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L. Jódar and E. López-Navarro

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Semilocal convergence for new Chebyshev-type iterative methods

Abhimanyu Kumar [♭], D.K. Gupta[♮], Eulalia Martínez[♯], Jose L. Hueso[♯] and Fabricio Cevallos[∗1]

(♭) Department of Mathematics,
Lalit Narayan Mithila University,
(♮) Department of Mathematics,
Indian Institute of Technology Kharagpur,
(♯) Instituto de Matemática Multidisciplinar,
Universitat Politècnica de València,
(∗) Facultad de Ciencias Económicas,
Universidad Laica Eloy Alfaro de Manabí.

1 Introduction

In this paper, the convergence of improved Chebyshev-Secant-type iterative methods are studied for solving nonlinear equations in Banach space settings. Its semilocal convergence is established using recurrence relations under weaker continuity conditions on first order divided differences. Convergence theorems are established for the existence-uniqueness of the solutions.

Consider approximating a locally unique solution ρ^* of

$$\mathrm{F}(x) = 0, \tag{1}$$

where F is a continuous nonlinear operator defined on a non-empty open convex subset D of a Banach space X with values in another Banach space Y. This is one of the most important problems in applied mathematics and engineering.

The next family of iterative methods used for the solution of (1) is known as the Chebyshev-Secant-type methods (CSTM).

$$\begin{aligned} y_k &= x_k - [x_{k-1}, x_k; \mathrm{F}]^{-1}\mathrm{F}(x_k), \\ z_k &= x_k + \alpha(y_k - x_k), \\ x_{k+1} &= x_k - [x_{k-1}, x_k; \mathrm{F}]^{-1}(\beta\mathrm{F}(x_k) + \gamma\mathrm{F}(z_k)), \end{aligned} \tag{2}$$

where $x_{-1},\ x_0 \in \mathrm{D}$ are two starting iterates and $[x, y; \mathrm{F}] \in \mathrm{L}(\mathrm{X}, \mathrm{Y})$ satisfies $[x, y; \mathrm{F}](x - y) = \mathrm{F}(x) - \mathrm{F}(y)$ for $x, y \in \mathrm{D}$ and $x \neq y$, for $x = y$, $[x, y; \mathrm{F}] = \mathrm{F}'(x)$. Here, α, β and γ are nonnegative real parameters carefully chosen so that the sequence $\{x_k\}$ converges to ρ^*.

[1] e-mail: alfa2205@gmail.com

The improved Chebyshev-Secant-type method (ICSTM) proposed by us is given for $k \geq 0$ by

$$\begin{aligned} x_{k+1} &= x_k - \mathrm{B}_k^{-1}\mathrm{F}(x_k), \quad \mathrm{B}_k = [x_k, y_k; \mathrm{F}], \\ z_k &= x_k + \alpha(x_{k+1} - x_k), \\ y_{k+1} &= x_k - \mathrm{B}_k^{-1}(\beta\mathrm{F}(x_k) + \gamma\mathrm{F}(z_k)), \end{aligned} \tag{3}$$

where $x_0, y_0 \in \mathrm{D}$ are two starting iterates and $\alpha,\ \beta$ and γ are nonnegative real parameters. Considering $\alpha = \beta = \gamma = 1$ we obtain the double step Secant method [1, 2] with order of convergence $1 + \sqrt{2}$. It can be easily seen that the number of functions evaluations and the corresponding divided differences used in CSTM and ICSTM are equal. The importance of the ICSTM lies in the fact that for $\alpha = \beta = \gamma = 1$, its convergence order is $1 + \sqrt{2}$, while the convergence order of the CSTM is 2.

2 Semilocal convergence of ICSTM

In this section, the semilocal convergence of ICSTM for solving (1) is established. Let $\mathcal{B}(x, r)$ and $\overline{\mathcal{B}}(x, r)$ denote open and closed balls with center at x and radius r, respectively. For suitably chosen initial approximations x_0 and y_0, we define a class $\mathrm{S}(\Theta, \delta, \eta, \sigma)$, where $\Theta > 0,\ \delta > 0,\ \eta > 0$ are some positive real numbers and σ is to be defined. The triplet $(\mathrm{F}, x_0, y_0) \in \mathrm{S}(\Theta, \delta, \eta, \sigma)$ if

[C_1] $\|x_0 - y_0\| \leq \Theta$ for $x_0, y_0 \in \mathrm{D}$.

[C_2] $\mathrm{B}_0^{-1} \in \mathrm{L}(\mathrm{Y}, \mathrm{X})$ such that $\|\mathrm{B}_0^{-1}\| \leq \delta$.

[C_3] $\|\mathrm{B}_0^{-1}\mathrm{F}(x_0)\| \leq \eta$.

[C_4] $\|([x, y; \mathrm{F}] - [u, v; \mathrm{F}])\| \leq \sigma(\|x - u\|, \|y - v\|)$, where $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and non decreasing function in its both arguments for $x, y, u, v \in \mathrm{D}$.

[C_5] $(1 - \beta) = (1 - \alpha)\gamma$ and $\alpha \in (0, 1]$.

[C_6] The equation
$$(1 - g(t))t - \eta = 0$$
where, $g(t) = \dfrac{M}{1 - \delta\sigma(t, t + \Theta)}$,
and $M = \max(\alpha\gamma\delta\sigma(\eta, \Theta), \alpha\delta\sigma(\eta, \Theta), \delta\sigma(\eta, \Theta), \alpha\gamma\delta\sigma(\eta, (1+p)\eta))$, where $p = \alpha\gamma\delta\sigma(\eta, \Theta)$, has at least one positive root. The smallest positive root is denoted by R.

[C_7] $g(R) \in (0, 0.618034...)$.

[C_8] $\overline{\mathcal{B}}(x_0, R) \subseteq \mathrm{D}$.

Lemma 1 *For the improved Chebyshev-Secant-type method (ICSTM) proposed in (3) it is verified:*

(i) $\mathrm{F}(z_k) = \alpha([z_k, x_k; \mathrm{F}] - \mathrm{B}_k)(x_{k+1} - x_k) + (1 - \alpha)\mathrm{F}(x_k)$.

(ii) $\mathrm{F}(x_{k+1}) = ([x_{k+1}, x_k; \mathrm{F}] - [x_k, y_k; \mathrm{F}])\,(x_{k+1} - x_k)$.

Proof: The proof follows obviously by (3) and the application of the usual property of the divided difference operator, $[x, y, \mathrm{F}](x-y) = \mathrm{F}(x) - \mathrm{F}(y)$, hence omitted here. $\square$

Lemma 2 *For method ICSTM proposed in (3) under conditions* $[\mathrm{C}_1]-[\mathrm{C}_8]$ *and for* $(\mathrm{F}, x_0, y_0) \in \mathrm{S}(\Theta, \delta, \eta, \sigma)$*, we obtain the following bounds:*

(i) *There exists* B_k^{-1} *satisfying* $\|\mathrm{B}_k^{-1}\| \leq \frac{\delta}{1-\delta\sigma(R,R+\Theta)}$,

(ii) $\|x_{k+1} - x_k\| \leq g(R)\|x_k - x_{k-1}\|$,

(iii) $\|y_{k+1} - x_k\| \leq (1 + g(R))\|x_{k+1} - x_k\|$,

(iv) $\|y_{k+1} - x_{k+1}\| \leq g(R)\|x_{k+1} - x_k\|$,

(v) $\|x_{k+1} - x_0\| \leq \sum_{j=0}^{k} g(R)^j \eta < R$,

(vi) $\|y_{k+1} - x_0\| \leq \sum_{j=0}^{k+1} g(R)^j \eta < R$,

(vii) $\|z_k - x_0\| \leq \sum_{j=0}^{k} g(R)^j \eta < R$.

Proof: The above inequalities can be proved by using mathematical induction. Using Lemma 1 and the definition of class $\mathrm{S}(\Theta, \delta, \eta, \sigma)$, we get $\|x_1 - x_0\| \leq \eta$, $\|z_0 - x_0\| \leq \eta$ and

$$\begin{aligned}\|y_1 - x_0\| &= \|x_1 - x_0 - \alpha\gamma \mathrm{B}_0^{-1}\sigma(\|z_0 - x_0\|, \|x_0 - y_0\|)(x_1 - x_0)\| \\ &\leq (1 + \alpha\gamma\delta\sigma(\eta, \Theta))\,\|x_1 - x_0\| < (1 + g(R))\eta < R.\end{aligned}$$

with $\|y_1 - x_1\| \leq \alpha\gamma\delta\sigma(\eta, \Theta)\|x_1 - x_0\| \leq g(R)\|x_1 - x_0\|$. Thus, lemma holds for $n = 0$. Suppose that it holds for some $n \leq k$. Now,

$$\|I - \mathrm{B}_0^{-1}\mathrm{B}_k\| \leq \delta\sigma(\|x_k - x_0\|, \|y_k - x_0\| + \|y_0 - x_0\|) \leq \delta\sigma(R, R + \Theta) < 1.$$

So using Banach's lemma on invertible operators [3], it is verified

$$\|\mathrm{B}_k^{-1}\| \leq \frac{\delta}{1 - \delta\sigma(R, R + \Theta)}.$$

Using Lemma 1 once more, we get

$$\begin{aligned}\|x_{k+1} - x_k\| &\leq \|\mathrm{B}_k^{-1}\|\|\mathrm{F}(x_k)\| \\ &\leq \frac{\delta\sigma(\|x_k - x_{k-1}\|, \|x_{k-1} - y_{k-1}\|)}{1 - \delta\sigma(R, R + \Theta)}\|x_k - x_{k-1}\| \\ &\leq g(R)\|x_k - x_{k-1}\|.\end{aligned}$$

Now,

$$\begin{aligned}\|y_{k+1} - x_k\| &\leq \|x_{k+1} - x_k - \alpha\gamma\mathrm{B}_k^{-1}\sigma(\|z_k - x_k\|, \|x_k - y_k\|)(x_{k+1} - x_k)\| \\ &\leq \left(1 + \frac{\alpha\gamma\delta\sigma(\|z_k - x_k\|, \|x_k - y_k\|)}{1 - \delta\sigma(R, R + \Theta)}\right)\|x_{k+1} - x_k\| \\ &\leq (1 + g(R))\|x_{k+1} - x_k\|.\end{aligned}$$

This gives

$$\|y_{k+1} - x_{k+1}\| \leq \|\alpha\gamma \mathrm{B}_k^{-1}\sigma(\|z_k - x_k\|, \|x_k - y_k\|)(x_{k+1} - x_k)\| \leq g(R)\|x_{k+1} - x_k\|.$$

Thus this proves (i)-(iv). (v), (vi) and (vii) can easily be obtained with the recursive use of (i)-(iv). Hence, this proves the lemma. □

Theorem 1 *Let $F : D \subseteq X \to Y$ be a continuous nonlinear operator, and consider the triplet $(\mathrm{F}, x_0, y_0) \in \mathrm{S}(\Theta, \delta, \eta, \sigma)$ defined in section 2, with $x_0, y_0 \in \mathrm{D}$ verifying conditions $[\mathrm{C}_1] - [\mathrm{C}_8]$. Then, by taking x_0, y_0 as starting points, the sequences $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ generated by (3) are well defined and belong to $\mathcal{B}(x_0, R) \subseteq \mathrm{D}$. Also, the iterate x_k, y_k and z_k converge to $\rho^* \in \overline{\mathcal{B}}(x_0, R) \subseteq \mathrm{D}$, where ρ^* is the unique solution of (1) in $\overline{\mathcal{B}}(x_0, R) \cap \mathrm{D}$.*

Proof: Using Lemma 1 and Lemma 2, we see that the iterates x_k and y_k are well defined and belong to $\mathcal{B}(x_0, R)$. It is sufficient to show that $\{x_k\}$ is a Cauchy sequence. For fixed k and $m \geq 1$, we get

$$\begin{aligned}
\|x_{k+m} - x_k\| &\leq \|x_{k+m} - x_{k+m-1}\| + \ldots + \|x_{k+1} - x_k\| \\
&\leq \Big(g(R)^{m-1} + g(R)^{m-2} + \ldots + g(R) + 1\Big)\|x_{k+1} - x_k\| \\
&\leq \Big(g(R)^{m-1} + g(R)^{m-2} + \ldots + g(R) + 1\Big)\|x_{k+1} - x_k\| \\
&\leq \left(\frac{1 - g(R)^m}{1 - g(R)}\right) g(R)^k \|x_1 - x_0\|.
\end{aligned}$$

Therefore $x_k \to \rho^*$ as $k \to \infty$. Now, we show that ρ^* is a solution of (1). From Lemma 1, we get

$$\|\mathrm{F}(x_{k+1})\| \leq \|[x_{k+1}, x_k; \mathrm{F}] - [x_k, y_k; \mathrm{F}]\|\|x_{k+1} - x_k\| \to 0 \text{ as } k \to \infty.$$

From the continuity of F, it is assured that $\mathrm{F}(\rho^*) = 0$. To show the uniqueness of ρ^*, let $\widehat{\rho}$ be another solution of (1) in $\overline{\mathcal{B}}(x_0, R)$ such that $\mathrm{F}(\widehat{\rho}) = 0$. For $\mathrm{B}^* = [\rho^*, \widehat{\rho}; \mathrm{F}]$, we get

$$\|I - \mathrm{B}_0^{-1}\mathrm{B}^*\| \leq \delta\sigma(\|x_k - x_0\|, \|y_k - x_0\| + \|y_0 - x_0\|) \leq \delta\sigma(R, R + \Theta) < 1.$$

This shows that B^* is invertible and from the identity $[\rho^*, \widehat{\rho}; \mathrm{F}](\rho^* - \widehat{\rho}) = \mathrm{F}(\rho^*) - \mathrm{F}(\widehat{\rho})$, taking norms on both sides, we get $\rho^* = \widehat{\rho}$. This implies the uniqueness of ρ^*. □

Keywords: Nonlinear equations, Divided differences, Semilocal convergence.

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