{2}||x(t;u)-x^1||^2+\frac{\beta}{2}||u||^2,$$

where $\beta \geq 0$ allows us to penalize using too much costly control.

On the other hand, the parameters that appear in this kind of formulations are generally set via experimental data. Therefore, since these values are obtained from certain measurements and samplings, they contain an intrinsic uncertainty. This situation allows us to consider inputs as random variables or stochastic processes rather than constants or deterministic functions, respectively.

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2 Probabilistic solution

As it has been pointed previously, a control problem is defined through a set of ordinary or partial differential equations depending on the dimensionality of the system, finite or infinite dimensional, respectively. In this contribution, given its interest, we consider the finite dimensional linear control system

$$\begin{array}{rcl} x'(t) &=& \mathbf{A}x(t) + \mathbf{B}\,u(t), \quad 0 < t < T, \\ x(0) &=& x^0. \end{array} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $x^0 \in \mathbb{R}^n$ is the initial data, $\mathbf{A}$ is a $n \times n$ matrix of the free dynamics part, $\mathbf{B}$ is a $n \times m$ matrix, with $m \in \mathbb{N}$ and $m \leq n$, called the control operator and $u(t)$ the m-dimensional control vector.

We study, from a probabilistic point of view, the randomized control problem

$$\begin{array}{rcl} x'(t,\omega) &=& \mathbf{A}(\omega)x(t,\omega) + \mathbf{B}(\omega)\,u(t,\omega), \quad 0 < t < T, \\ x(0,\omega) &=& x^0(\omega). \end{array} \tag{2}$$

where all the input parameters $A_{ij}(\omega)$, $B_{ik}(\omega)$, $0 \leq i,j \leq n$ and $0 \leq k \leq m$, the starting seed $x^0(\omega) = [x^0_1(\omega), \ldots, x^0_n(\omega)]^\top$ and the final target $x^1(\omega) = [x^1_1(\omega), \ldots, x^1_n(\omega)]^\top$ are assumed to be absolutely continuous random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assuming that the system is controllable, we can obtain a solution stochastic process of the initial value problem (2) [2, 3],

$$x(t,\omega) = \left(\mathrm{e}^{\mathbf{A}(\omega)t} - H(t;\mathbf{A}(\omega),\mathbf{B}(\omega))\mathrm{e}^{\mathbf{A}(\omega)T}\right) x^0(\omega) + H(t;\mathbf{A}(\omega),\mathbf{B}(\omega))x^1(\omega),$$

where

$$H(t;\mathbf{A}(\omega),\mathbf{B}(\omega)) = \int_0^t \mathrm{e}^{\mathbf{A}(\omega)(t-s)}\mathbf{B}(\omega)\mathbf{B}^*(\omega)\mathrm{e}^{\mathbf{A}^*(\omega)(T-s)}\,\mathrm{d}s\,\Lambda^{-1}(T;\mathbf{A}(\omega),\mathbf{B}(\omega)).$$

and

$$\Lambda(x;\mathbf{A}(\omega),\mathbf{B}(\omega)) = \int_0^x \mathrm{e}^{\mathbf{A}(\omega)(T-t)}\mathbf{B}(\omega)\mathbf{B}^*(\omega)\mathrm{e}^{\mathbf{A}^*(\omega)(T-t)}\mathrm{d}t.$$

Now, we apply the Random Variable Transformation method (see Reference [1]) to obtain the first probability density function of the solution stochastic process

$$\boxed{\begin{array}{rcl} f_1(x,t) &=& \displaystyle\int_{\mathbb{R}^{h_1}} f_{x^0,x^1,\mathbf{A},\mathbf{B}}\left(\left(\mathrm{e}^{\mathbf{A}t} - H(t;\mathbf{A},\mathbf{B})\mathrm{e}^{\mathbf{A}T}\right)^{-1}(x - H(t;\mathbf{A},\mathbf{B})x^1), x^1, \mathbf{A}, \mathbf{B}\right) \\ && \det\left(\left(\mathrm{e}^{\mathbf{A}t} - H(t;\mathbf{A},\mathbf{B})\mathrm{e}^{\mathbf{A}T}\right)^{-1}\right)\mathrm{d}x^1\,\mathrm{d}\mathbf{A}\,\mathrm{d}\mathbf{B}, \end{array}}$$

3 Numerical example

In this example we consider that A, B and x^1 are deterministic matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad x^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In addition, we assume that the random vector $x^0(\omega)$ follows a multivariate Normal distribution with mean $\mu = (1, 1)$ and variance-covariance matrix

$$\Sigma = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}, \qquad \text{i.e.} \qquad x^0 = (x_1^0, x_2^0) \sim \mathrm{N}(\mu, \Sigma)$$

In Figure (1) the first probability density function is plotted for the time instant $t = 0.1$. Phase portrait is represented in Figure (2). In the phase portrait the expectation and 50% and 90% confidence intervals are shown at the time instants $t \in \{0, 0.1, 0.5, 0.9\}$. We observe that the solution tends to the point $x^1 = (0, 0)$. Notice that as x^1 is deterministic, then, the variability vanishes as time increases.

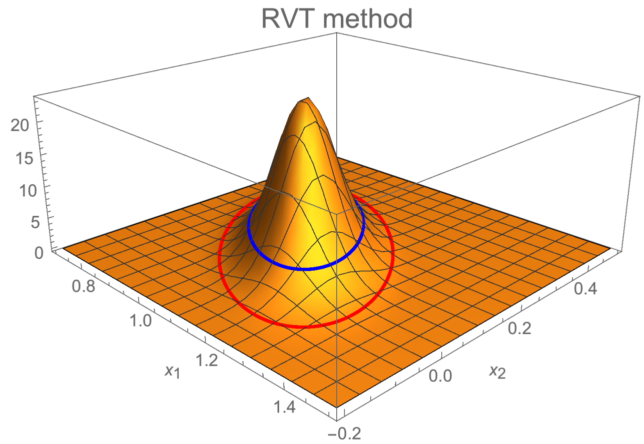

Figure 1: First probability density function of the solution stochastic process at the time instant $t = 0.1$. 50% (blue curve) and 90% (red curve) confidence regions.

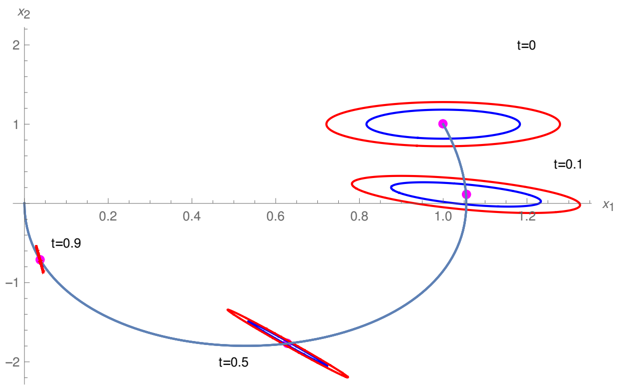

Figure 2: Continuous spiral line represents the expectation of the solution. 50% (blue curve) and 90% (red curve) confidence regions are plotted at the time instants $t \in \{0, 0.1, 0.5, 0.9\}$.

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