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Sampling of pairwise comparisons in decision-making

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1 Introduction

Various decision-making techniques rely on pairwise comparisons (PCs) between the involved elements. Traditionally, PCs are provided by experts or relevant actors, and compiled into pairwise comparison matrices (PCMs).

In highly complex problems, the number of elements to be compared may be very large. One of the issues limiting PC applicability to large-scale decision problems is the so-called curse of dimensionality, that is, many PCs need to be elicited from an actor, or built from a body of information.

In general, when applied to a set of n elements to be compared, the number of PCs that have to be made is $n(n-1)/2$. When the information in the comparison matrix is complete, the priorities can be obtained. This is the case of decision-making with complete information. However, if there are missing entries due to uncertainty or lack of information, decision-making must be performed from the available incomplete information. The authors have addressed the issue of incomplete information in [5, 6], and have characterized the consistent completion of a PCM using graph theory in [6].

In this contribution, we claim that less than that number of comparisons may be suitable to develop sound decision-making. There is a trivial solution providing a lower bound for the sample size: just produce $n-1$ PCs, for example comparing one element with the others. It can be shown that this is equivalent to give directly the priority vector. Here we reduce the number of pairwise comparisons in a decision-making problem by selecting just a sample of n PCs that are able to provide balanced and unbiased (incomplete) information that still produces consistent and robust decisions. Both the size of the sample and its distribution within the PCM are of interest.

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We address this research within a linearization theory developed by the authors [2] based on optimizing the consistency of reciprocal matrices.

2 Problem statement and solution

If there are n alternatives, the expert must build an $n \times n$ reciprocal matrix, and therefore, produce $n(n-1)/2$ PCs. If n is large, $n(n-1)/2$ is also large and the expert can be easily tired and lose the necessary concentration. For example, if $n = 10$ (which is not very large), then $n(n-1)/2 = 45$, and a survey of 45 questions may be tedious, strenous and time-consuming. In contrast, if the expert is asked to fill fewer entries, the survey will become more friendly and, arguably, more reliable.

2.1 Problem

Here we focus on the incomplete $n \times n$ reciprocal matrix B, where only entries $b_{12}, b_{23}, \ldots, b_{n-1,n}$, b_{n1} are known. For example, for size 6×6,

$$B = \begin{bmatrix} 1 & b_{12} & \star & \star & \star & b_{n1}^{-1} \\ b_{12}^{-1} & 1 & b_{23} & \star & \star & \star \\ \star & b_{23}^{-1} & 1 & b_{34} & \star & \star \\ \star & \star & b_{34}^{-1} & 1 & b_{45} & \star \\ \star & \star & \star & b_{45}^{-1} & 1 & b_{56} \\ b_{n1} & \star & \star & \star & b_{56}^{-1} & 1 \end{bmatrix}. \tag{1}$$

The next result characterizes when B can be completed to be consistent.

Theorem 1 *Let $B \in M_n$ be a reciprocal incomplete matrix with known entries b_{12}, b_{23}, $\ldots$, $b_{n-1,n}$, b_{n1}.*

(i) Matrix B admits a consistent completion if and only if

$$b_{12}\ b_{23} \cdots b_{n-1,n}\ b_{n1} = 1. \tag{2}$$

(ii) If B admits a consistent completion, then it is unique, say C, and C satisfies the following condition: if $(B)_{ij}$ is unspecified and $i < j$, then $(C)_{ij} = b_{i,i+1} b_{i+1,i+2} \cdots b_{j-1,j}$.

Let's check the performance of this approach. Let A be a (fully known) reciprocal matrix. We can find X_A, the closest consistent matrix to A by using the formula given in [4]. Also, from the incomplete matrix B defined as in the statement of Theorem 1, supposing that B satisfies the criterion given in this theorem, we can easily compute C. The next example compares matrices X_A and C and calculates the distance between both.

Example 1 *Let*

$$A = \begin{bmatrix} 1 & 2 & 2 & 8 \\ 1/2 & 1 & 4 & 1/2 \\ 1/2 & 1/4 & 1 & 1 \\ 1/8 & 2 & 1 & 1 \end{bmatrix}. \tag{3}$$

This matrix A is not consistent (e.g., rank$(A) > 1$, see [3, Theorem 1]). The Perron eigenvalue is $\lambda_{\max} \simeq 4.84$, and (see [7]) $\mathrm{CI}(A) = (\lambda_{\max} - 4)/(4-1) \simeq 0.279$ and $\mathrm{CI}(A)/\mathrm{RI}_4 \simeq 0.314 > 0.1$; the consistency of A is not acceptable (Saaty's criterion); here $\mathrm{RI}_4 = 0.89$ is the random index for 4×4 matrices.

Using the formula given in [4], we have

$$X_A \simeq \begin{bmatrix} 1 & 2.38 & 4 & 3.36 \\ 0.42 & 1 & 1.68 & 1.41 \\ 0.25 & 0.59 & 1 & 0.84 \\ 0.29 & 0.71 & 1.19 & 1 \end{bmatrix}.$$

To apply Theorem 1, let us consider the following incomplete reciprocal matrix

$$B = \begin{bmatrix} 1 & 2 & \star & 8 \\ 1/2 & 1 & 4 & \star \\ \star & 1/4 & 1 & 1 \\ 1/8 & \star & 1 & 1 \end{bmatrix}.$$

Since (2) holds, then there exists a unique consistent completion, namely

$$C = \begin{bmatrix} 1 & 2 & b_{12}b_{23} & 8 \\ 1/2 & 1 & 4 & b_{23}b_{34} \\ (b_{12}b_{23})^{-1} & 1/4 & 1 & 1 \\ 1/8 & (b_{23}a_{34})^{-1} & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 8 & 8 \\ 1/2 & 1 & 4 & 4 \\ 1/8 & 1/4 & 1 & 1 \\ 1/8 & 1/4 & 1 & 1 \end{bmatrix}.$$

Now the distance between X_A and C is $d(X_A - C) = \|X_A - C\|_F = tr(X_A C^T) \simeq 2.4992$, $\mathrm{tr}(\cdot)$ being the trace operator.

If (2) does not hold, the matrix B defined in Theorem 1 has no consistent completion. If we denote by $\mathcal{C}_n$ the set of $n \times n$ consistent matrices, then we must find $D \in M_n$, a reciprocal completion of B, such that

$$\mathrm{d}(D, \mathcal{C}_n) \leq \mathrm{d}(D', \mathcal{C}_n)$$

for any $D' \in M_n$ reciprocal completion of B.

We summarize the obtained results in the following theorem.

Theorem 2 *Let $B \in M_n$ be a reciprocal incomplete matrix with known entries b_{12}, b_{23}, ..., $b_{n-1,n}$, b_{n1}.*

(i) There is a unique reciprocal completion of B, say D, such that $\mathrm{d}(D, \mathcal{C}_n) \leq \mathrm{d}(D', \mathcal{C}_n)$ for all $D' \in M_n$ reciprocal completion of B.

(ii) There is a unique $Z \in \mathcal{C}_n$ such that $\mathrm{d}(D, Z) = \mathrm{d}(D, \mathcal{C}_n)$.

(iii) $Z = E[\phi_n(\mathcal{L}^\dagger \mathcal{Q} \boldsymbol{\rho})]$, where $\boldsymbol{\rho} = (\log b_{12}, \ldots, \log b_{n-1,n}, \log b_{n1})^T$, and matrices $\mathcal{Q}, \mathcal{L}$ are the Laplacian matrix and the incidence matrix, respectively, of the graph associated to B.

(iv) If (i, j) is an unknown entry of B, then the (i, j) entry of D and Z are equal.

Note that the oriented graph with n vertices and edges associated to B is

$$\{1 \to 2, 2 \to 3, \ldots, n-1 \to n, n \to 1\}.$$

Let's define the following matrix J directly associated to the structure of B

$$J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{4}$$

A number of considerations enable us to prove the following result.

Theorem 3 *Let $B \in M_n$ be a reciprocal incomplete matrix with known entries b_{12}, b_{23}, $\ldots$, $b_{n-1,n}$, b_{n1}. Under the notation of Theorem 2, one has*

$$Z = E\left[\phi_n\left(\frac{1}{2n}\sum_{k=0}^{n-1}(n-2k-1)J^k\boldsymbol{\rho}\right)\right],$$

where the matrix J is given in (4) and ϕ_n is the linear mapping $\phi_n : \mathbb{R}^n \to M_n$ given by $(\phi_n(\mathbf{v}))_{ij} = v_i - v_j$.

This expression shows that neither inverses nor pseudo-inverses have to be computed. Also, $\boldsymbol{\rho}, J\boldsymbol{\rho}, J^2\boldsymbol{\rho}, \ldots, J^{n-1}\boldsymbol{\rho}$ are trivial to compute. For example, for $n = 4$, one has $J^0 = J^4 = I_4$,

$$J^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad J^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad J^3 = J^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

If $\boldsymbol{\rho} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ and $\widehat{\boldsymbol{\rho}} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$, then $J^0\boldsymbol{\rho}$ are the $1, 2, 3, 4$ entries of $\widehat{\boldsymbol{\rho}}$; $J^1\boldsymbol{\rho}$ are the $2, 3, 4, 5$ entries of $\widehat{\boldsymbol{\rho}}$; $J^2\boldsymbol{\rho}$ are the $3, 4, 5, 6$ entries of $\widehat{\boldsymbol{\rho}}$; and $J^3\boldsymbol{\rho}$ are the $4, 5, 6, 7$ entries of $\widehat{\boldsymbol{\rho}}$.

3 Conclusions

Making too many comparisons may be strenuous and time-consuming, and lead to wrong and harmful conclusions. It is indispensable to focus on its reduction [1]. There is not a general solution to the problem of finding an optimal sample of PCs to be issued so that card(sample) $< n(n-1) = 2$ and still produce sound DM. We have given a solution in which one compares just the elements of a balanced and unbiased subset of items. The solution, according to Theorem 3, is obtained through elementary, simple calculations.

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